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Calculus of Variations

Existence results for Toda systems with sign-changing prescribed functions: Part I

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Abstract

Let (M, g) be a compact Riemann surface with area 1, we shall study the Toda system

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(h_1 e^{u_1} - 1\right) - \rho_2 \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1\right) - \rho_1 \left(h_1 e^{u_1} - 1\right), \end{cases} \tag{0.1}$$

on (M, g) with $\rho_1 = 4\pi$, $\rho_2 \in (0, 4\pi)$, h_1 and h_2 are two smooth functions on M. In Jost-Lin-Wang's celebrated article (Comm. Pure Appl. Math., 59 (2006), no. 4, 526–558), they obtained a sufficient condition for the existence of Eq. (0.1) when h_1 and h_2 are both positive. In this paper, we shall improve this result to the case h_1 and h_2 can change signs. We shall pursue a variational method and use the standard blowup analysis. Among other things, the main contribution in our proof is to show that the blowup can only happen at one point where h_1 is positive.

1 Introduction

Let (M, g) be a compact Riemann surface with area 1, $h_i \in C^\infty(M)$ and ρ_i be positive constant for $i = 1, 2$. The critical point (u_1, u_2) of the functional

$$J_{\rho_1,\rho_2}(u_1, u_2) = \frac{1}{3}\int_M \left(|\nabla u_1|^2 + \nabla u_1 \nabla u_2 + |\nabla u_2|^2\right) + \rho_1 \int_M u_1 + \rho_2 \int_M u_2$$

on the Hilbert space

$$\mathcal{H} = \left\{(u_1, u_2) \in H^1(M) \times H^1(M) : \int_M h_1 e^{u_1} = \int_M h_2 e^{u_2} = 1\right\}$$

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satisfies

$$\begin{cases}-\Delta u_1 = 2\rho_1\left(h_1 e^{u_1}-1\right)-\rho_2\left(h_2 e^{u_2}-1\right),\\ -\Delta u_2 = 2\rho_2\left(h_2 e^{u_2}-1\right)-\rho_1\left(h_1 e^{u_1}-1\right).\end{cases} \tag{1.1}$$

In the literal, people calls (1.1) as Toda system. It can be seen as the Frenet frame of holomorphic curves in $\mathbb{CP}^2$ (see [9]) in geometry, and also arises in physics in the study of the nonabelian Chern-Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential; see [5, 22, 24]. One can easily find that Toda system (1.1) is a generalization of the mean field equation

$$-\Delta u = \rho(he^u - 1). \tag{1.2}$$

If u is a solution of Eq. (1.2), then one has $\int_M he^u = 1$. Therefore, people solves Eq. (1.2) in Hilbert space

$$X = \left\{u \in H^1(M) : \int_M he^u = 1\right\}.$$

Since Eq. (1.2) has a variational structure, thanks to the Moser-Trudinger inequality (cf. [4, 6])

$$\log \int_M e^u \le \frac{1}{16\pi}\int_M |\nabla u|^2 + \int_M u + C,$$

it has a minimal type solution in X when $\rho \in (0, 8\pi)$. However, when $\rho = 8\pi$, the situation becomes subtle. The famous Kazdan-Warner problem [12] states that under what kind of condition on h, the equation

$$-\Delta u = 8\pi(he^u - 1) \tag{1.3}$$

has a solution. Necessarily, one needs $\max_M h > 0$. By using blowup argument and a variational method, Ding, Jost, Li and Wang [4] attacked this problem successfully. Assuming h is positive, if

$$\Delta \log h(p_0) + 8\pi - 2K(p_0) > 0, \tag{1.4}$$

where K is the Gauss curvature of M, p_0 is the maximum point of $2\log h(p) + A_p$ on M, $A_p = \lim_{x\to p}\left(G_p(x) + 4\log \mathrm{dist}(x, p)\right)$ and G_p is the Green function which satisfies

$$\begin{cases}-\Delta G_p = 8\pi(\delta_p - 1),\\ \int_M G_p = 0,\end{cases}$$

then Eq. (1.3) has a minimal type solution. Yang and the second author [25] generalized this existence result to the case h is nonnegative. With different arguments, the first author and Zhu [20] and the second author [27] proved respectively the Ding-Jost-Li-Wang condition (1.4) is still sufficient for the existence of Eq. (1.3) when h changes signs. The mentioned works are all based on variational method. We remark that these results were also obtained by using flow method [15, 16, 19, 23].

To well understand the analytic properties of the Toda system, Jost-Wang [10] derived the Moser-Trudinger inequality for it:

$$\inf_{(u_1,u_2)\in\mathcal{H}} J_{\rho_1,\rho_2} \ge -C \quad \text{iff} \quad \rho_1, \rho_2 \in (0, 4\pi]. \tag{1.5}$$

From this inequality, we know that J_{ρ_1,ρ_2} is coercive and hence attains its infimum when $\rho_1, \rho_2 \in (0, 4\pi)$. However, when ρ_1 or ρ_2 equals 4π, the existence problem also becomes subtle. In this paper, we shall put our attention on minimal type solution. Hence, throughout this paper, we assume $\rho_i \leq 4\pi$, $i = 1, 2$.

Let us review the existence result when one of ρ_1 and ρ_2 equals 4π, which was obtained by Jost, Lin and Wang when h_1 and h_2 are both positive.

Theorem 1.1 (Jost-Lin-Wang [11]) *Let (M, g) be a compact Riemann surface with Gauss curvature K. Let $h_1, h_2 \in C^2(M)$ be two positive functions and $\rho_2 \in (0, 4\pi)$. Suppose that*

$$\Delta \log h_1(x) + (8\pi - \rho_2) - 2K(x) > 0, \quad \forall x \in M, \tag{1.6}$$

then $J_{4\pi,\rho_2}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies

$$\begin{cases} -\Delta u_1 = 8\pi \left(h_1 e^{u_1} - 1\right) - \rho_2 \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1\right) - 4\pi \left(h_1 e^{u_1} - 1\right). \end{cases} \tag{1.7}$$

When $\rho_1 = \rho_2 = 4\pi$ and both h_1 and h_2 are positive, we have

Theorem 1.2 (Li-Li [14], Jost-Lin-Wang [11]) *Let (M, g) be a compact Riemann surface with Gauss curvature K. Let $h_1, h_2 \in C^2(M)$ be two positive functions. Suppose that*

$$\min\{\Delta \log h_1(x), \Delta \log h_2(x)\} + 4\pi - 2K(x) > 0, \quad \forall x \in M, \tag{1.8}$$

then $J_{4\pi,4\pi}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies

$$\begin{cases} -\Delta u_1 = 8\pi \left(h_1 e^{u_1} - 1\right) - 4\pi \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 8\pi \left(h_2 e^{u_2} - 1\right) - 4\pi \left(h_1 e^{u_1} - 1\right). \end{cases}$$

We remark that Li-Li obtained Theorem 1.2 when $h_1 = h_2 = 1$ and Jost-Lin-Wang obtained it for general positive h_1 and h_2.

Motivated mostly by works in [4, 20, 25, 27], we would like to release conditions (1.6) and (1.8) as much as possible. Comparing with the sufficient conditions in [4, 20, 25, 27], we believe that conditions (1.6) and (1.8) can release to h_i may change signs and the conditions only need hold on maximum points of the prescribed functions, namely h_1 and h_2. In the first step to this aim, we are successful to release (1.6) when h_1 and h_2 can change signs. To state our result, we introduce two Green functions first. Let $G_1(\cdot, p)$ and $G_2(\cdot, p)$ satisfy

$$\begin{cases} -\Delta G_1(\cdot, p) = 8\pi(\delta_p - 1) - \rho_2(h_2 e^{G_2(\cdot,p)} - 1), \\ -\Delta G_2(\cdot, p) = 2\rho_2(h_2 e^{G_2(\cdot,p)} - 1) - 4\pi(\delta_p - 1), \\ \int_M G_1(\cdot, p) = 0, \ \int_M h_2 e^{G_2(\cdot,p)} = 1, \ \sup_M G_2(\cdot, p) \leq C, \end{cases} \tag{1.9}$$

where δ_p is the Dirac distribution. It was proved in [14] (page 708) that in a small neighborhood around p,

$$G_1(\cdot, p) = -4 \log r + A_1(p) + f, \quad G_2(\cdot, p) = 2 \log r + A_2(p) + g, \tag{1.10}$$

where $r = \text{dist}(\cdot, p)$, $A_i(p)$ $(i = 1, 2)$ are constants and f, g are two smooth functions which are zero at p. Now we are prepared to state our main theorem.

Theorem 1.3 *Let (M, g) be a compact Riemann surface with Gauss curvature K. Let $h_1, h_2 \in C^2(M)$ which are positive somewhere and $\rho_2 \in (0, 4\pi)$. Denote $M_+ = \{x \in M : h_1(x) > 0\}$. Suppose that*

$$2 \log h_1(p) + A_1(p) = \max_{x \in M_+} \left(2 \log h_1(x) + A_1(x)\right),$$

where $A_1(p)$ is defined in (1.10). If

$$\Delta \log h_1(p) + (8\pi - \rho_2) - 2K(p) > 0, \tag{1.11}$$

then $J_{4\pi,\rho_2}$ has a minimizer $(u_1, u_2) \in \mathcal{H}$ which satisfies (1.7).

In the proof of Theorem 1.3, the function A_1 is used to locate the possible point of blow-up. This localization appears also in blow-up analysis and constructions in [3, 13, 18]. We greatly appreciate the reviewer for pointing out these articles to us.

At the end of the introduction, we would like to mention some related works which deal with sign-changing potential in the critical case with respect to Moser-Trudinger type inequalities ([17, 26]). For the generalization of Theorem 1.2, we can also release the condition and we have given the details in the paper [21].

The outline of the rest of the paper is following: In Sect. 2, we do some analysis on the minimizing sequence; In Sect. 3, we estimate the lower bound for $J_{4\pi,\rho_2}$; Finally, we complete the proof of Theorem 1.3 in the last section. Throughout the whole paper, the constant C is varying from line to line and even in the same line, we do not distinguish sequence and its subsequences since we just consider the existence result.

2 Analysis on the minimizing sequence

To show the functional $J_{4\pi,\rho_2}$ is bounded from below, we consider the perturbed functional $J_{4\pi-\epsilon,\rho_2}$. Since the infimum of the functional $J_{4\pi-\epsilon,\rho_2}$ in $\mathcal{H}$ can be attained by $\left(u_1^\epsilon, u_2^\epsilon\right)$, we call $\left(u_1^\epsilon, u_2^\epsilon\right)$ the minimizing sequence and analysis it in this section.

For $\rho_2 \in (0, 4\pi)$, in view of inequality (1.5), one knows for any $\epsilon \in (0, 4\pi)$ there exists a $\left(u_1^\epsilon, u_2^\epsilon\right) \in \mathcal{H}$ such that $J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right) = \inf_{(u_1,u_2)\in\mathcal{H}} J_{4\pi-\epsilon,\rho_2}(u_1, u_2)$. Direct calculation shows on M,

$$\begin{cases} -\Delta u_1^\epsilon = (8\pi - 2\epsilon)\left(h_1 e^{u_1^\epsilon} - 1\right) - \rho_2\left(h_2 e^{u_2^\epsilon} - 1\right), \\ -\Delta u_2^\epsilon = 2\rho_2\left(h_2 e^{u_2^\epsilon} - 1\right) - (4\pi - \epsilon)\left(h_1 e^{u_1^\epsilon} - 1\right). \end{cases} \tag{2.1}$$

Denote $\overline{u_i^\epsilon} = \int_M u_i^\epsilon$ and $m_i^\epsilon = \max_M u_i^\epsilon = u_i^\epsilon\left(x_i^\epsilon\right)$ for some $x_i^\epsilon \in M$. Since $\left(u_1^\epsilon, u_2^\epsilon\right)$ minimizes $J_{4\pi-\epsilon,\rho_2}$ in $\mathcal{H}$, we have $\int_M e^{u_i^\epsilon}$ $(i = 1, 2)$ is bounded from below and above by two positive constants. Namely,

Lemma 2.1 *There exist two positive constants C_1 and C_2 such that*

$$C_1 \le \int_M e^{u_i^\epsilon} \le C_2, \quad i = 1, 2.$$

Proof For $i = 1, 2$, the lower bound is easy since $\int_M h_i e^{u_i^\epsilon} = 1$ and $\max_M h_i > 0$. Since $\mathcal{H}$ is not empty, we can choose $(v_1, v_2) \in \mathcal{H}$, then

$$J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right) = \inf_{(u_1,u_2)\in\mathcal{H}} J_{4\pi-\epsilon,\rho_2}(u_1, u_2) \le J_{4\pi-\epsilon,\rho_2}(v_1, v_2) \to J_{4\pi,\rho_2}(v_1, v_2) \le C.$$

This together with the Moser-Trudinger inequality (1.5) and Jensen's inequality yields

$$\begin{aligned}\log\int_M e^{u_1^\epsilon}+\log\int_M e^{u_2^\epsilon}\le&\frac{1}{12\pi}\int_M\left(\left|\nabla u_1^\epsilon\right|^2+\nabla u_1^\epsilon\nabla u_2^\epsilon+\left|\nabla u_2^\epsilon\right|^2\right)+\overline{u_1^\epsilon}+\overline{u_2^\epsilon}+C\\=&\frac{1}{4\pi}J_{4\pi-\epsilon,\rho_2}\left(u^\epsilon\right)+\frac{\epsilon}{4\pi}\overline{u_1^\epsilon}+\frac{4\pi-\rho_2}{4\pi}\overline{u_2^\epsilon}+C\\\le&\frac{\epsilon}{4\pi}\log\int_M e^{u_1^\epsilon}+\frac{4\pi-\rho_2}{4\pi}\log\int_M e^{u_2^\epsilon}+C.\end{aligned}$$

This combining with $\int_M e^{u_i^\epsilon}$ is bounded from below by some $C_1>0$ shows that $\int_M e^{u_i^\epsilon}\le C_2$ for some $C_2>0$. This completes the proof. □

Lemma 2.2 *For any $s\in(1,2)$, $\|\nabla u_i^\epsilon\|_{L^s(M)}\le C$ for $i=1,2$.*

Proof Let $s'=1/s>2$, we know by definition that

$$\|\nabla u_1^\epsilon\|_{L^s(M)}=\sup\left\{\left|\int_M\nabla u_1^\epsilon\nabla\phi\right|:\phi\in W^{1,s'}(M),\int_M\phi=0,\|\phi\|_{W^{1,s'}(M)}=1\right\}.$$

The Sobolev embedding theorem shows that $\|\phi\|_{L^\infty(M)}\le C$ for some constant C. Then it follows by Eq. (2.1) and Lemma 2.1 that

$$\begin{aligned}\left|\int_M\nabla u_1^\epsilon\nabla\phi\right|=&\left|\int_M\phi(-\Delta u_1^\epsilon)\right|\\=&\left|\int_M\phi\left[(8\pi-2\epsilon)\left(h_1e^{u_1^\epsilon}-1\right)-\rho_2\left(h_2e^{u_2^\epsilon}-1\right)\right]\right|\\\le&C.\end{aligned}$$

Therefore we have $\|\nabla u_1^\epsilon\|_{L^s(M)}\le C$. Similarly, we have $\|\nabla u_2^\epsilon\|_{L^s(M)}\le C$. This ends the proof. □

Concerning $\left(u_1^\epsilon,u_2^\epsilon\right)$, we have the following equivalent characterizations.

Lemma 2.3 *The following three items are equivalent:*

(i) $m_1^\epsilon+m_2^\epsilon\to+\infty$ *as* $\epsilon\to0$;
(ii) $\int_M\left(|\nabla u_1^\epsilon|^2+\nabla u_1^\epsilon\nabla u_2^\epsilon+|\nabla u_2^\epsilon|^2\right)\to+\infty$ *as* $\epsilon\to0$;
(iii) $\overline{u_1^\epsilon}+\overline{u_2^\epsilon}\to-\infty$ *as* $\epsilon\to0$.

Proof $(ii)\Leftrightarrow(iii)$: Since $J_{4\pi-\epsilon,\rho_2}$ is bounded, (ii) is equivalent to

$$(4\pi-\epsilon)\overline{u_1^\epsilon}+\rho_2\overline{u_2^\epsilon}\to-\infty\quad\text{as}\quad\epsilon\to0.\tag{2.2}$$

Using Lemma 2.1 and Jensen's inequality, we have $\overline{u_i^\epsilon}\le C$ for $i=1,2$. Therefore, (2.2) is equivalent to (iii) and then (ii) is equivalent to (iii).

$(i)\Rightarrow(ii)$: Suppose not, we have

$$\int_M\left|\nabla u_1^\epsilon\right|^2+\int_M\left|\nabla u_2^\epsilon\right|^2\le2\int_M\left(|\nabla u_1^\epsilon|^2+\nabla u_1^\epsilon\nabla u_2^\epsilon+|\nabla u_2^\epsilon|^2\right)\le C.$$

Meanwhile, by $(ii)\Leftrightarrow(iii)$ one knows $\overline{u_1^\epsilon}+\overline{u_2^\epsilon}\ge-C$. So $\overline{u_i^\epsilon}$ is bounded for $i=1,2$. By Poincaré's inequality, we have for $i=1,2$ that

$$\int_M\left(u_i^\epsilon\right)^2-\overline{u_i^\epsilon}^2=\int_M(u_i^\epsilon-\overline{u_i^\epsilon})^2\le C\int_M|\nabla u_i^\epsilon|^2\le C.$$

So $\left(u_i^\epsilon\right)$ is bounded in $L^2(M)$. Since $\|\nabla u_i^\epsilon\|_{L^2(M)}$ and $\overline{u_i^\epsilon}$ are both bounded, we have by the Moser-Trudinger inequality that $\left(e^{u_i^\epsilon}\right)$ is bounded in $L^s(M)$ for any $s \geq 1$. Then by using elliptic estimates to (2.1) we obtain that $\left(u_i^\epsilon\right)$ is bounded in $W^{2,2}(M)$ and then $\|u_i^\epsilon\|_{L^\infty(M)}$ is bounded. Therefore, $m_i^\epsilon \leq C$ for $i = 1, 2$. This contradicts (i).

$(ii) \Rightarrow (i)$: If not, then we have $m_1^\epsilon + m_2^\epsilon \leq C$. Using Lemma 2.1, we have $m_i^\epsilon \geq C$ for $i = 1, 2$. So m_i^ϵ is bounded for $i = 1, 2$. Then $\left(e^{u_i^\epsilon}\right)$ is bounded. Since by Lemma 2.2, $u_i^\epsilon - \overline{u_i^\epsilon}$ is bounded in $L^s(M)$ for any $s > 1$, we have by using elliptic estimates to (2.1) that $u_i^\epsilon - \overline{u_i^\epsilon}$ is bounded. Since $(ii) \Leftrightarrow (iii)$, we have $\overline{u_1^\epsilon} + \overline{u_2^\epsilon} \to -\infty$. Notice that $\overline{u_i^\epsilon} \leq C$, we have $\overline{u_1^\epsilon}$ or $\overline{u_2^\epsilon}$ tends to $-\infty$. Without loss of generality, suppose $\overline{u_1^\epsilon}$ tends to $-\infty$. Then

$$1 = \int_M h_1 e^{u_1^\epsilon} = \int_M h_1 e^{u_1^\epsilon - \overline{u_1^\epsilon}} e^{\overline{u_1^\epsilon}} \to 0 \text{ as } \epsilon \to 0.$$

This is a contradiction. □

Definition 2.1 *(Blow up)* We call $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up, if one of the three items in Lemma 2.3 holds.

When $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up, there holds

Lemma 2.4 *Let $\left(u_1^\epsilon, u_2^\epsilon\right)$ minimize $J_{4\pi-\epsilon,\rho_2}$ in $\mathcal{H}$. If $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up, then*

$$\overline{u_1^\epsilon} \to -\infty \text{ as } \epsilon \to 0 \text{ and } \overline{u_2^\epsilon} \geq -C.$$

Proof Since $J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right)$ is bounded, we have by (1.5) that

$$\begin{aligned} C \geq & J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right) \\ \geq & \frac{1}{3}\int_M \left(\left|\nabla u_1^\epsilon\right|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + \left|\nabla u_2^\epsilon\right|^2\right) + (4\pi - \epsilon)\overline{u_1^\epsilon} + \rho_2 \overline{u_2^\epsilon} \\ \geq & C - \epsilon \overline{u_1^\epsilon} - (4\pi - \rho_2)\overline{u_2^\epsilon}. \end{aligned}$$

Since $\overline{u_1^\epsilon} \leq C$ and $\rho_2 < 4\pi$, we have

$$\overline{u_2^\epsilon} \geq -C.$$

If $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up, it follows from Lemma 2.3 that $\overline{u_1^\epsilon} \to -\infty$ as $\epsilon \to 0$. This finishes the proof. □

If $\left(u_1^\epsilon, u_2^\epsilon\right)$ does not blow up, then by Lemma 2.3, one can show that $\left(u_1^\epsilon, u_2^\epsilon\right)$ converges to $\left(u_1^0, u_2^0\right)$ in $\mathcal{H}$ and $\left(u_1^0, u_2^0\right)$ minimizes $J_{4\pi,\rho_2}$. The proof of Theorem 1.3 terminates in this case. Therefore, we assume $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up in the rest of this paper.

By Lemma 2.2, there exist G_i, $i = 1, 2$ such that $u_1^\epsilon - \overline{u_1^\epsilon} \rightharpoonup G_1$ and $u_2^\epsilon \rightharpoonup G_2$ weakly in $W^{1,s}(M)$ for any $1 < s < 2$ as $\epsilon \to 0$. Since $\left(e^{u_i^\epsilon}\right)$ is bounded in $L^1(M)$ we may extract a subsequence (still denoted $e^{u_i^\epsilon}$) such that $e^{u_i^\epsilon}$ converges in the sense of measures on M to some nonnegative bounded measure μ_i for $i = 1, 2$. We set

$$\gamma_1 = 8\pi h_1 \mu_1 - \rho_2 h_2 \mu_2, \quad \gamma_2 = 2\rho_2 h_2 \mu_2 - 4\pi h_1 \mu_1$$

and

$$S_i = \{x \in M : |\gamma_i(\{x\})| \geq 4\pi\}, \ i = 1, 2.$$

Let $S = S_1 \cup S_2$. By Theorem 1 in [1], it is easy to show that for any $\Omega \subset\subset M \setminus S$,

$$u_i^\epsilon - \overline{u_i^\epsilon} \text{ is uniformly bounded in } \Omega,\ i = 1, 2. \tag{2.3}$$

Since $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up, we know S is not empty (Or else, with a finite covering argument, we have by (2.3) that $\|u_i^\epsilon - \overline{u_i^\epsilon}\|_{L^\infty(M)} \le C$, then $m_i^\epsilon \le C$, this contradicts with Lemma 2.3 (i)). Meanwhile, by the definition of S, we have for any $x \in S$,

$$\mu_1(\{x\}) \ge \frac{1}{4\max_M |h_1|} \ \text{ or } \ \mu_2(\{x\}) \ge \frac{\pi}{\rho_2 \max_M |h_2|}.$$

In view of μ_1 and μ_2 are bounded, S is a finite set. We denote $S = \{x_l\}_{l=1}^L$. It follows from (2.3) and Fatou's lemma that

$$\mu_1 = \sum_{l=1}^L \mu_1(\{x_l\})\,\delta_{x_l} \ \text{ and } \ \mu_2 = e^{G_2} + \sum_{l=1}^L \mu_2(\{x_l\})\,\delta_{x_l}.$$

Lemma 2.5 suppμ_1 *is a single point set.*

Proof It follows from Lemma 2.1 that supp$\mu_1 \ne \emptyset$. If there are two different points in suppμ_1, then by Lemma 2.1 and the improved Moser-Trudinger inequality (cf. [2], Theorem 2.1), for any $\epsilon' > 0$, there exists some $C = C(\epsilon') > 0$ such that

$$C \le \log \int_M e^{u_1^\epsilon} \le \left(\frac{1}{32\pi} + \epsilon'\right) \int_M \left|\nabla u_1^\epsilon\right|^2 + \overline{u_1^\epsilon} + C. \tag{2.4}$$

Since

$$\begin{aligned} C \ge& J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right) \\ =& \frac{1}{3}\int_M \left(\left|\nabla u_1^\epsilon\right|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + \left|\nabla u_2^\epsilon\right|^2\right) + (4\pi - \epsilon)\,\overline{u_1^\epsilon} + \rho_2 \overline{u_2^\epsilon} \\ =& \frac{1}{4}\int_M \left|\nabla u_1^\epsilon\right|^2 + (4\pi-\epsilon)\,\overline{u_1^\epsilon} + \frac{1}{3}\int_M \left|\nabla\left(u_2^\epsilon + \frac{1}{2}u_1^\epsilon\right)\right|^2 + \rho_2\overline{u_2^\epsilon} \\ \ge& \frac{1}{4}\int_M \left|\nabla u_1^\epsilon\right|^2 + (4\pi-\epsilon)\,\overline{u_1^\epsilon} - C, \end{aligned} \tag{2.5}$$

then we have by combining (2.4) and (2.5) that

$$\overline{u_1^\epsilon} \ge -C.$$

In view of Lemmas 2.3 and 2.4, this is a contradiction. Therefore, suppμ_1 is a single point set. This completes the proof. ◻

Since by (2.3) we know supp$\mu_1 \subset S$, we can assume without loss of generality that supp$\mu_1 = \{x_1\}$. By noticing that $\int_M h_1 e^{u_1^\epsilon} = 1$, we have $h_1\mu_1 = \delta_{x_1}$.

Lemma 2.6 *There holds* $\gamma_2(\{x_l\}) \le -4\pi$ *if* $x_l \ne x_1$ *and* $\gamma_2(\{x_1\}) < 4\pi$.

Proof Since of Lemma 2.4 and (2.3), we know $\overline{u_2^\epsilon} \ge -C$. For any $x_l \in S$, choosing $r > 0$ sufficiently small, we have $u_2^\epsilon\,|_{\partial B_r(x_l)} \ge -C_0$ for some constant C_0. Let w_2^ϵ be the solution of

$$\begin{cases} -\Delta w_2^\epsilon = 2\rho_2\left(h_2 e^{u_2^\epsilon} - 1\right) - (4\pi-\epsilon)\left(h_1 e^{u_1^\epsilon} - 1\right) & \text{in } B_r(x_l), \\ w_2^\epsilon = -C_0 \ \text{ on } \partial B_r(x_l). \end{cases}$$

By the maximum principle $w_2^\epsilon \le u_2^\epsilon$ in $B_r(x_l)$. Since $2\rho_2 h_2 e^{u_2^\epsilon} - (4\pi - \epsilon) h_1 e^{u_1^\epsilon}$ is bounded in $L^1(B_r(x_l))$, $w_2^\epsilon \rightharpoonup w_2$ weakly in $W^{1,s}(B_r(x_l))$ for any $1 < s < 2$, where w_2 is the solution of

$$\begin{cases} -\Delta w_2 = 2\rho_2 \left(h_2 e^{G_2} - 1\right) + 4\pi + \gamma_2 \left(\{x_l\}\right) \delta_{x_l} \ \text{ in } B_r(x_l), \\ w_2 = -C_0 \ \text{ on } \partial B_r(x_l). \end{cases}$$

Since $h_1 \mu_1 = \delta_{x_1}$, if $\gamma_2 \left(\{x_l\}\right) > 0$, then $h_2(x_l) > 0$ and one has

$$2\rho_2 \left(h_2 e^{G_2} - 1\right) + 4\pi \ge -C \ \text{ near } \ x_l.$$

Then we have $-\Delta w_2 \ge \gamma_2 \left(\{x_l\}\right) \delta_{x_l} - C$ in $B_r(x_l)$ (Here, for simplicity, we assume r is small enough to ensure $h_2(x_l) > 0$ in $B_r(x_l)$). Therefore

$$w_2 \ge -\frac{1}{2\pi} \gamma_2 \left(\{x_l\}\right) \log |x - x_l| - C \ \text{ in } B_r(x_l).$$

Thus $e^{w_2} \ge C / |x - x_l|^{\frac{\gamma_2(\{x_l\})}{2\pi}}$. Note that it follows by Fatou's lemma that

$$\int_{B_r(x_l)} e^{w_2} \le \lim_{\epsilon \to 0} \int_{B_r(x_l)} e^{w_2^\epsilon} \le \lim_{\epsilon \to 0} \int_{B_r(x_l)} e^{u_2^\epsilon} \le C.$$

Then we have

$$\gamma_2 \left(\{x_l\}\right) < 4\pi, \quad \forall l = 1, 2, \cdots, L.$$

If $x_l \ne x_1$, we have $\gamma_2 \left(\{x_l\}\right) \le -4\pi$. In fact, if $\gamma_2 \left(\{x_l\}\right) > -4\pi$, then since $x_l \ne x_1$, one has

$$\gamma_1 \left(\{x_l\}\right) = -\rho_2 h_2(x_l) \mu_2 \left(\{x_l\}\right) = -\frac{1}{2} \gamma_2 \left(\{x_l\}\right) \in (-2\pi, 2\pi).$$

Then $x_l \notin S$. A contradiction. This ends the proof. □

Now we are prepared to prove the following lemma, which can be seen as a key in the proof of our main theorem. We remark that this lemma is obtained much more directly with the help of Proposition 2.4 in [11] when the prescribed functions h_1 and h_2 are positive. However, when h_1 or h_2 changes signs, we do not have the counterpart of Proposition 2.4 in [11] in the hand, and therefore more effort is needed in our situation.

Lemma 2.7 *We have $u_2^\epsilon \le C$.*

Proof By Lemma 2.6, we divide the whole proof into two cases.

Case 1 $\gamma_2 \left(\{x_l\}\right) \le -4\pi$ $(x_l \ne x_1)$.

In this case, we have $2\rho_2 h_2(x_l) \mu_2 \left(\{x_l\}\right) \le -4\pi$, then $h_2(x_l) < 0$ and $\mu_2 \left(\{x_l\}\right) > 0$. We can choose r sufficiently small such that $h_2(x) < 0$ in $B_r(x_l)$. Consider

$$\begin{cases} -\Delta v_1^\epsilon = -(4\pi - \epsilon) h_1 e^{u_1^\epsilon} \ \text{ in } \ B_r(x_l), \\ v_1^\epsilon = 0 \ \text{ on } \ \partial B_r(x_l). \end{cases}$$

We define $v_2^\epsilon = u_2^\epsilon - \overline{u_2^\epsilon} - v_1^\epsilon$. Then $-\Delta v_2^\epsilon = -2\rho_2 + (4\pi - \epsilon) + 2\rho_2 h_2 e^{u_2^\epsilon} \leq -2\rho_2 + (4\pi - \epsilon)$. By Theorem 8.17 in [8] (or Theorem 4.1 in [7]) and Lemma 2.2, we have

$$\begin{aligned}\sup_{B_{r/2}(x_l)} v_2^\epsilon \leq & C \left(\|(v_2^\epsilon)^+\|_{L^s(B_r(x_l))} + C\right) \\ \leq & C \left(\|u_2^\epsilon - \overline{u_2^\epsilon}\|_{L^s(M)} + \|v_1^\epsilon\|_{L^s(B_r(x_l))} + C\right) \\ \leq & C \left(\|\nabla u_2^\epsilon\|_{L^s(M)} + \|v_1^\epsilon\|_{L^s(B_r(x_l))} + C\right) \\ \leq & C \left(\|v_1^\epsilon\|_{L^s(B_r(x_l))} + C\right).\end{aligned}$$

Notice that $h_1\mu_1 = \delta_{x_1}$ and $x_l \neq x_1$, one has $\int_{B_r(x_l)} |h_1| e^{u_1^\epsilon} \to 0$ as $\epsilon \to 0$ for sufficiently small r. It then follows from Theorem 1 in [1] that $\int_{B_r(x_l)} e^{t|v_1^\epsilon|} \leq C$ for some $t > 1$, which yields that

$$\|v_1^\epsilon\|_{L^s(B_r(x_l))} \leq C.$$

Then we have

$$\sup_{B_{r/2}(x_l)} v_2^\epsilon \leq C.$$

Note that

$$\begin{aligned}\int_{B_{r/2}(x_l)} e^{su_2^\epsilon} = & \int_{B_{r/2}(x_l)} e^{s\overline{u_2^\epsilon}} e^{sv_2^\epsilon} e^{sv_1^\epsilon} \\ \leq & C \int_{B_{r/2}(x_l)} e^{s|v_1^\epsilon|} \\ \leq & C.\end{aligned}$$

Therefore, one has by Hölder's inequality that

$$\mu_2(\{x_l\}) = \lim_{r\to 0}\lim_{\epsilon\to 0} \int_{B_{r/2}(x_l)} e^{u_2^\epsilon} \leq \lim_{r\to 0}\lim_{\epsilon\to 0} \left(\int_{B_{r/2}(x_l)} e^{su_2^\epsilon}\right)^{1/s} \left(\mathrm{vol} B_{r/2}(x_l)\right)^{1-1/s} = 0,$$

this is a contradiction with $\mu_2(\{x_l\}) > 0$. Hence, we obtain $S = \{x_1\}$.

Case 2 $\gamma_2(\{x_1\}) < 4\pi$ (We shall divide this case into three subcases.)

Case 2.1 $h_2(x_1)\mu_2(x_1) = 0$.

Choosing $r > 0$ sufficiently small such that $h_1(x) > 0$ in $B_r(x_1)$. Let z_1^ϵ be the solution of

$$\begin{cases} -\Delta z_1^\epsilon = 2\rho_2 h_2 e^{u_2^\epsilon} & \text{in } B_r(x_1), \\ z_1^\epsilon = 0 & \text{on } \partial B_r(x_1). \end{cases}$$

Let $z_2^\epsilon = u_2^\epsilon - \overline{u_2^\epsilon} - z_1^\epsilon$ so that $-\Delta z_2^\epsilon \leq -2\rho_2 + (4\pi - \epsilon)$. By Theorem 8.17 in [8] (or Theorem 4.1 in [7]) and Lemma 2.2, we have

$$\begin{aligned}\sup_{B_{r/2}(x_1)} z_2^\epsilon \leq & C \left(\|(z_2^\epsilon)^+\|_{L^s(B_r(x_1))} + C\right) \\ \leq & C \left(\|u_2^\epsilon - \overline{u_2^\epsilon}\|_{L^s(M)} + \|z_1^\epsilon\|_{L^s(B_r(x_1)))} + C\right) \\ \leq & C \left(\|\nabla u_2^\epsilon\|_{L^s(M)} + \|z_1^\epsilon\|_{L^s(B_r(x_1))} + C\right) \\ \leq & C \left(\|z_1^\epsilon\|_{L^s(B_r(x_1))} + C\right).\end{aligned}$$

Since $\int_{B_r(x_1)} |h_2| e^{u_2^\epsilon} \to 0$ as $\epsilon \to 0$ for sufficiently small r, it follows from Theorem 1 in [1] that $\int_{B_r(x_1)} e^{t|z_1^\epsilon|} \leq C$ for some $t > 1$, which yields that

$$\|z_1^\epsilon\|_{L^s(B_r(x_1))} \leq C.$$

Then we have

$$\sup_{B_{r/2}(x_1)} z_2^\epsilon \leq C.$$

Note that

$$\begin{aligned}\int_{B_{r/2}(x_1)} e^{tu_2^\epsilon} =& \int_{B_{r/2}(x_1)} e^{t\overline{u_2^\epsilon}} e^{tz_2^\epsilon} e^{tz_1^\epsilon} \\ \leq& C \int_{B_{r/2}(x_1)} e^{t|z_1^\epsilon|} \\ \leq& C.\end{aligned}$$

By the standard elliptic estimates, we have

$$\|z_1^\epsilon\|_{L^\infty(B_{r/4}(x_1))} \leq C.$$

Therefore, we obtain that

$$u_2^\epsilon - \overline{u_2^\epsilon} \leq C \ \text{ in } \ B_{r/4}(x_1).$$

Case 2.2 $h_2(x_1)\mu_2(x_1) > 0$.
Consider the equation

$$\begin{cases} -\Delta v_1^\epsilon = 2\rho_2 h_2 e^{u_2^\epsilon} - (4\pi - \epsilon)\, h_1 e^{u_1^\epsilon} := f_\epsilon \quad \text{in } \ B_\delta(x_1), \\ v_1^\epsilon = 0 \ \text{ on } \ \partial B_\delta(x_1). \end{cases}$$

Define $v_2^\epsilon = u_2^\epsilon - \overline{u_2^\epsilon} - v_1^\epsilon$, then $-\Delta v_2^\epsilon = (4\pi - \epsilon) - 2\rho_2$ in $B_\delta(x_1)$. By Theorem 4.1 in [7] and Lemma 2.2, we have

$$\begin{aligned}\sup_{B_{\delta/2}(x_1)} |v_2^\epsilon| \leq& C \left(\|v_2^\epsilon\|_{L^1(B_\delta(x_1))} + C\right) \\ \leq& C \left(\|u_2^\epsilon - \overline{u_2^\epsilon}\|_{L^1(M)} + \|v_1^\epsilon\|_{L^1(B_\delta(x_1))} + C\right) \\ \leq& C \left(\|\nabla u_2^\epsilon\|_{L^s(M)} + \|v_1^\epsilon\|_{L^1(B_\delta(x_1))} + C\right) \\ \leq& C \left(\|v_1^\epsilon\|_{L^1(B_\delta(x_1))} + C\right).\end{aligned}$$

Since in this case $\|f_\epsilon\|_{L^1(B_\delta(x_1))} < 4\pi$ for sufficiently small $\epsilon > 0$, it then follows from Theorem 1 in [1] that $e^{s|v_1^\epsilon|}$ is bounded in $B_\delta(x_1)$ for some $s > 1$, which yields that

$$\|v_1^\epsilon\|_{L^1(B_\delta(x_1))} \leq C.$$

Then we have

$$\sup_{B_{\delta/2}(x_1)} v_2^\epsilon \leq C.$$

Note that

$$
\begin{aligned}
\int_{B_{\delta/2}(x_1)} e^{su_2^\epsilon} = & \int_{B_{\delta/2}(x_1)} e^{s\overline{u_2^\epsilon}} e^{sv_2^\epsilon} e^{sv_1^\epsilon} \\
\leq & C \int_{B_{\delta/2}(x_1)} e^{s|v_1^\epsilon|} \\
\leq & C.
\end{aligned} \tag{2.6}
$$

Therefore, one has by Hölder's inequality that

$$
\mu_2(\{x_1\}) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \int_{B_{r/2}(x_l)} e^{u_2^\epsilon} \leq \lim_{\delta\to 0}\lim_{\epsilon\to 0} \left(\int_{B_{r/2}(x_l)} e^{su_2^\epsilon} \right)^{1/s} \left(\mathrm{vol} B_{r/2}(x_l)\right)^{1-1/s} = 0,
$$

this is a contradiction with $\mu_2(\{x_1\}) > 0$. This shows that this subcase will not happen.

Case 2.3 $h_2(x_1)\mu_2(x_1) < 0$.

Since $S = \{x_1\}$, it follows by (2.3) that u_2^ϵ is locally uniformly bounded in $M \setminus \{x_1\}$. But in this subcase, we have $\mu_2(\{x_1\}) > 0$, then $\max_{B_r(x_1)} u_2^\epsilon = \max_M u_2^\epsilon \to +\infty$ as $\epsilon \to 0$. We assume $u_2^\epsilon(x_2^\epsilon) = \max_{B_r(x_1)} u_2^\epsilon$, it is obvious that $x_2^\epsilon \to x_1$ as $\epsilon \to 0$. At the maximum point x_2^ϵ, we have

$$
-\Delta u_2^\epsilon(x_2^\epsilon) = 2\rho_2(h_2(x_2^\epsilon)e^{u_2^\epsilon(x_2^\epsilon)} - 1) - (4\pi - \epsilon)\,(h_1(x_2^\epsilon)e^{u_1^\epsilon(x_2^\epsilon)} - 1) < 0.
$$

This is a contradiction. Therefore, this subcase will not happen either.

Concluding all the cases above, we finish the proof. □

Since $S = \{x_1\}$ and $h_1\mu_1 = \delta_{x_1}$, we have $x_1^\epsilon \to x_1$ as $\epsilon \to 0$ by (2.3). Let $(\Omega; (x^1, x^2))$ be an isothermal coordinate system around x_1 and we assume the metric to be

$$
g|_\Omega = e^\phi \left(\left(dx^1\right)^2 + \left(dx^2\right)^2 \right), \quad \phi(0) = 0.
$$

We have

$$
u_1^\epsilon(x_1^\epsilon + r_1^\epsilon x) - m_1^\epsilon \to -2\log(1 + \pi h_1(x_1)|x|^2), \tag{2.7}
$$

where $r_1^\epsilon = e^{-m_1^\epsilon/2}$. Recalling that for any $s \in (1, 2)$, we have $u_1^\epsilon - \overline{u_1^\epsilon} \to G_1$ weakly in $W^{1,s}(M)$ and strongly in $C^2_{\mathrm{loc}}(M \setminus \{x_1\})$, $u_2^\epsilon \to G_2$ weakly in $W^{1,s}(M)$ and strongly in $C^2_{\mathrm{loc}}(M \setminus \{x_1\})$, where $G_1 = G_1(x, x_1)$ and $G_2 = G_2(x, x_1)$ are defined in (1.9).

3 The lower bound for $J_{4\pi,\rho_2}$

In this section, we shall give the first step in proving Theorem 1.3: deriving an explicit lower bound of $J_{4\pi,\rho_2}$ when $\left(u_1^\epsilon, u_2^\epsilon\right)$ blows up.

Define $v_2^\epsilon = \frac{1}{3}(2u_2^\epsilon + u_1^\epsilon) - \frac{1}{3}(2\overline{u_2^\epsilon} + \overline{u_1^\epsilon})$, we have

$$
\begin{cases}
-\Delta v_2^\epsilon = (4\pi - \epsilon)\,(h_2 e^{u_2^\epsilon} - 1), \\
\int_M v_2^\epsilon = 0.
\end{cases}
$$

Notice that $u_2^\epsilon \le C$, it follows from the standard elliptic estimates that $\|v_2^\epsilon\|_{C^1(M)} \le C$. Then we obtain that

$$\begin{aligned}\frac{1}{3}\int_{B_\delta(x_1^\epsilon)}\left(\left|\nabla u_1^\epsilon\right|^2+\nabla u_1^\epsilon\nabla u_2^\epsilon+\left|\nabla u_2^\epsilon\right|^2\right)=&\frac{1}{4}\int_{B_\delta(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2+\frac{3}{4}\int_{B_\delta(x_1^\epsilon)}|\nabla v_2^\epsilon|^2\\ =&\frac{1}{4}\int_{B_\delta(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2+O(\delta^2).\end{aligned}$$

Denote $w(x)=-2\log(1+\pi h_1(x_1)|x|^2)$, we have by (2.7) that

$$\begin{aligned}\frac{1}{4}\int_{B_\delta(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2=&\frac{1}{4}\int_{B_L}|\nabla w|^2\\ &+\frac{1}{4}\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2+o_\epsilon(1)+O(\delta^2).\end{aligned}$$

To estimate $\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2$, we shall follow [14] closely. Let

$$a_1^\epsilon=\inf_{\partial B_{Lr_1^\epsilon}(x_1^\epsilon)}u_1^\epsilon,\quad b_1^\epsilon=\sup_{\partial B_{Lr_1^\epsilon}(x_1^\epsilon)}u_1^\epsilon.$$

We set $a_1^\epsilon-b_1^\epsilon=m_1^\epsilon-\overline{u_1^\epsilon}+d_1^\epsilon$. Then

$$d_1^\epsilon=w(L)-\sup_{\partial B_\delta(x_1)}G_1+o_\epsilon(1).$$

Define $f_1^\epsilon=\max\{\min\{u_1^\epsilon,a_1^\epsilon\},b_1^\epsilon\}$. We have

$$\begin{aligned}\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}\left|\nabla u_1^\epsilon\right|^2\ge&\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla f_1^\epsilon|^2\\ =&\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)}|\nabla_{\mathbb{R}^2}f_1^\epsilon|^2\\ \ge&\inf_{\Psi|_{\partial B_{Lr_1^\epsilon}(0)}=a_1^\epsilon,\Psi|_{\partial B_\delta(0)}=b_1^\epsilon}\int_{B_\delta(0)\setminus B_{Lr_1^\epsilon}(0)}|\nabla_{\mathbb{R}^2}\Psi|^2.\end{aligned}$$

By the Dirichlet's principle, we know

$$\inf_{\Psi|_{\partial B_{Lr_1^\epsilon}(0)}=a_1^\epsilon,\Psi|_{\partial B_\delta(0)}=b_1^\epsilon}\int_{B_\delta(0)\setminus B_{Lr_1^\epsilon}(0)}|\nabla_{\mathbb{R}^2}\Psi|^2$$

is uniquely attained by the following harmonic function

$$\begin{cases}-\Delta_{\mathbb{R}^2}\phi=0,\\ \phi|_{\partial B_{Lr_1^\epsilon}(0)}=a_1^\epsilon,\phi|_{\partial B_\delta(0)}=b_1^\epsilon.\end{cases}$$

Thus,

$$\phi=\frac{a_1^\epsilon-b_1^\epsilon}{-\log Lr_1^\epsilon+\log\delta}\log r-\frac{a_1^\epsilon\log\delta-b_1^\epsilon\log Lr_1^\epsilon}{-\log Lr_1^\epsilon+\log\delta},$$

and then

$$\int_{B_\delta(0)\setminus B_{Lr_1^\epsilon}(0)}|\nabla_{\mathbb{R}^2}\phi|^2=\frac{4\pi(a_1^\epsilon-b_1^\epsilon)^2}{-\log(Lr_1^\epsilon)^2+\log\delta^2}.$$

Concluding, we have

$$\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} \left|\nabla u_1^\epsilon\right|^2 \geq \frac{4\pi(a_1^\epsilon - b_1^\epsilon)^2}{-\log(Lr_1^\epsilon)^2 + \log\delta^2}.$$

Since $-\log(r_1^\epsilon)^2 = m_1^\epsilon$, we obtain

$$\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} \left|\nabla u_1^\epsilon\right|^2 \geq 4\pi \frac{(m_1^\epsilon - \overline{u_1^\epsilon} + d_1^\epsilon)^2}{m_1^\epsilon - \log L^2 + \log\delta^2}. \tag{3.1}$$

By (2.5), one has

$$\frac{1}{4}\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} \left|\nabla u_1^\epsilon\right|^2 + (4\pi-\epsilon)\,\overline{u_1^\epsilon} \leq \frac{1}{4}\int_M \left|\nabla u_1^\epsilon\right|^2 + (4\pi-\epsilon)\,\overline{u_1^\epsilon} \leq C. \tag{3.2}$$

It follows form (3.1) and (3.2) that

$$\pi \frac{(m_1^\epsilon - \overline{u_1^\epsilon} + d_1^\epsilon)^2}{m_1^\epsilon - \log L^2 + \log\delta^2} + (4\pi-\epsilon)\,\overline{u_1^\epsilon} \leq C. \tag{3.3}$$

Recalling that $\overline{u_1^\epsilon} \to -\infty$ and $m_1^\epsilon \to +\infty$, we get from (3.3)

$$\frac{\overline{u_1^\epsilon}}{m_1^\epsilon} = -1 + o_\epsilon(1) \tag{3.4}$$

by dividing both sides by m_1^ϵ and letting ϵ tend to 0. Taking (3.4) into (3.1), we have

$$\int_{B_\delta(x_1^\epsilon)\setminus B_{Lr_1^\epsilon}(x_1^\epsilon)} \left|\nabla u_1^\epsilon\right|^2 \geq 4\pi \frac{(m_1^\epsilon - \overline{u_1^\epsilon})^2}{m_1^\epsilon} + 16\pi\left(d_1^\epsilon + \log L^2 - \log\delta^2 + o_\epsilon(1)\right).$$

Then

$$\begin{aligned}
&\frac{1}{3}\int_{B_\delta(x_1^\epsilon)} \left(\left|\nabla u_1^\epsilon\right|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + \left|\nabla u_2^\epsilon\right|^2\right) + (4\pi-\epsilon)\,\overline{u_1^\epsilon} + \rho_2 \overline{u_2^\epsilon} \\
\geq & -4\pi - 4\pi\log(\pi h_1(x_1)) - 4\pi A_1(x_1) + 8\pi\log\delta \\
&+ \rho_2 \int_M G_2 + o_\epsilon(1) + o_L(1) + o_\delta(1).
\end{aligned} \tag{3.5}$$

Using (1.9) and (1.10), one has

$$\begin{aligned}
&\frac{1}{3}\int_{M\setminus B_\delta(x_1^\epsilon)} \left(\left|\nabla u_1^\epsilon\right|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon + \left|\nabla u_2^\epsilon\right|^2\right) \\
=&\frac{\rho_2}{2}\int_M G_2\left(h_2 e^{G_2} - 1\right) - 8\pi\log\delta + 2\pi A_1(x_1) + o_\epsilon(1) + o_\delta(1).
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned}
J_{4\pi-\epsilon,\rho_2}\left(u_1^\epsilon, u_2^\epsilon\right) \geq & -4\pi - 4\pi\log(\pi h_1(x_1)) - 2\pi A_1(x_1) + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2} + 1) \\
& + o_\epsilon(1) + o_L(1) + o_\delta(1).
\end{aligned}$$

By letting $\epsilon \to 0$ first, then $L \to +\infty$ and then $\delta \to 0$, we obtain finally that

$$\begin{aligned}\inf_{u\in\mathcal{H}} J_{4\pi,\rho_2}(u) \geq & -4\pi - 4\pi\log(\pi h_1(x_1)) - 2\pi A_1(x_1) + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2}+1)\\ \geq & -4\pi - 4\pi\log\pi - 2\pi \max_{x\in M_+}(2\log h_1(x) + A_1(x))\\ & + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2}+1). \end{aligned} \tag{3.7}$$

4 Completion of the proof of Theorem 1.3

In this section, we first outline the rest proof, then construct the blowup sequences like in [14] and present our calculations.

4.1 Outline of the rest proof

Let ϕ_1^ϵ and ϕ_2^ϵ be defined as [14] (see section 6). If the condition (1.11) is satisfied on M_+, we can follow [14] step by step to show that for sufficiently small ϵ

$$\begin{aligned}J_{4\pi,\rho_2}(\phi_1^\epsilon,\phi_2^\epsilon) < & -4\pi - 4\pi\log\pi - 2\pi \max_{x\in M_+}(2\log h_1(x) + A_1(x))\\ & + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2}+1).\end{aligned}$$

It is easy to check that $\int_M h_1 e^{\phi_1^\epsilon} > 0$ and $\int_M h_2 e^{\phi_2^\epsilon} > 0$, we define

$$\widetilde{\phi_i^\epsilon} = \phi_i^\epsilon - \log\int_M h_i e^{\phi_i^\epsilon}, \quad i = 1, 2.$$

Then $(\phi_1^\epsilon, \phi_2^\epsilon) \in \mathcal{H}$. Since $J_{4\pi,\rho_2}(u_1+c_1, u_2+c_2) = J_{4\pi,\rho_2}(u_1, u_2)$ for any $c_1, c_2 \in \mathbb{R}$, we have for sufficiently small ϵ that

$$\begin{aligned}\inf_{u\in\mathcal{H}} J_{4\pi,\rho_2}(u) \leq & J_{4\pi,\rho_2}(\widetilde{\phi_1^\epsilon},\widetilde{\phi_2^\epsilon}) = J_{4\pi,\rho_2}(\phi_1^\epsilon,\phi_2^\epsilon)\\ & < -4\pi - 4\pi\log\pi - 2\pi \max_{x\in M_+}(2\log h_1(x) + A_1(x))\\ & + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2}+1).\end{aligned} \tag{4.1}$$

Combining (3.7) and (4.1), one knows that $\left(u_1^\epsilon, u_2^\epsilon\right)$ does not blow up. So $\left(u_1^\epsilon, u_2^\epsilon\right)$ converges to some $\left(u_1^0, u_2^0\right)$ which minimizes $J_{4\pi,\rho_2}$ in $\mathcal{H}$ and solves (1.7). The smooth of u_1^0 and u_2^0 follows from the standard elliptic estimates. Finally, we complete the proof of Theorem 1.3.

4.2 Test function

Suppose that $2\log h_1(p) + A_1(p) = \max_{x\in M_+}(2\log h_1(x) + A_1(x))$. Let $(\Omega; (x^1, x^2))$ be an isothermal coordinate system around p and we assume the metric to be

$$g|_\Omega = e^\phi\left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right),$$

and

$$\phi = b_1(p)x^1 + b_2(p)x^2 + c_1(p)\left(x^1\right)^2 + c_2(p)\left(x^2\right)^2 + c_{12}(p)x^1x^2 + O(r^3),$$

where $r(x^1, x^2) = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2}$. Moreover we assume near p that

$$G_i = a_i \log r + A_i(p) + \lambda_i(p)x^1 + \nu_i(p)x^2 + \alpha_i(p)\left(x^1\right)^2 + \beta_i(p)\left(x^2\right)^2 + \xi_i(p)x^1x^2 + \ell_i(x^1, x^2) + O(r^4), i = 1, 2,$$

where $a_1 = -4$, $a_2 = 2$. It is well known that

$$K(p) = -(c_1(p) + c_2(p)),$$
$$|\nabla u|^2 dV_g = |\nabla u|^2 dx^1 dx^2,$$

and

$$\frac{\partial u}{\partial n} dS_g = \frac{\partial u}{\partial r} r d\theta.$$

For α_i and β_i, we have the following lemma:

Lemma 4.1 *We have*

$$\alpha_1(p) + \beta_1(p) = 4\pi - \frac{\rho_2}{2}, \quad \alpha_2(p) + \beta_2(p) = \rho_2 - 2\pi.$$

Proof We have near p that

$$2\alpha_1(p) + 2\beta_1(p) + O(r) = \Delta_{\mathbb{R}^2} G_1 = e^{-\phi}\left[8\pi + \rho_2\left(h_2 e^{G_2} - 1\right)\right],$$
$$2\alpha_2(p) + 2\beta_2(p) + O(r) = \Delta_{\mathbb{R}^2} G_2 = e^{-\phi}\left[-2\rho_2\left(h_2 e^{G_2} - 1\right) - 4\pi\right],$$

then the lemma is proved since $e^{G_2} = O(r^2)$ near p. □

We choose as in [14] that

$$\phi_1^\epsilon = \begin{cases} w(\frac{x}{\epsilon}) + \lambda_1(p) r\cos\theta + \nu_1(p)\sin\theta, & x \in B_{L\epsilon}(p), \\ G_1 - \eta H_1 + 4\log(L\epsilon) - 2\log\left(1 + \pi L^2\right) - A_1(p), & x \in B_{2L\epsilon}(p) \setminus B_{L\epsilon}(p), \\ G_1 + 4\log(L\epsilon) - 2\log\left(1 + \pi L^2\right) - A_1(p), & \text{otherwise} \end{cases}$$

and

$$\phi_2^\epsilon = \begin{cases} -\frac{w(\frac{x}{\epsilon}) + 2\log\left(1+\pi L^2\right)}{2} + 2\log(L\epsilon) & \\ \quad + \lambda_2(p) r\cos\theta + \nu_2(p) r\sin\theta + A_2(p), & x \in B_{L\epsilon}(p), \\ G_2 - \eta H_2, & x \in B_{2L\epsilon}(p) \setminus B_{L\epsilon}(p), \\ G_2, & \text{otherwise.} \end{cases}$$

Here,

$$H_i = G_i - a_i \log r - A_i(p) - \lambda_i(p) r\cos\theta - \nu_i(p) r\sin\theta, \quad i = 1, 2$$

and η is a cut-off function which equals 1 in $B_{L\epsilon}(p)$, equals 0 in $M \setminus B_{2L\epsilon}(p)$ and satisfies $|\nabla\eta| \leq \frac{C}{L\epsilon}$.

Using Lemma 5.2 in [14] and Lemma 4.1, we have

$$\begin{aligned}\int_M |\nabla\phi_1^\epsilon|^2 =& \int_{B_{L\epsilon}(p)} |\nabla\phi_1^\epsilon|^2 + \int_{M\setminus B_{L\epsilon}(p)} |\nabla G_1|^2 \\ &- 2\int_{M\setminus B_{L\epsilon}(p)} \nabla G_1 \nabla(\eta H_1) + \int_{M\setminus B_{L\epsilon}(p)} |\nabla(\eta H_1)|^2 \\ =& \int_{B_L(0)} |\nabla_{\mathbb{R}^2} w|^2 + \pi(\lambda_1^2(p) + \nu_1^2(p))\,(L\epsilon)^2 - 8\pi\left(4\pi - \frac{\rho_2}{2}\right)(L\epsilon)^2 \\ &+ \int_{M\setminus B_{L\epsilon}(p)} |\nabla G_1|^2 + O\left((L\epsilon)^4\right),\end{aligned}$$

$$\begin{aligned}\int_M |\nabla\phi_2^\epsilon|^2 =& \int_{B_{L\epsilon}(p)} |\nabla\phi_2^\epsilon|^2 + \int_{M\setminus B_{L\epsilon}(p)} |\nabla G_2|^2 \\ &- 2\int_{M\setminus B_{L\epsilon}(p)} \nabla G_2 \nabla(\eta H_2) + \int_{M\setminus B_{L\epsilon}(p)} |\nabla(\eta H_2)|^2 \\ =& \frac{1}{4}\int_{B_L(0)} |\nabla_{\mathbb{R}^2} w|^2 + \pi(\lambda_2^2(p) + \nu_2^2(p))\,(L\epsilon)^2 + 4\pi(\rho_2 - 2\pi)\,(L\epsilon)^2 \\ &+ \int_{M\setminus B_{L\epsilon}(p)} |\nabla G_2|^2 + O\left((L\epsilon)^4\right)\end{aligned}$$

and

$$\begin{aligned}\int_M \nabla\phi_1^\epsilon\nabla\phi_2^\epsilon =& \int_{B_{L\epsilon}(p)} \nabla\phi_1^\epsilon\nabla\phi_2^\epsilon + \int_{M\setminus B_{L\epsilon}(p)} \nabla G_1\nabla G_2 \\ &- \int_{M\setminus B_{L\epsilon}(p)} (\nabla G_1\nabla(\eta H_2) + \nabla G_2\nabla(\eta H_1)) + \int_{M\setminus B_{L\epsilon}(p)} \nabla(\eta H_1)\nabla(\eta H_2) \\ =& -\frac{1}{2}\int_{B_L(0)} |\nabla_{\mathbb{R}^2} w|^2 + \pi(\lambda_1(p)\lambda_2(p) + \nu_1(p)\nu_2(p))\,(L\epsilon)^2 \\ &- 4\pi(\rho_2 - 2\pi)\,(L\epsilon)^2 + 2\pi\left(4\pi - \frac{\rho_2}{2}\right)(L\epsilon)^2 \\ &+ \int_{M\setminus B_{L\epsilon}(p)} \nabla G_1\nabla G_2 + O\left((L\epsilon)^4\right).\end{aligned}$$

Noticing that

$$\begin{aligned}&\int_{M\setminus B_{L\epsilon}(p)} \left(|\nabla G_1|^2 + |\nabla G_2|^2 + \nabla G_1\nabla G_2\right) \\ =& \int_{M\setminus B_{L\epsilon}(p)} \left(|\nabla G_1|^2 + |\nabla G_2|^2 + \frac{\nabla G_1\nabla G_2 + \nabla G_2\nabla G_1}{2}\right) \\ =& 6\pi\int_{B_{L\epsilon}(p)} G_1 + \frac{3}{2}\rho_2\int_M G_2\left(h_2 e^{G_2} - 1\right) + \frac{3}{2}\rho_2\int_{B_{L\epsilon}(p)} G_2 \\ &- \int_{\partial B_{L\epsilon}(p)} \left(G_1\frac{\partial G_1}{\partial n} + G_2\frac{\partial G_2}{\partial n} + \frac{G_1\frac{\partial G_2}{\partial n} + G_2\frac{\partial G_1}{\partial n}}{2}\right) \\ &+ O\left((L\epsilon)^4\log(L\epsilon)\right).\end{aligned}$$

Calculating directly, we have

$$\int_{B_{L\epsilon}(p)} G_1 = -4\pi\,(L\epsilon)^2\log(L\epsilon) + 2\pi\,(L\epsilon)^2 + \pi A_1(p)\,(L\epsilon)^2 + O\left((L\epsilon)^4\log(L\epsilon)\right)$$

and

$$\int_{B_{L\epsilon}(p)} G_2 = 2\pi\,(L\epsilon)^2\log(L\epsilon) - \pi\,(L\epsilon)^2 + \pi A_2(p)\,(L\epsilon)^2 + O\left((L\epsilon)^4\log(L\epsilon)\right).$$

For the boundary terms, we use Lemma 5.2 in [14] and Lemma 4.1 to calculate. Precisely, we have

$$\begin{aligned}
\int_{\partial B_{L\epsilon}(p)} G_1\frac{\partial G_1}{\partial n} =& 32\pi\log(L\epsilon) - 4\pi\left(4\pi - \frac{\rho_2}{2}\right)(L\epsilon)^2 + \pi(\lambda_1^2(p) + \nu_1^2(p))\,(L\epsilon)^2\\
& - 8\pi A_1(p) + 2\pi\left(4\pi - \frac{\rho_2}{2}\right)A_1(p)\,(L\epsilon)^2\\
& - 8\pi\left(4\pi - \frac{\rho_2}{2}\right)(L\epsilon)^2\log(L\epsilon)\\
& + O\left((L\epsilon)^4\log(L\epsilon)\right),\\
\int_{\partial B_{L\epsilon}(p)} G_2\frac{\partial G_2}{\partial n} =& 8\pi\log(L\epsilon) + 2\pi(\rho_2 - 2\pi)\,(L\epsilon)^2 + \pi(\lambda_2^2(p) + \nu_2^2(p))\,(L\epsilon)^2\\
& + 4\pi A_2(p) + 4\pi(\rho_2 - 2\pi)A_2(p)\,(L\epsilon)^2\\
& + 4\pi(\rho_2 - 2\pi)\,(L\epsilon)^2\log(L\epsilon)\\
& + O\left((L\epsilon)^4\log(L\epsilon)\right),\\
\int_{\partial B_{L\epsilon}(p)} G_1\frac{\partial G_2}{\partial n} =& -16\pi\log(L\epsilon) - 4\pi(\rho_2 - 2\pi)\,(L\epsilon)^2\\
& + \pi(\lambda_1(p)\lambda_2(p) + \nu_1(p)\nu_2(p))\,(L\epsilon)^2\\
& - 8\pi A_2(p) + 2\pi\left(4\pi - \frac{\rho_2}{2}\right)A_2(p)\,(L\epsilon)^2\\
& + 4\pi(\rho_2 - 2\pi)\,(L\epsilon)^2\log(L\epsilon)\\
& + O\left((L\epsilon)^4\log(L\epsilon)\right),\\
\int_{\partial B_{L\epsilon}(p)} G_2\frac{\partial G_1}{\partial n} =& -16\pi\log(L\epsilon) + 2\pi\left(4\pi - \frac{\rho_2}{2}\right)(L\epsilon)^2\\
& + \pi(\lambda_2(p)\lambda_1(p) + \nu_2(p)\nu_1(p))\,(L\epsilon)^2\\
& + 4\pi A_1(p) + 2\pi(\rho_2 - 2\pi)A_1(p)\,(L\epsilon)^2\\
& - 8\pi(\rho_2 - 2\pi)\,(L\epsilon)^2\log(L\epsilon)\\
& + O\left((L\epsilon)^4\log(L\epsilon)\right).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
&\frac{1}{3}\int_M \left(|\nabla\phi_1^\epsilon|^2 + |\nabla\phi_2^\epsilon|^2 + \nabla\phi_1^\epsilon\nabla\phi_2^\epsilon\right)\\
=& 4\pi\log\left(1 + \pi L^2\right) - \frac{4\pi^2 L^2}{1 + \pi L^2} - 8\pi\log(L\epsilon) + 2\pi A_1(p)\\
& + \frac{1}{2}\rho_2\int_M G_2\left(h_2 e^{G_2} - 1\right) + O\left((L\epsilon)^4\log(L\epsilon)\right).
\end{aligned} \tag{4.2}$$

Do calculations, we have

$$\int_M \phi_1^\epsilon = \epsilon^2 \int_{B_L(0)} we^{\phi(\epsilon x^1,\epsilon x^2)} + 4\log(L\epsilon) + 2\pi\,(L\epsilon)^2 \log\left(1+\pi L^2\right) \\ - 2\pi\,(L\epsilon)^2 - A_1(p) - 2\log\left(1+\pi L^2\right) + O\left((L\epsilon)^4 \log(L\epsilon)\right) \tag{4.3}$$

and

$$\int_M \phi_2^\epsilon = -\frac{\epsilon^2}{2} \int_{B_L(0)} we^{\phi(\epsilon x^1,\epsilon x^2)} - \pi\,(L\epsilon)^2 \log\left(1+\pi L^2\right) \\ + \pi\,(L\epsilon)^2 + \int_M G_2 + O\left((L\epsilon)^4 \log(L\epsilon)\right). \tag{4.4}$$

Since

$$\int_{B_L(0)} we^{\phi(\epsilon x^1,\epsilon x^2)} = 2\pi L^2 - 2\log\left(1+\pi L^2\right) - 2\pi L^2 \log\left(1+\pi L^2\right) + O\left(L^2\epsilon^2 \log L\right),$$

we obtain that by instituting this into (4.3) and (4.4) respectively

$$\int_M \phi_1^\epsilon = 4\log(L\epsilon) - A_1(p) - 2\log\left(1+\pi L^2\right) \\ - 2\epsilon^2 \log\left(1+\pi L^2\right) + O\left((L\epsilon)^4 \log(L\epsilon)\right) \tag{4.5}$$

and

$$\int_M \phi_2^\epsilon = \epsilon^2 \log\left(1+\pi L^2\right) + \int_M G_2 + O\left((L\epsilon)^4 \log(L\epsilon)\right). \tag{4.6}$$

Denoting $\mathcal{M} = \frac{1}{\pi}\left(-\frac{K(p)}{2} + \frac{(b_1(p)+\lambda_1(p))^2+(b_2(p)+\nu_1(p))^2}{4}\right)$ and using $\alpha_1(p)+\beta_1(p) = 4\pi - \frac{\rho_2}{2}$, we have

$$\int_{B_{L\epsilon}(p)} e^{\phi_1^\epsilon} = \epsilon^2 \left(1 - \frac{1}{1+\pi L^2} + \mathcal{M}\epsilon^2 \log\left(1+\pi L^2\right) + O\left(\epsilon^2\right) + O\left(\epsilon^3 \log L\right)\right), \tag{4.7}$$

$$\int_{B_\delta(p)\setminus B_{L\epsilon}(p)} e^{\phi_1^\epsilon} = \epsilon^2 \left(\frac{\pi L^2}{\left(1+\pi L^2\right)^2} - \left(\mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi}\right)\epsilon^2 \log(L\epsilon)^2 \\ + O\left(\epsilon^2\right) + O\left(\frac{1}{L^4}\right)\right), \tag{4.8}$$

and

$$\int_{M\setminus B_\delta(p)} e^{\phi_1^\epsilon} = O\left(\epsilon^4\right). \tag{4.9}$$

By combining (4.7), (4.8) and (4.9), one has

$$\int_M e^{\phi_1^\epsilon} = \epsilon^2 \left(1 + \mathcal{M}\epsilon^2 \log\left(1+\pi L^2\right) - \left(\mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi}\right)\epsilon^2 \log(L\epsilon)^2 \\ + O\left(\epsilon^2\right) + O\left(\frac{1}{L^4}\right) + O\left(\epsilon^3 \log L\right)\right). \tag{4.10}$$

Suppose that in $B_\delta(p)$

$$h_1(x) - h_1(p) = k_1 r\cos\theta + k_2 r\sin\theta + k_3 r^2\cos^2\theta + 2k_4\cos\theta\sin\theta + k_5 r^2\sin^2\theta + O(r^3).$$

It follows from a simple computation that

$$\int_{B_{L\epsilon}(p)}(h_1 - h_1(p))e^{\phi_1^\epsilon} = \frac{1}{2\pi}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^4\log\left(1+\pi L^2\right) + O\left(\epsilon^4\right), \tag{4.11}$$

$$\int_{B_\delta(p)\setminus B_{L\epsilon}(p)}(h_1 - h_1(p))e^{\phi_1^\epsilon} = -\frac{1}{2\pi}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^4\log(L\epsilon)^2 + O\left(\epsilon^4\right), \tag{4.12}$$

and

$$\int_{M\setminus B_\delta(p)}(h_1 - h_1(p))e^{\phi_1^\epsilon} = O\left(\epsilon^4\right). \tag{4.13}$$

By (4.10), (4.11), (4.12) and (4.13), we know that

$$\begin{aligned}\int_M h_1 e^{\phi_1^\epsilon} =& h_1(p)\int_M e^{\phi_1^\epsilon} + \int_M (h_1 - h_1(p))e^{\phi_1^\epsilon}\\ =& h_1(p)\epsilon^2\left(1 + \mathcal{M}\epsilon^2\log\left(1+\pi L^2\right) - \left(\mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi}\right)\epsilon^2\log(L\epsilon)^2\right)\\ &+ \frac{1}{2\pi}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^4\log\left(1+\pi L^2\right)\\ &- \frac{1}{2\pi}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^4\log(L\epsilon)^2\\ &+ O\left(\epsilon^4\right) + O\left(\frac{\epsilon^2}{L^4}\right) + O\left(\epsilon^5\log L\right).\end{aligned}$$

Then we have

$$\begin{aligned}\log\int_M h_1 e^{\phi_1^\epsilon} =& \log h_1(p) + \log\epsilon^2\\ &+ \mathcal{M}\epsilon^2\log\left(1+\pi L^2\right) - \left(\mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi}\right)\epsilon^2\log(L\epsilon)^2\\ &+ \frac{1}{2\pi h_1(p)}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^2\log\left(1+\pi L^2\right)\\ &- \frac{1}{2\pi h_1(p)}[k_3 + k_5 + k_1(b_1+\lambda_1) + k_2(b_2+\nu_1)]\epsilon^2\log(L\epsilon)^2\\ &+ O\left(\epsilon^2\right) + O\left(\frac{1}{L^4}\right).\end{aligned} \tag{4.14}$$

Direct calculation shows that

$$\int_{B_{2L\epsilon}(p)} e^{\phi_2^\epsilon} = O\left((L\epsilon)^4\right), \quad \int_{B_{2L\epsilon}(p)} e^{G_2} = O\left((L\epsilon)^4\right).$$

Since $\int_M h_2 e^{G_2} = 1$, we obtain that

$$\log \int_M h_2 e^{\phi_2^\epsilon} = \log\left(1 - O\left((L\epsilon)^4\right)\right) = O\left((L\epsilon)^4\right). \tag{4.15}$$

Taking (4.2), (4.5), (4.6), (4.14) and (4.15) into the functional, we obtain that

$$\begin{aligned} J_{4\pi,\rho_2}(\phi_1^\epsilon, \phi_2^\epsilon) =& -4\pi - 4\pi\log\pi - 4\pi\log h_1(p) - 2\pi A_1(p) + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2} + 1) \\ & - 4\pi\left[\mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi} + \frac{k_3 + k_5 + k_1(b_1 + \lambda_1) + k_2(b_2 + \nu_1)}{2\pi h_1(p)}\right] \\ & \times \epsilon^2\left[\log\left(1 + \pi L^2\right) - \log(L\epsilon)^2\right] \\ & + O\left(\epsilon^2\right) + O\left(\frac{1}{L^4}\right) + O\left((L\epsilon)^4 \log(L\epsilon)\right) + O\left(\epsilon^3 \log L\right). \end{aligned}$$

Note that under the assumption (1.11), we have

$$\begin{aligned} \mathcal{N} :=& \mathcal{M} + \frac{4\pi - \frac{\rho_2}{2}}{2\pi} + \frac{k_3 + k_5 + k_1(b_1 + \lambda_1) + k_2(b_2 + \nu_1)}{2\pi h_1(p)} \\ =& -\frac{K(p)}{2\pi} + \frac{(b_1 + \lambda_1)^2 + (b_2 + \mu_1)^2}{4\pi} \\ & + \frac{4\pi - \frac{\rho_2}{2}}{2\pi} + \frac{\frac{1}{2}\Delta h_1(p) + k_1(b_1 + \lambda_1) + k_2(b_2 + \nu_1)}{2\pi h_1(p)} \\ =& \frac{1}{4\pi}\left[\Delta \log h_1(p) + 8\pi - \rho_2 - 2K(p)\right] + \frac{1}{4\pi}\left[(b_1 + \lambda_1 + k_1)^2 + (b_2 + \nu_1 + k_2)^2\right] \\ >& 0, \end{aligned}$$

where we have used $\Delta h_1(p) = \frac{1}{2}(k_3 + k_5)$ and $\nabla h_1(p) = (k_1, k_2)$.

By choosing $L^4\epsilon^2 = \frac{1}{\log(-\log\epsilon)}$, we have

$$\begin{aligned} J_{4\pi,\rho_2}(\phi_1^\epsilon, \phi_2^\epsilon) =& -4\pi - 4\pi\log\pi - 4\pi\log h_1(p) - 2\pi A_1(p) + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2} + 1) \\ & - 4\pi\mathcal{N}\epsilon^2(-\log\epsilon^2) + o(\epsilon^2(-\log\epsilon^2)). \end{aligned}$$

Since $\mathcal{N} > 0$, we have for sufficiently small ϵ that

$$J_{4\pi,\rho_2}(\phi_1^\epsilon, \phi_2^\epsilon) < -4\pi - 4\pi\log\pi - 4\pi\log h_1(p) - 2\pi A_1(p) + \frac{\rho_2}{2}\int_M G_2(h_2 e^{G_2} + 1).$$

This finishes the proof of Theorem 1.3. □

Data Availability Data sharing is not applicable to this article as obviously no datasets were generated or analyzed during the current study.

Declarations

Conflicts of Interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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