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Existence Results for Toda Systems With Sign-Changing Prescribed Functions: Part II

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Abstract

Let (M, g) be a compact Riemann surface with area 1. We investigate the Toda system

$$\begin{cases} -\Delta u_1 = 2\rho_1(h_1 e^{u_1} - 1) - \rho_2(h_2 e^{u_2} - 1), \\ -\Delta u_2 = 2\rho_2(h_2 e^{u_2} - 1) - \rho_1(h_1 e^{u_1} - 1), \end{cases} \tag{0.1}$$

on (M, g) where $\rho_1, \rho_2 \in (0, 4\pi]$, and h_1 and h_2 are two C^2 functions on M. When some ρ_i equals 4π, Eq. (0.1) becomes critical with respect to the Moser-Trudinger inequality for the Toda system, making the existence problem significantly more challenging. In their seminal article (Comm. Pure Appl. Math., 59 (2006), no. 4, 526–558), Jost, Lin, and Wang established sufficient conditions for the existence of solutions to Eq. (0.1) when $\rho_1 = 4\pi$, $\rho_2 \in (0, 4\pi)$ or $\rho_1 = \rho_2 = 4\pi$, assuming that h_1 and h_2 are both positive. In our previous paper we extended these results to allow h_1 and h_2 to change signs in the case $\rho_1 = 4\pi$, $\rho_2 \in (0, 4\pi)$. In this paper we further extend the study to prove that Jost-Lin-Wang's sufficient conditions remain valid even when h_1 and h_2 can change signs and $\rho_1 = \rho_2 = 4\pi$. Our proof relies on an improved version of the Moser-Trudinger inequality for the Toda system, along with dedicated analyses similar to Brezis-Merle type and the use of Pohozaev identities.

1 Introduction

Let (M, g) be a compact Riemann surface with area 1 , and let $h(x)$ be a smooth function on M . The celebrated Kazdan-Warner problem [20] seeks to understand under what conditions on the prescribed function h the following sub-linear elliptic

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partial differential equation has a solution:

$$-\Delta u = 8\pi (he^u - 1). \tag{1.1}$$

This problem is often referred to as the “Nirenberg problem" when M is the standard sphere, and it has been extensively studied [3–8, 13, 20, 26, 27, 34], among others. When M is a general Riemann surface, Eq. (1.1) arises in the context of the so-called Chern-Simons Higgs theory [2, 14, 17, 32], among others. The coefficient 8π in Eq. (1.1) is critical with respect to the Moser-Trudinger inequality (cf. [9, 11]):

$$\log \int_M e^u dv_g \le \frac{1}{16\pi} \int_M |\nabla u|^2 dv_g + \int_M u dv_g + C. \tag{1.2}$$

Thus, the existence problem for Eq. (1.1) becomes intricate. Ding-Jost-Li-Wang [9] addressed this problem using a variational approach by minimizing the functional

$$I(u) = \frac{1}{2} \int_M |\nabla u|^2 dv_g + 8\pi \int_M u dv_g \tag{1.3}$$

on the set

$$X = \left\{ u \in H^1(M) : \int_M he^u dv_g = 1 \right\}. \tag{1.4}$$

Assuming h is positive, they showed that if

$$\Delta \log h(p) + 8\pi - 2K(p) > 0, \tag{1.5}$$

where K is the Gaussian curvature of M , and p is any maximum point of the sum of $2 \log h$ and the regular part of the Green function, then I attains its infimum in X and Eq. (1.1) has a minimal solution. Yang and the second author [36] relaxed the positivity condition on h to nonnegativity. Recently, the first author and Zhu [29] and the second author [38] independently proved that the condition (1.5) remains sufficient even for sign-changed prescribed functions. All these works are based on the variational approach. These results were also obtained using the flow method [22, 24, 28, 33].

In this paper, we continue to investigate the Toda system (0.1), which can be viewed as the Frenet frame of holomorphic curves in $\mathbb{CP}^2$ (see [16]) from a geometric perspective, and also arises in physics in the study of the nonabelian Chern-Simons theory in the self-dual case, where a scalar Higgs field is coupled to a gauge potential; see [10, 31, 35]. Our focus is on the existence result, and we aim to explore the variational approach developed in [9, 18, 19, 21]. Recall that (0.1) represents the critical point of the functional

$$J_{\rho_1,\rho_2}(u_1,u_2) = \frac{1}{3}\int_M \left(|\nabla u_1|^2 + \nabla u_1 \nabla u_2 + |\nabla u_2|^2\right) dv_g + \rho_1 \int_M u_1 dv_g + \rho_2 \int_M u_2 dv_g$$

on the set

$$\mathcal{H} = \left\{(u_1,u_2) \in H^1(M) \times H^1(M) : \int_M h_1 e^{u_1} dv_g = \int_M h_2 e^{u_2} dv_g = 1\right\}.$$

From the Moser-Trudinger inequality for the Toda system

$$\inf_{(u_1,u_2)\in\mathcal{H}} J_{\rho_1,\rho_2} \geq -C \quad \text{iff} \quad \rho_1, \rho_2 \in (0, 4\pi], \tag{1.6}$$

derived by Jost-Wang [18], it is known that J_{ρ_1,ρ_2} is coercive and attains its infimum when $\rho_1, \rho_2 \in (0, 4\pi)$. However, when either ρ_1 or ρ_2 equals 4π, the existence problem becomes more intricate. In this paper, we shall focus on minimal type solutions. Consequently, we assume $\rho_i \leq 4\pi$, $i = 1, 2$, throughout the discussion.

Let us review the existence result when one of ρ_1 and ρ_2 equals 4π, which was obtained by Jost, Lin, and Wang when h_1 and h_2 are both positive.

Theorem 1.1 *(Jost-Lin-Wang [19]) Let (M, g) be a compact Riemann surface with Gaussian curvature K. Let $h_1, h_2 \in C^2(M)$ be two positive functions and $\rho_2 \in (0, 4\pi)$. Suppose that*

$$\Delta \log h_1(x) + (8\pi - \rho_2) - 2K(x) > 0, \quad \forall x \in M, \tag{1.7}$$

then $J_{4\pi,\rho_2}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies

$$\begin{cases} -\Delta u_1 = 8\pi(h_1 e^{u_1} - 1) - \rho_2(h_2 e^{u_2} - 1), \\ -\Delta u_2 = 2\rho_2(h_2 e^{u_2} - 1) - 4\pi(h_1 e^{u_1} - 1). \end{cases} \tag{1.8}$$

When $\rho_1 = \rho_2 = 4\pi$ and both h_1 and h_2 are positive, we have:

Theorem 1.2 *(Li-Li [21], Jost-Lin-Wang [19]) Let (M, g) be a compact Riemann surface with Gaussian curvature K. Let $h_1, h_2 \in C^2(M)$ be two positive functions. Suppose that*

$$\min\{\Delta \log h_1(x), \Delta \log h_2(x)\} + 4\pi - 2K(x) > 0, \quad \forall x \in M, \tag{1.9}$$

then $J_{4\pi,4\pi}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies

$$\begin{cases} -\Delta u_1 = 8\pi(h_1 e^{u_1} - 1) - 4\pi(h_2 e^{u_2} - 1), \\ -\Delta u_2 = 8\pi(h_2 e^{u_2} - 1) - 4\pi(h_1 e^{u_1} - 1). \end{cases} \tag{1.10}$$

We remark that Li-Li obtained Theorem 1.2 when $h_1 = h_2 = 1$ and Jost-Lin-Wang obtained it for general positive h_1 and h_2.

Motivated mostly by works in [9, 29, 36, 38], we would like to relax the positivity of h_1 and h_2 in conditions (1.7) and (1.9). In our former paper [30], we successfully relaxed the positivity of h_1 and h_2 in (1.7) and proved the following theorem.

Theorem 1.3 *(Sun-Zhu [30]) Let (M, g) be a compact Riemann surface with the Gaussian curvature K. Let $h_1, h_2 \in C^2(M)$ which are positive somewhere and $\rho_2 \in (0, 4\pi)$. Denote $M_1^+ = \{x \in M : h_1(x) > 0\}$. If*

$$\Delta \log h_1(x) + (8\pi - \rho_2) - 2K(x) > 0, \quad \forall x \in M_1^+,$$

then $J_{4\pi,\rho_2}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies (1.8).

In this paper, we shall show that the positivity of h_1 and h_2 in (1.9) in Theorem 1.2 can also be relaxed. Precisely,

Theorem 1.4 *Let (M, g) be a compact Riemann surface with the Gaussian curvature K. Let $h_1, h_2 \in C^2(M)$ which are positive somewhere. Denote $M_i^+ = \{x \in M : h_i(x) > 0\}$ for $i = 1, 2$. If*

$$\Delta \log h_i(x) + 4\pi - 2K(x) > 0, \quad \forall x \in M_i^+, \quad i = 1, 2, \tag{1.11}$$

then $J_{4\pi,4\pi}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies (1.10).

At the end of the introduction, we would like to outline the proof of Theorem 1.4. For any $\epsilon \in (0, 4\pi)$, we assume that $J_{4\pi-\epsilon,4\pi-\epsilon}(u_1^\epsilon, u_2^\epsilon) = \inf_{\mathcal{H}} J_{4\pi-\epsilon,4\pi-\epsilon}$, then $(u_1^\epsilon, u_2^\epsilon)$ satisfies a Toda type system. If $(u_1^\epsilon, u_2^\epsilon)$ converges to some $(u_1, u_2) \in \mathcal{H}$ as $\epsilon \to 0$, then $J_{4\pi,4\pi}(u_1, u_2) = \inf_{\mathcal{H}} J_{4\pi,4\pi}$, and we are done. Otherwise, if $(u_1^\epsilon, u_2^\epsilon)$ does not converge in $\mathcal{H}$, we say that $(u_1^\epsilon, u_2^\epsilon)$ blows up. We show that there are three characterizations of the definition of blow-up, one of which is that $\overline{u_1^\epsilon} + \overline{u_2^\epsilon} \to -\infty$ as $\epsilon \to 0$. Here, $\overline{u_i^\epsilon}$ denotes the mean value of u_i^ϵ on M, for $i = 1, 2$. Based on this characterization, we divide the proof into three cases:

- Case 1: $\overline{u_1^\epsilon} \to -\infty$ and $\overline{u_2^\epsilon} \geq -C$ as $\epsilon \to 0$.
- Case 2: $\overline{u_1^\epsilon} \geq -C$ and $\overline{u_2^\epsilon} \to -\infty$ as $\epsilon \to 0$.
- Case 3: $\overline{u_1^\epsilon} \to -\infty$ and $\overline{u_2^\epsilon} \to -\infty$ as $\epsilon \to 0$.

Case 1 is similar to the situation where $\rho_1 = 4\pi$ and $\rho_2 \in (0, 4\pi)$, which has been proved by us in [30]. Case 2 follows from Case 1. Suppose we are in Case 3. Since h_1 and h_2 can change signs, we do not have directly the characterization of the blow-up set (Proposition 2.4 in [19]). More effort is needed to understand the blow-up set, which is one of the main contributions in this paper. Since the L^1 norm of $e^{u_i^\epsilon}$ is bounded, $e^{u_i^\epsilon} dv_g$ converges to some nonnegative measure μ_i, and $\mathrm{supp}\mu_i \neq \emptyset$, for $i = 1, 2$. By Fatou's lemma, $\mathrm{supp}\mu_i$ is a finite set, for $i = 1, 2$. With the help of the improved Moser-Trudinger inequality for the Toda system, we know that at least one $\mathrm{supp}\mu_i$ is a single point set. By the Pohozaev identity, we derive that $\mathrm{supp}\mu_j$ $(j \neq i)$

is also a single point set, which is different from suppμ_i. Then we can show that $h_1\mu_1 = \delta_{x_1}$ and $h_2\mu_2 = \delta_{x_2}$ with $x_1 \neq x_2$. Based on this, we derive a dedicated lower bound for $J_{4\pi,4\pi}$. Finally, we use the test functions $(\phi_1^\epsilon, \phi_2^\epsilon)$ constructed in [21] to show that under condition (1.11), $J_{4\pi,4\pi}(\phi_1^\epsilon, \phi_2^\epsilon)$ are strictly less than the lower bound derived before. This contradiction tells us that $(u_1^\epsilon, u_2^\epsilon)$ does not blow up, which proves Theorem 1.4.

There are some related works which deal with sign-changing potential in the critical case with respect to Moser-Trudinger type inequalities (cf. [25, 37]). We believe that our techniques could be used to deal with other nonlinear existence problems with sign-changing prescribed functions.

The outline of the rest of the paper is following: In Sect. 2, we do some analysis on the minimizing sequence; In Sect. 3, we estimate the lower bound for $J_{4\pi,4\pi}$ in Case 3; Finally, we complete the proof of Theorem 1.4 in the last section. Throughout the whole paper, the constant C is varying from line to line and even in the same line, we do not distinguish sequence and its subsequences since we only care about the existence result.

2 Analysis on the minimizing sequence

In this section, we conduct an analysis on the minimizing sequence.

Given inequality (1.6), it is known that for any $\epsilon \in (0, 4\pi)$, there exists a pair $(u_1^\epsilon, u_2^\epsilon) \in \mathcal{H}$ such that

$$J_{4\pi-\epsilon,4\pi-\epsilon}(u_1^\epsilon, u_2^\epsilon) = \inf_{\mathcal{H}} J_{4\pi-\epsilon,4\pi-\epsilon}.$$

Direct calculations reveal the following equations on M:

$$\begin{cases} -\Delta u_1^\epsilon = (8\pi - 2\epsilon)(h_1 e^{u_1^\epsilon} - 1) - (4\pi - \epsilon)(h_2 e^{u_2^\epsilon} - 1), \\ -\Delta u_2^\epsilon = (8\pi - 2\epsilon)(h_2 e^{u_2^\epsilon} - 1) - (4\pi - \epsilon)(h_1 e^{u_1^\epsilon} - 1). \end{cases} \tag{2.1}$$

Let $\overline{u_i^\epsilon} = \int_M u_i^\epsilon dv_g$ and $m_i^\epsilon = \max_M u_i^\epsilon = u_i^\epsilon(x_i^\epsilon)$ for some $x_i^\epsilon \in M$. Assume $x_i^\epsilon \to p_i$ as $\epsilon \to 0$. The following three lemmas can be derived similarly to those in [30, section 2].

Lemma 2.1 *There exist two positive constants C_1 and C_2 such that*

$$C_1 \leq \int_M e^{u_i^\epsilon} dv_g \leq C_2, \quad i = 1, 2.$$

Lemma 2.2 *For any $s \in (1, 2)$, $\|\nabla u_i^\epsilon\|_{L^s(M)} \leq C$ for $i = 1, 2$.*

Lemma 2.3 *The following statements are equivalent:*

(i) $m_1^\epsilon + m_2^\epsilon \to +\infty$ *as* $\epsilon \to 0$,
(ii) $\int_M (|\nabla u_1^\epsilon|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + |\nabla u_2^\epsilon|^2) dv_g \to +\infty$ *as* $\epsilon \to 0$,

(iii) $\overline{u_1^\epsilon} + \overline{u_2^\epsilon} \to -\infty$ *as* $\epsilon \to 0$.

Definition 2.1 (Blow-Up) We say $(u_1^\epsilon, u_2^\epsilon)$ blows up if any one of the conditions in Lemma 2.3 holds.

If $(u_1^\epsilon, u_2^\epsilon)$ does not blow up, then by Lemma 2.3, one can show that $(u_1^\epsilon, u_2^\epsilon)$ converges to some (u_1, u_2) in $\mathcal{H}$ which minimizes $J_{4\pi,4\pi}$. The proof of Theorem 1.4 terminates in this case. Therefore, without loss of generality we may assume $(u_1^\epsilon, u_2^\epsilon)$ blows up in the rest of this paper.

By Lemma 2.3 (iii), we divide the proof into the following three cases:

Case 1 $\overline{u_1^\epsilon} \to -\infty$, $\overline{u_2^\epsilon} \geq -C$ as $\epsilon \to 0$;

Case 2 $\overline{u_1^\epsilon} \geq -C$, $\overline{u_2^\epsilon} \to -\infty$ as $\epsilon \to 0$;

Case 3 $\overline{u_1^\epsilon} \to -\infty$, $\overline{u_2^\epsilon} \to -\infty$ as $\epsilon \to 0$.

Suppose we are in Case 1, by checking the proofs in [30] carefully, we find that $\rho_2 < 4\pi$ is used to show $\overline{u_2^\epsilon} \geq -C$ which happens to be the situation in Case 1. And at any other places $\rho_2 < 4\pi$ can be replaced by $\rho_2 = 4\pi$. By Theorem 1.3, if

$$\Delta \log h_1(x) + 4\pi - 2K(x) > 0 \ \text{ for } x \in M_1^+,$$

where $M_1^+ = \{x \in M : \ h_1(x) > 0\}$, $J_{4\pi,4\pi}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies (1.10).

Suppose we are in Case 2, similar as Case 1, we know that, if

$$\Delta \log h_2(x) + 4\pi - 2K(x) > 0 \ \text{ for } x \in M_2^+,$$

where $M_2^+ = \{x \in M : \ h_2(x) > 0\}$, then $J_{4\pi,4\pi}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies (1.10).

Suppose we are in Case 3, by Lemma 2.2, there exist G_i, $i = 1, 2$ such that $u_i^\epsilon - \overline{u_i^\epsilon} \rightharpoonup G_i$ weakly in $W^{1,s}(M)$ for any $1 < s < 2$ as $\epsilon \to 0$. Since $(e^{u_i^\epsilon})$ is bounded in $L^1(M)$ we may extract a subsequence (still denoted $e^{u_i^\epsilon}$) such that $e^{u_i^\epsilon}$ converges in the sense of measures on M to some nonnegative bounded measure μ_i for $i = 1, 2$. We set

$$\gamma_1 = 8\pi h_1 \mu_1 - 4\pi h_2 \mu_2, \quad \gamma_2 = 8\pi h_2 \mu_2 - 4\pi h_1 \mu_1$$

and

$$S_i = \{x \in M : |\gamma_i(\{x\})| \geq 4\pi\}, \ i = 1, 2.$$

Let $S = S_1 \cup S_2$. By Theorem 1 in [1], we have

Lemma 2.4 *For any* $\Omega \subset\subset M \setminus S$, *there holds*

$$u_i^\epsilon - \overline{u_i^\epsilon} \text{ is uniformly bounded in } \Omega, \ i = 1, 2. \tag{2.2}$$

Proof $\forall x \in M \setminus S$, we have $B_\delta(x) \subset\subset M \setminus S$ for sufficiently small $\delta > 0$. Consider the equation

$$\begin{cases} -\Delta w_1^\epsilon = (8\pi - 2\epsilon)h_1 e^{u_1^\epsilon} - (4\pi - \epsilon)h_2 e^{u_2^\epsilon} := f_\epsilon & \text{in } B_\delta(x), \\ w_1^\epsilon = 0 & \text{on } \partial B_\delta(x). \end{cases}$$

Since $|\gamma_1(\{x\})| < 4\pi$, we have $\|f_\epsilon\|_{L^1(B_\delta(x))} < 4\pi$ for sufficiently small $\epsilon > 0$ and $\delta > 0$. Fix such a δ. Define $w_2^\epsilon = u_1^\epsilon - \overline{u_1^\epsilon} - w_1^\epsilon$, then $-\Delta w_2^\epsilon = -(4\pi - \epsilon)$ in $B_\delta(x)$. By Theorem 4.1 in [12] and Lemma 2.2, we have

$$\begin{aligned} \sup_{B_{\delta/2}(x)} w_2^\epsilon \leq & C\left(\|w_2^\epsilon\|_{L^1(B_\delta(x))} + C\right) \\ \leq & C\left(\|u_1^\epsilon - \overline{u_1^\epsilon}\|_{L^1(M)} + \|v_1^\epsilon\|_{L^1(B_\delta(x))} + C\right) \\ \leq & C\left(\|\nabla u_1^\epsilon\|_{L^s(M)} + \|w_1^\epsilon\|_{L^1(B_\delta(x))} + C\right) \\ \leq & C\left(\|w_1^\epsilon\|_{L^1(B_\delta(x))} + C\right). \end{aligned}$$

It follows from Theorem 1 in [1] that $e^{s_1|w_1^\epsilon|}$ is bounded in $B_\delta(x)$ for some $s_1 > 1$, which yields that

$$\|w_1^\epsilon\|_{L^1(B_\delta(x))} \leq C.$$

So we have

$$\sup_{B_{\delta/2}(x)} w_2^\epsilon \leq C.$$

Then

$$\begin{aligned} \int_{B_{\delta/2}(x)} e^{s_1 u_1^\epsilon} dv_g = & \int_{B_{\delta/2}(x)} e^{s_1 \overline{u_1^\epsilon}} e^{s_1 w_2^\epsilon} e^{s_1 w_1^\epsilon} dv_g \\ \leq & C \int_{B_{\delta/2}(x)} e^{s_1 |w_1^\epsilon|} dv_g \\ \leq & C. \end{aligned}$$

Similarly, we have $\int_{B_{\delta/2}(x)} e^{s_2 u_2^\epsilon} dv_g \leq C$ for some $s_2 > 1$ and sufficiently small $\delta > 0$. Then (2.2) follows from the standard elliptic estimates and we finish the proof. □

Since $(u_1^\epsilon, u_2^\epsilon)$ blows up, we know S is not empty. Otherwise, by using a finite covering argument and inequality (2.2), we would have $\|u_i^\epsilon - \overline{u_i^\epsilon}\|_{L^\infty(M)} \leq C$, which implies $m_i^\epsilon \leq C$. This contradicts Lemma 2.3 (i). By the definition of S, for any $x \in S$,

$$\mu_1(\{x\}) \geq \frac{1}{4 \max_M |h_1|} \quad \text{or} \quad \mu_2(\{x\}) \geq \frac{1}{4 \max_M |h_2|}.$$

In view of μ_1 and μ_2 are bounded, S is a finite set. We denote $S = \{x_l\}_{l=1}^L$. It follows from (2.2) and Fatou's lemma that

Lemma 2.5 *We have*

$$\mu_i = \sum_{l=1}^{L} \mu_i(\{x_l\})\delta_{x_l}, \quad i = 1, 2, \tag{2.3}$$

where δ_x is the Dirac distribution.

Proof For any closed set $V \subset M$, we need to show

$$\mu_i(V) = \sum_{l=1}^{L} \mu_i(V \cap \{x_l\}),\ i = 1, 2. \tag{2.4}$$

In fact, we have $B_r(x_l) \cap B_r(x_m) = \emptyset$ for sufficiently small r and $l \neq m \in \{1, \cdots, L\}$. Then

$$\begin{aligned}\int_V e^{u_i^\epsilon} dv_g &= \int_{V \setminus \bigcup_{l=1}^L B_r(x_l)} e^{u_i^\epsilon} dv_g + \int_{V \cap \left(\bigcup_{l=1}^L B_r(x_l)\right)} e^{u_i^\epsilon} dv_g \\ &= \int_{V \setminus \bigcup_{l=1}^L B_r(x_l)} e^{u_i^\epsilon - \overline{u_i^\epsilon}} e^{\overline{u_i^\epsilon}} dv_g + \sum_{l=1}^{L} \int_{V \cap B_r(x_l)} e^{u_i^\epsilon} dv_g. \end{aligned} \tag{2.5}$$

Since $\overline{u_i^\epsilon} \to -\infty$ and (2.2), it follows from Fatou's lemma that

$$\liminf_{\epsilon \to 0} \int_{V \setminus \bigcup_{l=1}^L B_r(x_l)} e^{u_i^\epsilon - \overline{u_i^\epsilon}} e^{\overline{u_i^\epsilon}} dv_g = 0.$$

Letting $\epsilon \to 0$ in both sides of (2.5) first and then $r \to 0$, we obtain (2.4) and finish the proof. □

It follows from Lemma 2.1 and (2.3) that supp$\mu_i \neq \emptyset$, $i = 1, 2$. If there are at least two points in each suppμ_i, then by the improved Moser-Trudinger inequality for Toda system (cf. [23, Proposition 2.5]), for any $\epsilon' > 0$, there exists some $C = C(\epsilon') > 0$ such that

$$\begin{aligned}\log \int_M e^{u_1^\epsilon} dv_g + \log \int_M e^{u_2^\epsilon} dv_g \le \frac{1+\epsilon'}{24\pi} \int_M (|\nabla u_1^\epsilon|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + |\nabla u_2^\epsilon|^2) dv_g \\ + \overline{u_1^\epsilon} + \overline{u_2^\epsilon} + C.\end{aligned}$$

By choosing $\epsilon' = 1/3$ and using Lemma 2.1, we have

$$\frac{1}{3} \int_M (|\nabla u_1^\epsilon|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + |\nabla u_2^\epsilon|^2) dv_g \ge -6\pi(\overline{u_1^\epsilon} + \overline{u_2^\epsilon}) - C.$$

This, combining with the fact that $J_{4\pi-\epsilon,4\pi-\epsilon}(u_1^\epsilon, u_2^\epsilon)$ is bounded, shows that

$$\overline{u_1^\epsilon} + \overline{u_2^\epsilon} \geq -C,$$

which contradicts the assumption that $(u_1^\epsilon, u_2^\epsilon)$ blows up. Hence, either suppμ_1 or suppμ_2 has only one point. Without loss of generality, we assume that suppμ_1 has only one point and supp$\mu_1 = \{x_1\}$ since supp$\mu_1 \subset S$. By noticing that $\int_M h_1 e^{u_1^\epsilon} dv_g = 1$, we have

$$h_1\mu_1 = \delta_{x_1}. \tag{2.6}$$

The following result which is based on Pohozaev identities is very important in the understanding of blow-up set.

Lemma 2.6 *Denote by $h_i\mu_i = \sigma_i$ for $i = 1, 2$, we have*

$$\sigma_1^2(\{x_l\}) + \sigma_2^2(\{x_l\}) - \sigma_1(\{x_l\})\sigma_2(\{x_l\}) = \sigma_1(\{x_l\}) + \sigma_2(\{x_l\}), \tag{2.7}$$

where $l = 1, 2, \cdots, L$.

Proof Using (2.2), (2.3) and (2.6), we have G_1 and G_2 satisfy the following equation

$$\begin{cases} -\Delta G_1 = 8\pi(\delta_{x_1} - 1) - 4\pi(h_2 \sum_{l=1}^L \mu_2(\{x_l\})\delta_{x_l} - 1), \\ -\Delta G_2 = 8\pi(h_2 \sum_{l=1}^L \mu_2(\{x_l\})\delta_{x_l} - 1) - 4\pi(\delta_{x_1} - 1), \\ \int_M G_1 dv_g = \int_M G_2 dv_g = 0. \end{cases} \tag{2.8}$$

It follows from standard elliptic estimates that

$$u_i^\epsilon - \overline{u_i^\epsilon} \to G_i \ \text{ in } \ C^2_{\text{loc}}(M \setminus S), \ i = 1, 2. \tag{2.9}$$

Let $(B_\delta(x_l);\, (x^1, x^2))$ be a local coordinate system around x_l and we assume the metric to be

$$g|_\Omega = e^\phi((dx^1)^2 + (dx^2)^2)$$

with $\phi(0) = 0$ and $\nabla_{\mathbb{R}^2}\phi(0) = 0$. Here $\nabla_{\mathbb{R}^2} = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$. Similar as the proof of in [21, page 708], we know that

$$G_i = -\frac{\gamma_i(\{x_l\})}{2\pi}\log r + \psi_i, \ \ i = 1, 2, \tag{2.10}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2}$ and ψ_i is a smooth function near x_l. In this coordinate system, (2.1) can be reduced to

$$\begin{cases} -\Delta_{\mathbb{R}^2} u_1^\epsilon = (8\pi - 2\epsilon)e^\phi(h_1 e^{u_1^\epsilon} - 1) - (4\pi - \epsilon)e^\phi(h_2 e^{u_2^\epsilon} - 1), \\ -\Delta_{\mathbb{R}^2} u_2^\epsilon = (8\pi - 2\epsilon)e^\phi(h_2 e^{u_2^\epsilon} - 1) - (4\pi - \epsilon)e^\phi(h_1 e^{u_1^\epsilon} - 1) \end{cases} \tag{2.11}$$

for $|x| \leq \delta$, where $\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2}$ is the Laplacian in $\mathbb{R}^2$. We set

$$\hat{u}_i^\epsilon(x) = u_i^\epsilon(x) - (4\pi - \epsilon)\zeta(x),$$

where $\zeta(x)$ satisfies

$$\begin{cases} \Delta_{\mathbb{R}^2}\zeta = e^{\phi(x)} \ \text{ for } |x| \leq \delta, \\ \zeta(0) = 0 \ \text{ and } \ \nabla_{\mathbb{R}^2}\zeta(0) = 0. \end{cases}$$

It is clear that $\zeta(x) = O(|x|^2)$ for $|x| \leq \delta$. By (2.11) we know $\hat{u}_i^\epsilon$ satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2}\hat{u}_1^\epsilon = (8\pi - 2\epsilon)\hat{h}_1 e^{\hat{u}_1^\epsilon} - (4\pi - \epsilon)\hat{h}_2 e^{\hat{u}_2^\epsilon}, \\ -\Delta_{\mathbb{R}^2}\hat{u}_2^\epsilon = (8\pi - 2\epsilon)\hat{h}_2 e^{\hat{u}_2^\epsilon} - (4\pi - \epsilon)\hat{h}_1 e^{\hat{u}_1^\epsilon} \end{cases} \tag{2.12}$$

for $|x| \leq \delta$, where

$$\hat{h}_i(x) = e^{\phi(x)} h_i(x) e^{(4\pi - \epsilon)\zeta(x)}, \quad i = 1, 2. \tag{2.13}$$

It follows from the choice of $\phi(x)$ and (2.13) that

$$\hat{h}_i(0) = h_i(x_l) \ \text{ and } \ \nabla_{\mathbb{R}^2}\hat{h}_i(0) = \nabla h_i(x_l). \tag{2.14}$$

From equation (2.12) we have the Pohozaev identities as follows:

$$\begin{aligned} &-\delta \int_{\partial\mathbb{B}_\delta(0)} \left(\left(\frac{\partial \hat{u}_1^\epsilon}{\partial r} \right)^2 - \frac{1}{2}|\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon|^2 \right) ds \\ &= (8\pi - 2\epsilon)\delta \int_{\partial\mathbb{B}_\delta(0)} \hat{h}_1 e^{\hat{u}_1^\epsilon} ds - (8\pi - 2\epsilon) \int_{\mathbb{B}_\delta(0)} (2\hat{h}_1 e^{\hat{u}_1^\epsilon} + x \cdot \nabla_{\mathbb{R}^2}\hat{h}_1 e^{\hat{u}_1^\epsilon}) dx \\ &\quad - (4\pi - \epsilon) \int_{\mathbb{B}_\delta(0)} x \cdot \nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon \hat{h}_2 e^{\hat{u}_2^\epsilon} dx, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &-\delta \int_{\partial\mathbb{B}_\delta(0)} \frac{\partial \hat{u}_1^\epsilon}{\partial r} \frac{\partial \hat{u}_2^\epsilon}{\partial r} ds + \int_{\mathbb{B}_\delta(0)} \left(\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon \nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon + \sum_{j=1}^{2} x \cdot \left(\nabla_{\mathbb{R}^2} \frac{\partial \hat{u}_2^\epsilon}{\partial x^j} \right) \frac{\partial \hat{u}_1^\epsilon}{\partial x^j} \right) dx \\ &= -(4\pi - \epsilon)\delta \int_{\partial\mathbb{B}_\delta(0)} \hat{h}_2 e^{\hat{u}_2^\epsilon} ds + (8\pi - 2\epsilon) \int_{\mathbb{B}_\delta(0)} \left(\hat{h}_2 e^{\hat{u}_2^\epsilon} + x \cdot \nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon \hat{h}_1 e^{\hat{u}_1^\epsilon} \right) dx \\ &\quad + (4\pi - \epsilon) \int_{\mathbb{B}_\delta(0)} x \cdot \nabla_{\mathbb{R}^2}\hat{h}_2 e^{\hat{u}_2^\epsilon} dx, \end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
&-\delta\int_{\partial\mathbb{B}_\delta(0)}\frac{\partial\hat{u}_1^\epsilon}{\partial r}\frac{\partial\hat{u}_2^\epsilon}{\partial r}ds+\int_{\mathbb{B}_\delta(0)}\left(\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon\nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon+\sum_{j=1}^{2}x\cdot\left(\nabla_{\mathbb{R}^2}\frac{\partial\hat{u}_1^\epsilon}{\partial x^j}\right)\frac{\partial\hat{u}_2^\epsilon}{\partial x^j}\right)dx\\
&\quad=-(4\pi-\epsilon)\delta\int_{\partial\mathbb{B}_\delta(0)}\hat{h}_1e^{\hat{u}_1^\epsilon}ds+(8\pi-2\epsilon)\int_{\mathbb{B}_\delta(0)}\left(\hat{h}_1e^{\hat{u}_1^\epsilon}+x\cdot\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon\hat{h}_2e^{\hat{u}_2^\epsilon}\right)dx\\
&\qquad+(4\pi-\epsilon)\int_{\mathbb{B}_\delta(0)}x\cdot\nabla_{\mathbb{R}^2}\hat{h}_1e^{\hat{u}_1^\epsilon}dx,
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
&-\delta\int_{\partial\mathbb{B}_\delta(0)}\left(\left(\frac{\partial\hat{u}_2^\epsilon}{\partial r}\right)^2-\frac{1}{2}|\nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon|^2\right)ds\\
&\quad=(8\pi-2\epsilon)\delta\int_{\partial\mathbb{B}_\delta(0)}\hat{h}_2e^{\hat{u}_2^\epsilon}ds-(8\pi-2\epsilon)\int_{\mathbb{B}_\delta(0)}(2\hat{h}_2e^{\hat{u}_2^\epsilon}+x\cdot\nabla_{\mathbb{R}^2}\hat{h}_2e^{\hat{u}_2^\epsilon})dx\\
&\qquad-(4\pi-\epsilon)\int_{\mathbb{B}_\delta(0)}x\cdot\nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon\hat{h}_1e^{\hat{u}_1^\epsilon}dx.
\end{aligned}
\tag{2.18}
$$

Two times both sides of (2.15) and (2.18) and then plus each sides of them with (2.16) and (2.17), we have

$$
\begin{aligned}
&-2\delta\int_{\partial\mathbb{B}_\delta(0)}\left(\left(\frac{\partial\hat{u}_1^\epsilon}{\partial r}\right)^2+\left(\frac{\partial\hat{u}_2^\epsilon}{\partial r}\right)^2+\frac{\partial\hat{u}_1^\epsilon}{\partial r}\frac{\partial\hat{u}_2^\epsilon}{\partial r}\right)ds\\
&\quad+\delta\int_{\partial\mathbb{B}_\delta(0)}\left(|\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon|^2+|\nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon|^2+\nabla_{\mathbb{R}^2}\hat{u}_1^\epsilon\nabla_{\mathbb{R}^2}\hat{u}_2^\epsilon\right)ds\\
&=3(4\pi-\epsilon)\delta\int_{\partial\mathbb{B}_\delta(0)}\left(\hat{h}_1e^{\hat{u}_1^\epsilon}+\hat{h}_2e^{\hat{u}_2^\epsilon}\right)ds-6(4\pi-\epsilon)\int_{\mathbb{B}_\delta(0)}\left(\hat{h}_1e^{\hat{u}_1^\epsilon}+\hat{h}_2e^{\hat{u}_2^\epsilon}\right)dx\\
&\quad-3(4\pi-\epsilon)\int_{\mathbb{B}_\delta(0)}\left(x\cdot\nabla_{\mathbb{R}^2}\hat{h}_1e^{\hat{u}_1^\epsilon}+x\cdot\nabla_{\mathbb{R}^2}\hat{h}_2e^{\hat{u}_2^\epsilon}\right)dx.
\end{aligned}
\tag{2.19}
$$

Letting $\epsilon\to 0$ first and then $\delta\to 0$ in (2.19), by using (2.9), (2.10), (2.13) and (2.14) we conclude

$$
\begin{aligned}
&-2\pi\left[\left(\frac{\gamma_1(\{x_l\})}{2\pi}\right)^2+\left(\frac{\gamma_2(\{x_l\})}{2\pi}\right)^2+\frac{\gamma_1(\{x_l\})}{2\pi}\frac{\gamma_2(\{x_l\})}{2\pi}\right]\\
&=-24\pi\left[h_1(x_l)\mu_1(\{x_l\})+h_2(x_l)\mu_2(\{x_l\})\right].
\end{aligned}
\tag{2.20}
$$

Recalling that $h_i\mu_i=\sigma_i$ for $i=1,2$, then (2.20) reduces to (2.7), this ends the proof. □

Now we show by Lemma 2.6 that suppμ_2 also has one point which is different with x_1.

We know from (2.6) that $\sigma_1(\{x_1\}) = 1$ and $\sigma_1(\{x_l\}) = 0$ for any $l \geq 2$, taking this fact into (2.7) we obtain that

$$\sigma_2(\{x_1\}) = 0 \text{ or } \sigma_2(\{x_1\}) = 2;$$
$$\sigma_2(\{x_l\}) = 0 \text{ or } \sigma_2(\{x_l\}) = 1, \quad \forall l \geq 2.$$

Combining it with $\sigma_2(M) = \int_M h_2 e^{u_2^\epsilon} dv_g = 1$, we have

$$\sigma_2(\{x_m\}) = 1 \text{ for some } m \geq 2 \text{ and } \sigma_2(\{x_l\}) = 0 \ \forall l \in \{1, \cdots, L\} \setminus \{m\}.$$

Without loss of generality, we assume $m = 2$. Then we have

$$h_2\mu_2 = \sigma_2 = \delta_{x_2}. \tag{2.21}$$

We would like to collect (2.6) and (2.21) as the following lemma.

Lemma 2.7 *It holds that $h_1\mu_1 = \delta_{x_1}$ and $h_2\mu_2 = \delta_{x_2}$ with $x_1 \neq x_2$.*

To do blow-up analysis near x_i for $i = 1, 2$, one still needs the upper bound of u_j^ϵ for $j \in \{1, 2\} \setminus \{i\}$ near x_i. In fact, we have

Lemma 2.8 *Suppose r is a positive number which is less than $\mathrm{dist}(x_1, x_2)/2$ and makes $h_i > 0$ in $B_r(x_i)$ for $i = 1, 2$, there holds*

$$\sup_{B_{r/4}(x_i)} \left(u_j^\epsilon - \overline{u_j^\epsilon}\right) \leq C, \quad i, j \in \{1, 2\} \text{ and } i \neq j.$$

Proof For $i = 1$, we consider the solution of

$$\begin{cases} -\Delta v_1^\epsilon = (8\pi - 2\epsilon) h_2 e^{u_2^\epsilon} & \text{in } B_r(x_1), \\ v_1^\epsilon = 0 & \text{on } \partial B_r(x_1). \end{cases}$$

Denote by $v_2^\epsilon = u_2^\epsilon - \overline{u_2^\epsilon} - v_1^\epsilon$, then

$$-\Delta v_2^\epsilon = -(4\pi - \epsilon) - (4\pi - \epsilon) h_1 e^{u_1^\epsilon} \leq -(4\pi - \epsilon) \text{ in } B_r(x_1)$$

since $h_1 > 0$ in $B_r(x_1)$. By Theorem 8.17 in [15] (or Theorem 4.1 in [12]) and Lemma 2.2, we have

$$\begin{aligned} \sup_{B_{r/2}(x_1)} v_2^\epsilon \leq & C\left(\|(v_2^\epsilon)^+\|_{L^s(B_r(x_1))} + C\right) \\ \leq & C\left(\|u_2^\epsilon - \overline{u_2^\epsilon}\|_{L^s(M)} + \|v_1^\epsilon\|_{L^s(B_r(x_1))} + C\right) \\ \leq & C\left(\|\nabla u_2^\epsilon\|_{L^s(M)} + \|v_1^\epsilon\|_{L^s(B_r(x_1))} + C\right) \\ \leq & C\left(\|v_1^\epsilon\|_{L^s(B_r(x_1))} + C\right). \end{aligned}$$

Since $\int_{B_r(x_1)} |h_2| e^{u_2^\epsilon} dv_g \to 0$ as $\epsilon \to 0$, it follows from Theorem 1 in [1] that $\int_{B_r(x_1)} e^{t|v_1^\epsilon|} dv_g \leq C$ for some $t > 1$, which yields that

$$\|v_1^\epsilon\|_{L^s(B_r(x_1))} \leq C.$$

Then we have

$$\sup_{B_{r/2}(x_1)} v_2^\epsilon \leq C.$$

Note that

$$\begin{aligned}\int_{B_{r/2}(x_1)} e^{tu_2^\epsilon} dv_g =& \int_{B_{r/2}(x_1)} e^{t\overline{u_2^\epsilon}} e^{tv_2^\epsilon} e^{tv_1^\epsilon} dv_g \\ \leq& C \int_{B_{r/2}(x_1)} e^{t|v_1^\epsilon|} dv_g \\ \leq& C.\end{aligned}$$

By the standard elliptic estimates, we have

$$\|v_1^\epsilon\|_{L^\infty(B_{r/4}(x_1))} \leq C.$$

Therefore, we obtain that

$$u_2^\epsilon - \overline{u_2^\epsilon} \leq C \ \text{ in } \ B_{r/4}(x_1).$$

Similarly, we can prove

$$u_1^\epsilon - \overline{u_1^\epsilon} \leq C \ \text{ in } \ B_{r/4}(x_2).$$

This finishes the proof. □

Recalling that $\overline{u_i^\epsilon} \to -\infty$ and $\max_M u_i^\epsilon(x) = u_i^\epsilon(x_i^\epsilon)$, $i = 1, 2$, it follows from (2.9), Lemmas 2.7 and 2.8 that

$$x_i^\epsilon \to x_i \ \text{ as } \ \epsilon \to 0, \ i = 1, 2.$$

Let $(\Omega_i; (x^1, x^2))$ be an isothermal coordinate system around x_i and we assume the metric to be

$$g|_{\Omega_i} = e^{\phi_i}((dx^1)^2 + (dx^2)^2), \ \ \phi_i(0) = 0.$$

Similar as Case 1 in [21] and Lemma 2.5 in [9], we have

$$u_i^\epsilon(x_i^\epsilon + r_i^\epsilon x) - m_i^\epsilon \to -2\log(1 + \pi h_i(x_i)|x|^2), \ \ i = 1, 2,$$

where $m_i^\epsilon = \max_M u_i^\epsilon$ and $r_i^\epsilon = e^{-m_i^\epsilon/2}$.

By taking (2.21) into (2.8), we have

$$\begin{cases} -\Delta G_1 = 8\pi(\delta_{x_1} - 1) - 4\pi(\delta_{x_2} - 1), \\ -\Delta G_2 = 8\pi(\delta_{x_2} - 1) - 4\pi(\delta_{x_1} - 1), \\ \int_M G_1 dv_g = \int_M G_2 dv_g = 0. \end{cases}$$

Recalling that for any $s \in (1, 2)$, for $i = 1, 2$, we have $u_i^\epsilon - \overline{u_i^\epsilon} \to G_i$ weakly in $W^{1,s}(M)$ and strongly in $C^2_{\text{loc}}(M \setminus \{x_1, x_2\})$ as $\epsilon \to 0$.

It was proved by Li-Li in [21, page 708] that, in Ω_1,

$$G_1(x, x_1) = -4 \log r + A_1(x_1) + f_1, \quad G_2(x, x_1) = 2 \log r + A_2(x_1) + g_1,$$

where $r^2 = x_1^2 + x_2^2$, $A_i(x_1)$ $(i = 1, 2)$ are constants and f_1, g_1 are two smooth functions which are zero at x_1. In Ω_2,

$$G_1(x, x_2) = 2 \log r + A_1(x_2) + f_2, \quad G_2(x, x_2) = -4 \log r + A_2(x_2) + g_2,$$

where $A_i(x_2)$ $(i = 1, 2)$ are constants and f_2, g_2 are two smooth functions which are zero at x_2.

3 The lower bound for $J_{4\pi,4\pi}$ in Case 3

In this section, we shall derive an explicit lower bound of $J_{4\pi,4\pi}$ under the assumptions $(u_1^\epsilon, u_2^\epsilon)$ blows up and Case 3 happens.

Following closely the calculations in [21, Section 3], we have

$$\begin{aligned} J_{4\pi-\epsilon,4\pi-\epsilon}(u_1^\epsilon, u_2^\epsilon) \geq & -4\pi - 4\pi \log(\pi h_1(x_1)) - 2\pi A_1(x_1) \\ & -4\pi - 4\pi \log(\pi h_2(x_2)) - 2\pi A_2(x_2) \\ & + o_\epsilon(1) + o_L(1) + o_\delta(1). \end{aligned}$$

By letting $\epsilon \to 0$ first, then $L \to +\infty$ and then $\delta \to 0$, we obtain finally that

$$\begin{aligned} \inf_{\mathcal{H}} J_{4\pi,4\pi} \geq & -4\pi - 4\pi \log(\pi h_1(x_1)) - 2\pi A_1(x_1) \\ & -4\pi - 4\pi \log(\pi h_2(x_2)) - 2\pi A_2(x_2) \\ \geq & -8\pi - 8\pi \log \pi - 2\pi \max_{x \in M_1^+} (2 \log h_1(x) + A_1(x)) \\ & -2\pi \max_{x \in M_2^+} (2 \log h_2(x) + A_2(x)). \end{aligned} \tag{3.1}$$

4 Completion of the proof of Theorem 1.4

In this section, we shall use the test functions constructed in [21] to finish the proof of our main theorem.

Let ϕ_1^ϵ and ϕ_2^ϵ be defined as [21, Section 5]. Suppose that

$$2\log h_i(p_i) + A_i(p_i) = \max_{x\in M_i^+} (2\log h_i(x) + A_i(x)) \text{ for } i = 1, 2.$$

Following directly the calculations in [21, Section 5] and [30, Section 4], we obtain

$$\begin{aligned} J_{4\pi,4\pi}(\phi_1^\epsilon, \phi_2^\epsilon) \leq & - 8\pi - 8\pi \log\pi - 4\pi \log h_1(p_1) - 2\pi A_1(p_1) \\ & - 4\pi \log h_2(p_2) - 2\pi A_2(p_2) \\ & - \left[\Delta \log h_1(p_1) + 4\pi - 2K(p_1)\right] \epsilon^2(-\log\epsilon^2) \\ & - \left[\Delta \log h_2(p_2) + 4\pi - 2K(p_2)\right] \epsilon^2(-\log\epsilon^2) \\ & + o(\epsilon^2(-\log\epsilon^2)). \end{aligned}$$

Then under the condition (1.11), we have for sufficiently small ϵ that

$$\begin{aligned} J_{4\pi,4\pi}(\phi_1^\epsilon, \phi_2^\epsilon) < & - 8\pi - 8\pi \log\pi - 4\pi \log h_1(p_1) - 2\pi A_1(p_1) \\ & - 4\pi \log h_2(p_2) - 2\pi A_2(p_2). \end{aligned}$$

It is easy to check that $\int_M h_i e^{\phi_i^\epsilon} dv_g > 0$ for $i = 1, 2$, we define

$$\widetilde{\phi_i^\epsilon} = \phi_i^\epsilon - \log \int_M h_i e^{\phi_i^\epsilon} dv_g, \quad i = 1, 2.$$

Then $(\widetilde{\phi_1^\epsilon}, \widetilde{\phi_2^\epsilon}) \in \mathcal{H}$. Since $J_{4\pi,4\pi}(u_1 + c_1, u_2 + c_2) = J_{4\pi,4\pi}(u_1, u_2)$ for any $c_1, c_2 \in \mathbb{R}$, we have for sufficiently small ϵ that

$$\begin{aligned} \inf_{\mathcal{H}} J_{4\pi,4\pi} \leq & J_{4\pi,4\pi}(\widetilde{\phi_1^\epsilon}, \widetilde{\phi_2^\epsilon}) = J_{4\pi,4\pi}(\phi_1^\epsilon, \phi_2^\epsilon) \\ & < - 8\pi - 8\pi \log\pi - 2\pi \max_{x\in M_+} (2\log h_1(x) + A_1(x)) \\ & \quad - 2\pi \max_{x\in M_2^+} (2\log h_2(x) + A_2(x)) . \end{aligned} \tag{4.1}$$

Combining (3.1) and (4.1), one knows that $(u_1^\epsilon, u_2^\epsilon)$ does not blow up. So $(u_1^\epsilon, u_2^\epsilon)$ converges to some (u_1, u_2) which minimizes $J_{4\pi,4\pi}$ in $\mathcal{H}$ and solves (1.10). The smooth of u_1 and u_2 follows from the standard elliptic estimates. Finally, we complete the proof of Theorem 1.4. □

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Declarations

Conflicts of Interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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