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Calculus of Variations

An existence result for the Kazdan–Warner equation with a sign-changing prescribed function

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Abstract

In this paper, we study the following Kazdan–Warner equation with a sign-changing prescribed function h

$$-\Delta u = 8\pi \left(\frac{he^u}{\int_\Sigma he^u} - 1 \right)$$

on a closed Riemann surface Σ whose area equals one. The solutions are the critical points of the functional $J_{8\pi}$ which is defined by

$$J_{8\pi}(u) = \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 + \fint_\Sigma u - \ln \left| \int_\Sigma he^u \right|, \quad u \in H^1(\Sigma).$$

We prove the existence of the minimizer of $J_{8\pi}$ by assuming

$$\Delta \ln h^+ + 8\pi - 2\kappa > 0$$

at each maximum point of $2\ln h^+ + A$, where κ is the Gaussian curvature, h^+ is the positive part of h and A is the regular part of the Green function. This generalizes the existence result of Ding et al. (Asian J Math 1:230–248, 1997) to the sign-changing prescribed function case. We are also interested in the blow-up behavior of a sequence u_ε of critical points of $J_{8\pi-\varepsilon}$ with $\int_\Sigma he^{u_\varepsilon} = 1$, $\lim\limits_{\varepsilon \searrow 0} J_{8\pi-\varepsilon}(u_\varepsilon) < \infty$ and obtain the following identity during the blow-up process

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$$-\varepsilon = \frac{16\pi}{(8\pi - \varepsilon)h(p_\varepsilon)} \left[\Delta \ln h(p_\varepsilon) + 8\pi - 2\kappa(p_\varepsilon)\right] \lambda_\varepsilon e^{-\lambda_\varepsilon} + O\left(e^{-\lambda_\varepsilon}\right),$$

where u_ε takes its maximum value λ_ε at p_ε. Moreover, p_ε converges to the blow-up point which is a critical point of the function $2 \ln h^+ + A$.

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1 Introduction

Let Σ be a closed Riemann surface whose area equals one. Let h be a nonzero smooth function on Σ such that $\max_\Sigma h > 0$. For each positive number ρ, we consider the following functional

$$J_\rho(u) = \frac{1}{2\rho} \int_\Sigma |\nabla u|^2 + \fint_\Sigma u - \ln \left| \int_\Sigma h e^u \right|, \quad u \in H^1(\Sigma).$$

The critical points of J_ρ are solutions to the following mean field equation

$$-\Delta u = \rho \left(\frac{h e^u}{\int_\Sigma h e^u} - 1 \right) \tag{1.1}$$

where Δ is the Laplace operator on Σ.

Mean field equation has a strong relationship with Kazdan–Warner equation. Forty years ago, Kazdan and Warner [15] considered the solvability of the equation

$$-\Delta u = h e^u - \rho,$$

where ρ is a constant and h is some smooth prescribed function. When $\rho > 0$, the equation above is equivalent to the mean field Eq. (1.1). The special case $\rho = 8\pi$ is sometimes called the Kazdan–Warner equation. In particular, when Σ is the standard sphere $\mathbb{S}^2$, it is called the Nirenberg problem, which comes from the conformal geometry. It has been studied by Moser [21], Kazdan and Warner [15], Chen and Ding [6], Chang and Yang [3], and others. The mean field Eq. (1.1) appears in various context such as the abelian Chern-Simons-Higgs models. The existence of solutions of (1.1) and its evolution problem has been widely studied in recent decades (see for example [1, 2, 4, 5, 9–12, 16, 19, 20, 22, 23] and the references therein).

In this paper, we consider the existence theory of Kazdan–Warner equation ($\rho = 8\pi$) with sign-changing prescribed function. The key is to analyze the asymptotic behavior of the blow-up solutions u_ε (see (1.2)) and the functional $J_{8\pi}$. We prove the following identity near the blow-up point, whose analogue was proved by Chen and Lin in [4] when the prescribed function h is positive.

Theorem 1.1 *Let h be positive somewhere on Σ and u_ε a blow-up sequence satisfying*

$$-\Delta u_\varepsilon = (8\pi - \varepsilon)\left(h e^{u_\varepsilon} - 1\right), \quad in\ \Sigma \tag{1.2}$$

and

$$\lim_{\varepsilon \searrow 0} J_{8\pi-\varepsilon}(u_\varepsilon) < \infty. \tag{1.3}$$

Then up to a subsequence, for $p_\varepsilon \in \Sigma$ with

$$\lambda_\varepsilon = \max_\Sigma u_\varepsilon = u_\varepsilon\left(p_\varepsilon\right),$$

we have

$$-\varepsilon = \frac{16\pi}{(8\pi - \varepsilon)h(p_\varepsilon)}[\Delta \ln h^+(p_\varepsilon) + 8\pi - 2\kappa(p_\varepsilon)]\lambda_\varepsilon e^{-\lambda_\varepsilon} + O\left(e^{-\lambda_\varepsilon}\right), \quad as\ \varepsilon \searrow 0,$$

where κ denotes the Gaussian curvature of Σ.

This yields a uniform bound of minimizers as $\varepsilon \searrow 0$ provided that $\Delta \ln h^+ + 8\pi - 2\kappa > 0$ at all blow-up points. Let $G(q, p)$ be the Green function on Σ with singularity at p, i.e.,

$$\Delta G(\cdot, p) = 1 - \delta_p, \quad \int_\Sigma G(\cdot, p) = 0.$$

Under a local normal coordinate x centering at p, we have

$$8\pi G(x, p) = -4\ln|x| + A(p) + b_1x_1 + b_2x_2 + c_1x_1^2 + 2c_2x_1x_2 + c_3x_2^2 + O\left(|x|^3\right). \tag{1.4}$$

By Lemma 2.4, we know the blow-up point has to be a critical point of $2\ln h^+(p) + A(p)$. Thus, we get an existence result. That is, we have the following

Corollary 1.2 *Let Σ be a compact Riemann surface and κ be its Gaussian curvature. Suppose h is a smooth function which is positive somewhere on Σ. If we have the following for all critical points of $2\ln h^+ + A$*

$$\Delta \ln h^+ + 8\pi - 2\kappa > 0,$$

then Eq. (1.1) has a solution for $\rho = 8\pi$.

Furthermore, if u_ε is a minimizer of $J_{8\pi-\varepsilon}$, we can show the blow-up point is actually the maximum point of $2\ln h^+ + A$.

Theorem 1.3 *If u_ε is a minimizer of $J_{8\pi-\varepsilon}$ and blows up as $\varepsilon \searrow 0$, then the blow-up point p_0 is a maximum point of the function $2\ln h^+ + A$. Moreover,*

$$\inf_{u\in H^1(\Sigma)} J_{8\pi} = -1 - \ln\pi - \left(\ln h(p_0) + \frac{1}{2}A(p_0)\right),$$

and there is a sequence $\phi_\varepsilon \in H^1(\Sigma)$ such that

$$J_{8\pi}\left(\phi_\varepsilon\right) = -1 - \ln\pi - \left(\ln h(p_0) + \frac{1}{2}A(p_0)\right) - \frac{1}{4}\left(\Delta \ln h(p_0) + 8\pi - 2\kappa(p_0)\right)\varepsilon\ln\varepsilon^{-1} + o\left(\varepsilon\ln\varepsilon^{-1}\right).$$

Hence, we obtain a minimizing solution of the functional $J_{8\pi}$. In other words, we obtain the following

Theorem 1.4 *Let Σ be a compact Riemann surface and κ be its Gaussian curvature. Suppose h is a smooth function which is positive somewhere on Σ. If the following holds at the maximum points of $2\ln h^+ + A$*

$$\Delta \ln h + 8\pi - 2\kappa > 0,$$

then Eq. (1.1) has a minimizing solution for $\rho = 8\pi$.

Remark 1.5 The condition mentioned in Theorem 1.4 can not hold on 2-sphere with arbitrary metric. Assume $g = e^{2\phi} g_0$ and solve

$$-\Delta_{g_0}\psi = \frac{1}{|\Sigma|_{g_0}} - \frac{e^{2\phi}}{|\Sigma|_g}, \quad \int_\Sigma \psi \mathrm{d}\mu_{g_0} = 0,$$

where $|\Sigma|_g$ stands for the area of Σ with respect to the metric g. Set $h_0 = he^{2\phi+\rho\psi}$. Then

$$J_{\rho,h,g}(u) = J_{\rho,h_0,g_0}(u - \rho\psi) - \frac{\rho}{2}\int_\Sigma |\mathrm{d}\psi|^2_{g_0} \mathrm{d}\mu_{g_0},$$

where $J_{\rho,g,h}(u) = \frac{1}{2\rho}\int_\Sigma |\nabla_g u|^2_g \mathrm{d}\mu_g + \frac{1}{|\Sigma|_g}\int_\Sigma u \mathrm{d}\mu_g - \ln\left|\int_\Sigma he^u \mathrm{d}\mu_g\right|$. If the condition mentioned in Theorem 1.4 holds, then there is a minimizer of $J_{8\pi,h,g}$. Hence, there is also a minimizer of $J_{8\pi,h_0,g_0}$. If Σ is a 2-sphere, we choose g_0 such that the Gaussian curvature is constant, then h_0 must be a constant (see [14]). Thus h is a positive function and

$$\Delta_g \ln h + \frac{8\pi}{|\Sigma|_g} - 2\kappa_g = e^{-2\phi}\left(\Delta_{g_0} \ln h_0 + \frac{8\pi}{|\Sigma|_{g_0}} - 2\kappa_{g_0}\right) = 0$$

which is a contradiction.

Remark 1.6 Zhu [25] also obtained the infimum of the functional $J_{8\pi}$ if there is no minimizer (when h is non-negative). He pointed out the blow-up point must be the positive point of h and used the maximum principle to estimate the lower bound of the functional $J_{8\pi}$ when h is non-negative. In our case, the maximum principle does not work since h is sign-changed. We will use the method of energy estimate to give the lower bound of the functional $J_{8\pi}$. Such a method also can be used to consider the flow case (cf. [16, 23]) and the Palais-Smale sequence.

Remark 1.7 The method in the proof of Theorem 1.4 can be used to prove the convergence of the Kazdan–Warner flow. In other words, under the same condition mentioned in Theorem 1.4, there exists an initial date u_0 such that the following flow

$$\frac{\partial e^u}{\partial t} = \Delta u + 8\pi\left(\frac{he^u}{\int_\Sigma he^u} - 1\right), \quad u(0) = u_0$$

converges to a minimizer of $J_{8\pi}$. This gives a generalization of the previous results [16] (positive prescribed function case) and [23] (non-negative prescribed function case).

After we release the first version of this paper on arXiv (see arXiv:2012.12840), more articles have appeared on this topic. For example, Wang and Yang give more details about our Remark 1.7 in [24]. Chen, Li, Li and Xu [8] consider another flow approach to the Gaussian curvature flow on sphere and reproved the existence result for sign-changing prescribed function which was obtained by Han [14].

2 Preliminary

Recall the strong Trudinger–Moser inequality (cf. [13, Theorem 1.7])

$$\sup_{u\in H^1(\Sigma), \int_\Sigma |\nabla u|^2 \le 1, \int_\Sigma u = 0} \int_\Sigma \exp\left(4\pi u^2\right) < \infty.$$

which implies the Trudinger–Moser inequality

$$\ln \int_{\Sigma} e^{u} \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^{2} + \int_{\Sigma} u + c \tag{2.1}$$

where c is a uniform constant depends only on the geometry of Σ.

We may assume h is positive somewhere. If $0 < \rho < 8\pi$, then applying the Trudinger–Moser inequality (2.1) Kazdan and Warner ([15, Theorem 7.2]) proved that the Kazdan–Warner Eq. (1.1) admits a solution u which minimizes the functional J_{ρ} and satisfies

$$\int_{\Sigma} he^{u} = 1.$$

We consider the critical case $\rho = 8\pi$. For every $\varepsilon \in (0, 8\pi)$, let u_{ε} be a minimizer of $J_{8\pi-\varepsilon}$ which satisfies

$$\int_{\Sigma} he^{u_{\varepsilon}} = 1.$$

Thus u_{ε} satisfies (1.2). It is clear that the function

$$\rho \mapsto \inf_{u \in H^{1}(\Sigma)} J_{\rho}(u)$$

is a decreasing function on $(0, +\infty)$. In particular, u_{ε} satisfies (1.3). By the Trudinger–Moser inequality (2.1), we have

$$J_{8\pi-\varepsilon}(u_{\varepsilon}) \geq \ln \int_{\Sigma} e^{u_{\varepsilon}} - c. \tag{2.2}$$

Thus (1.3) and (2.2) gives

$$\int_{\Sigma} e^{u_{\varepsilon}} \leq C, \quad \forall \varepsilon \in (0, 4\pi). \tag{2.3}$$

One can check that

$$\lim_{\varepsilon \searrow 0} J_{8\pi-\epsilon}(u_{\varepsilon}) = \inf_{u \in H^{1}(\Sigma)} J_{8\pi}(u).$$

If

$$\limsup_{\varepsilon \searrow 0} \max_{\Sigma} u_{\varepsilon} < +\infty,$$

then up to a subsequence u_{ε} converges smoothly to a minimizer of $J_{8\pi}$.

In the rest of this section, we only assume u_{ε} is a solution to (1.2) and satisfies the condition (2.3).

Assume now $\{u_{\varepsilon}\}$ is a blow-up sequence, i.e.,

$$\limsup_{\varepsilon \searrow 0} \max_{\Sigma} u_{\varepsilon} = +\infty.$$

Denote $h^{+} = \max\{h, 0\}$ and $h^{-} = (-h)^{+}$ by the positive and negative part of h respectively. Without loss of generality, we may assume $h^{\pm} e^{u_{\varepsilon}} \mathrm{d}\mu_{\Sigma}$ converges to a nonzero Radon measure $\mu^{\pm}$ as $\varepsilon \to 0$. As in [1, Page 1240], let us define the singular set S of the sequence $\{u_{\varepsilon}\}$ by

$$S=\left\{x\in\Sigma:|\mu|\left(\{x\}\right)\geq\frac{1}{2}\right\},$$

where $|\mu|=\mu^{+}+\mu^{-}$. By the Fatou Lemma, it follows from (2.3) that S is a finite set. Applying Brezis-Merle's estimate [1, Theorem 1], one can obtain that for each compact subset $K\subset\Sigma\setminus S$ (cf. [9, Lemma 2.8])

$$\left\|u_{\varepsilon}-\int_{\Sigma}u_{\varepsilon}\right\|_{L^{\infty}(K)}\leq C_{K}. \tag{2.4}$$

Then one obtains a characterization of S by the blow-up sets of $\{u_{\varepsilon}\}$ (cf. [1, Page 1240])

$$S=\left\{p\in\Sigma:\exists\, p_{\varepsilon}\in\Sigma,\ s.t.\ \lim_{\varepsilon\to 0}p_{\varepsilon}=p,\ \lim_{\varepsilon\to 0}u_{\varepsilon}\left(p_{\varepsilon}\right)=+\infty.\right\}$$

In fact, on one hand, by (2.4), we know that

$$\left\{p\in\Sigma:\exists\, p_{\varepsilon}\in\Sigma,\ s.t.\ \lim_{\varepsilon\to 0}p_{\varepsilon}=p,\ \lim_{\varepsilon\to 0}u_{\varepsilon}\left(p_{\varepsilon}\right)=+\infty\right\}\subset S.$$

On the other hand, for $p_0\in S$, we may assume $B_{2r}(p_0)\cup S=\{p_0\}$ and choose $p_{\varepsilon}\in\overline{B_r}$ with $\lambda_{\varepsilon}:=u_{\varepsilon}(p_{\varepsilon})=\max\limits_{\overline{B_r(p_0)}}u_{\varepsilon}$. One can show that $\lim\limits_{\varepsilon\to 0}\lambda_{\varepsilon}=+\infty$ and $\lim\limits_{\varepsilon\to 0}p_{\varepsilon}=p_0$ (cf. [23, Theorem 3.4]). In particular,

$$S\subset\left\{p\in\Sigma:\exists\, p_{\varepsilon}\in\Sigma,\ s.t.\ \lim_{\varepsilon\to 0}p_{\varepsilon}=p,\ \lim_{\varepsilon\to 0}u_{\varepsilon}\left(p_{\varepsilon}\right)=+\infty.\right\}$$

Moreover, S is nonempty and

$$\lim_{\varepsilon\to 0}\int_{\Sigma}u_{\varepsilon}=-\infty,$$

which implies that u_{ε} goes to $-\infty$ uniformly on each compact subsets $K\subset\Sigma\setminus S$. Thus, $|\mu|$ is a Dirac measure. By using blow-up analysis (cf. [18, Lemma 1]) and the classification result of Chen-Li [7, Theorem 1] as in the proof of [23, Lemma 3.5], one can show that

$$S=\left\{p\in\Sigma:\mu^{+}\left(\{p\}\right)\geq 1, h(p)>0\right\}$$

and then $\mu^{-}=0$. In fact, fixed $p_0\in S$, let λ_{ε} and p_{ε} are given before. By choose a conformal coordinate y centered at x_0, we consider the blow-up sequence

$$\tilde{u}_{\varepsilon}(y)=u_{\varepsilon}\left(p_{\varepsilon}+e^{-\lambda_{\varepsilon}/2}y\right)-\lambda_{\varepsilon}.$$

One can show that $\tilde{u}_{\varepsilon}$ will converges to a solution w to the following PDE

$$-\Delta_{\mathbb{R}^2}w=h(p_0)Ce^{w},\quad\int_{\mathbb{R}^2}e^{w}<\infty,$$

for some positive number C. By a classification theorem of Chen-Li [7, Theorem 1], we know that $h(p_0)>0$ which means $\mu^{-}=0$. Then according the Fatou Lemma, we know that $\mu^{+}(p_0)\geq 1$. Hence, $he^{u_{\varepsilon}}\mathrm{d}\mu_{\Sigma}$ converges to the nonzero Radon measure μ^{+} as $\varepsilon\to 0$. As in Lemma 3.5 in [23], we conclude that $S=\{p_0\}$ is a single point set and $|\mu|=\mu^{+}=\delta_{p_0}$. Thus

Lemma 2.1 (cf. Lemma 2.6 in [9]) *$u_\varepsilon - \int_\Sigma u_\varepsilon$ converges to $8\pi G(\cdot, p_0)$ weakly in $W^{1,q}(\Sigma)$ and strongly in $L^q(\Sigma)$ for every $q \in (1,2)$, and converges in $C^2_{loc}(\Sigma \setminus \{p_0\})$.*

For a fixed small $\delta_0 > 0$ and u_ε of $J_{8\pi}$, we define ρ_ε to be

$$\rho_\varepsilon = (8\pi - \varepsilon) \int\limits_{B_{\delta_0}(p_0)} he^{u_\varepsilon}$$

and

$$\lambda_\varepsilon = u_\varepsilon(p_\varepsilon) = \max_{B_\delta(p_0)} u_\varepsilon \to +\infty.$$

We may assume

$$h|_{B_{\delta_0}(p_0)} \geq \frac{1}{2} h(p_0) > 0, \quad \max_{\partial B_{\delta_0}(p_0)} u_\varepsilon - \min_{\partial B_{\delta_0}(p_0)} u_\varepsilon \leq C, \quad \int\limits_{B_{\delta_0}(p_0)} e^{u_\varepsilon} \leq C.$$

Li [17, Theorem 0.3] obtained the following local estimate

$$\left| u_\varepsilon(p) - \ln \frac{e^{\lambda_\varepsilon}}{1 + \frac{(8\pi-\varepsilon) h_{p_\varepsilon}}{8} e^{\lambda_\varepsilon} |p - p_\varepsilon|^2} \right| \leq C \tag{2.5}$$

for $p \in B_{\delta_0}(p_0)$, where $|p - p_\varepsilon|$ stands for the distance between p and p_ε. Together with Lemma 2.1, the above local estimate (2.5) gives the following

Lemma 2.2 (cf. Corollary 2.4 in [4]) *There exists a constant $C > 0$ such that*

$$|u_\varepsilon + \lambda_\varepsilon| \leq C, \quad \text{in } \Sigma \setminus B_{\delta_0}(p_0).$$

Lemma 2.3 (cf. Estimate A in [4]) *Set w_ε to be the error term defined by*

$$\omega_\varepsilon(q) = u_\varepsilon(q) - \rho_\varepsilon G(q, p_\varepsilon) - \bar{u}_\varepsilon, \quad \text{on } \Sigma \setminus B_{\delta_0/2}(p_0),$$

where $\bar{u}_\varepsilon = \int\limits_\Sigma u_\varepsilon$. Then we have

$$\|\omega_\varepsilon\|_{C^1(\Sigma \setminus B_{\delta_0}(p_0))} = O\left(e^{-\lambda_\varepsilon/2}\right).$$

Proof Notice that h maybe non-positive outside of $B_{\delta_0/2}(p_0)$ and in this case we also have the above estimate. We list a proof here. By Green representation formula, for every $q \in \Sigma \setminus B_{\delta_0}(p_0)$

$$\begin{aligned} u_\varepsilon(q) - \bar{u}_\varepsilon &= (8\pi - \varepsilon) \int\limits_\Sigma G(q, p) \left[h(p) e^{u_\varepsilon(p)} - 1 \right] \mathrm{d}\mu_\Sigma(p) \\ &= (8\pi - \varepsilon) \int\limits_\Sigma \left(G(q, p) - G(q, p_\varepsilon) \right) \left[h(p) e^{u_\varepsilon(p)} - 1 \right] \mathrm{d}\mu_\Sigma(p) \\ &= (8\pi - \varepsilon) \int\limits_{\Sigma \setminus B_{\delta_0/2}(p_0)} \left(G(q, p) - G(q, p_\varepsilon) \right) h(p) e^{u_\varepsilon(p)} \mathrm{d}\mu_\Sigma(p) \\ &\quad + (8\pi - \varepsilon) \int\limits_{B_{\delta_0/2}(p_0)} \left(G(q, p) - G(q, p_\varepsilon) \right) h(p) e^{u_\varepsilon(p)} \mathrm{d}\mu_\Sigma(p) \end{aligned}$$

$$
\begin{aligned}
&+ (8\pi - \varepsilon) G(q, p_\varepsilon) \\
&= (8\pi - \varepsilon) G(q, p_\varepsilon) + O\left(e^{-\lambda_\varepsilon/2}\right).
\end{aligned}
$$

Here we used estimate (2.4) and Li's local estimate (2.5). By definition,

$$
\rho_\varepsilon = (8\pi - \varepsilon) - (8\pi - \varepsilon) \int_{\Sigma \setminus B_{\delta_0}(p_0)} h e^{u_\varepsilon} = (8\pi - \varepsilon) + O\left(e^{-\lambda_\varepsilon}\right).
$$

Thus

$$
u_\varepsilon(q) - \bar{u}_\varepsilon - \rho_\varepsilon G(q, p_\varepsilon) = O\left(e^{-\lambda_\varepsilon/2}\right), \quad \forall q \in \Sigma \setminus B_{\delta_0}(p_0).
$$

Notice that

$$
\begin{aligned}
-\Delta\left(u_\varepsilon - \bar{u}_\varepsilon - \rho_\varepsilon G(\cdot, p_\varepsilon)\right) &= (8\pi - \varepsilon) h e^{u_\varepsilon} + \rho_\varepsilon - (8\pi - \varepsilon) \\
&= O\left(e^{-\lambda_\varepsilon}\right), \quad \text{in } \Sigma \setminus B_{\delta_0}(p_0)
\end{aligned}
$$

and

$$
u_\varepsilon - \bar{u}_\varepsilon - \rho_\varepsilon G(\cdot, p_\varepsilon) = O\left(e^{-\lambda_\varepsilon/2}\right), \quad \text{on } \partial B_{\delta_0}(p_0).
$$

The standard elliptic estimate gives

$$
\|u_\varepsilon - \bar{u}_\varepsilon - \rho_\varepsilon G(\cdot, p_\varepsilon)\|_{C^1(\Sigma \setminus B_{\delta_0}(p_0))} = O\left(e^{-\lambda_\varepsilon/2}\right).
$$

□

Based on these facts, we then have the following local estimates. The proofs are same as those in [4], so we omit them here.

Lemma 2.4 (cf. Estimate B in [4]) *By using the local normal coordinate x centering at p_ε, we set the regular part of Green function $G(x, p_\varepsilon)$ to be*

$$
\tilde{G}_\varepsilon(x) = G(x, p_\varepsilon) + \frac{1}{2\pi} \ln |x|,
$$

and set

$$
G_\varepsilon^*(x) = \rho_\varepsilon \tilde{G}_\varepsilon(x).
$$

Then we get

$$
\left|\nabla\left(\ln h^+ + G_\varepsilon^*\right)(p_\varepsilon)\right| = O\left(e^{-\lambda_\varepsilon/2}\right).
$$

Notice that the Green function is symmetric and we conclude that

$$
\left|\nabla\left(2 \ln h^+ + \frac{8\pi - \varepsilon}{8\pi} A\right)(p_\varepsilon)\right| = O\left(e^{-\lambda_\varepsilon/2}\right).
$$

In $B_{\delta_0}(p_\varepsilon)$, we define the following function as in [4]

$$
v_\varepsilon(p) = \ln \frac{e^{\lambda_\varepsilon}}{\left(1 + \frac{(8\pi - \varepsilon) h(p_\varepsilon)}{8} e^{\lambda_\varepsilon} |p - q_\varepsilon|^2\right)^2},
$$

where q_ε is chosen to satisfy

$$
\nabla v_\varepsilon(p_\varepsilon) = \nabla \ln h(p_\varepsilon),
$$

which implies $|p_\varepsilon - q_\varepsilon| = O\left(e^{-\lambda_\varepsilon}\right)$. We also set the error term as

$$\eta_\varepsilon(p) = u_\varepsilon(p) - v_\varepsilon(p) - (G^*_\varepsilon(p) - G^*_\varepsilon(p_\varepsilon))$$

and

$$R_\varepsilon = \left(\frac{(8\pi - \varepsilon)h(p_\varepsilon)}{8} e^{\lambda_\varepsilon}\right)^{\frac{1}{2}} \delta_0.$$

Then we have the following estimate for the scaled function $\tilde{\eta}_\varepsilon(z) = \eta_\varepsilon\left(\delta_0 R_\varepsilon^{-1} z\right)$ for $|z| \le R_\varepsilon$.

Lemma 2.5 (cf. Estimates C, D and E in [4]) *For any $\tau \in (0, 1)$, there exists a constant $C = C_\tau$ such that*

$$\eta_\varepsilon(p) = \left(4 - \frac{\rho_\varepsilon}{2\pi}\right) \ln|p - p_\varepsilon| + O\left(\lambda_\varepsilon e^{-\frac{\tau\lambda_\varepsilon}{2}} \sup_{\frac{\delta_0}{2} \le |p - p_\varepsilon| \le \delta_0} |\eta_\varepsilon| + e^{-\frac{\lambda_\varepsilon}{2}}\right)$$

and

$$|\tilde{\eta}_\varepsilon(z)| \le C\,(1 + |z|)^\tau \left(e^{-\tau\lambda_\varepsilon} + e^{-\frac{\tau}{2}\lambda_\varepsilon}|8\pi - \rho_\varepsilon|\right)$$

hold for $p \in \bar{B}_{\delta_0}(p_\varepsilon)\backslash B_{\delta_0/2}(p_\varepsilon)$ and $|z| \le R_\varepsilon$.

The following lemma shows the relationship between $\rho_\varepsilon - 8\pi$ and η_ε.

Lemma 2.6 (cf. Estimate F in [4])

$$\rho_\varepsilon - 8\pi = -\int_{\partial B_{\delta_0}(p_\varepsilon)} \frac{\partial \eta_\varepsilon}{\partial \nu} d\sigma + O\left(e^{-\lambda_\varepsilon}\right),$$

where ν denotes the unit outer normal of $\partial B_{\delta_0}(p_\varepsilon)$.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 as in [4].

Proof By Lemma 2.2, we have

$$\rho_\varepsilon = 8\pi - \varepsilon + O\left(e^{-\lambda_\varepsilon}\right). \tag{3.1}$$

This implies that we need to control $\rho_\varepsilon - 8\pi$, which is equivalent to compute $-\int_{\partial B_{\delta_0}(p_\varepsilon)} \frac{\partial \eta_\varepsilon}{\partial \nu} d\sigma$ by Lemma 2.6. To do so, we set

$$\psi = \frac{1 - a|x - y_\varepsilon|^2}{1 + a|x - y_\varepsilon|^2} \quad \text{for } x \in \mathbb{R}^2,$$

where $a = \frac{(8\pi - \varepsilon)h(p_\varepsilon)}{8} e^{\lambda_\varepsilon}$. Then ψ satisfies

$$\Delta_0 \psi + (8\pi - \varepsilon)h(p_\varepsilon)e^{v_\varepsilon}\psi = 0, \tag{3.2}$$

where Δ_0 is the standard Laplacian in $\mathbb{R}^2$. On the other hand, by (3.1), we have

$$\begin{aligned} \Delta_0 \eta_\varepsilon &= \Delta_0 u_\varepsilon - \Delta_0 v_\varepsilon - \Delta_0 G^*_\varepsilon \\ &= -(8\pi - \varepsilon)h(p_\varepsilon)e^{v_\varepsilon(x)} H(x, \eta_\varepsilon) + O(e^{-\lambda_\varepsilon}), \end{aligned} \tag{3.3}$$

where

$$H(x,t)=\frac{h^*(x)}{h(p_\varepsilon)}e^{t+G_\varepsilon^*(x)-G_\varepsilon^*(0)}-1$$

and $h^*(x)=h(x)e^{2\phi(x)}$, $\phi(x)$ comes from the metric $ds^2=e^{2\phi(x)}dx^2$ with $\phi(0)=0$ and $\nabla\phi(0)=0$. By using (3.2), (3.3) and integration by parts, we get

$$\begin{aligned}\int_{\partial B_{\delta_0}(p_\varepsilon)}\left(\psi\frac{\partial\eta_\varepsilon}{\partial\nu}-\eta_\varepsilon\frac{\partial\psi}{\partial\nu}\right)d\sigma &= \int_{B_{\delta_0}(p_\varepsilon)}(\psi\Delta_0\eta_\varepsilon-\eta_\varepsilon\Delta_0\psi)dx\\ &=-\int_{B_{\delta_0}(p_\varepsilon)}\psi(x)(8\pi-\varepsilon)h(p_\varepsilon)e^{v_\varepsilon(x)}(H(x,\eta_\varepsilon)-\eta_\varepsilon(x))\\ &\quad+O\left(e^{-\lambda_\varepsilon}\right).\end{aligned}$$

Since ψ satisfies

$$\psi(x)=-1+\frac{2}{1+a|x-y_\varepsilon|^2}=-1+O\left(e^{-\lambda_\varepsilon}\right)\text{ and }|\nabla\psi(x)|=O\left(e^{-\lambda_\varepsilon}\right)$$

for $x\in\partial B_{\delta_0}(p_\varepsilon)$, we have

$$-\int_{\partial B_{\delta_0}(p_\varepsilon)}\frac{\partial\eta_\varepsilon}{\partial\nu}d\sigma=-\int_{B_{\delta_0}(p_\varepsilon)}\psi(x)(8\pi-\varepsilon)h(p_\varepsilon)e^{v_\varepsilon(x)}(H(x,\eta_\varepsilon)-\eta_\varepsilon(x))+O\left(e^{-\lambda_\varepsilon}\right).$$

Recall

$$\begin{aligned}H(x,\eta_\varepsilon)-\eta_\varepsilon(x)&=\frac{h^*(x)}{h(p_\varepsilon)}e^{\eta_\varepsilon+G_\varepsilon^*(x)-G_\varepsilon^*(0)}-1-\eta_\varepsilon(x)\\ &=H(x,0)+H(x,0)\eta_\varepsilon+O(1)|\eta_\varepsilon|^2,\end{aligned}$$

where

$$\begin{aligned}H(x,0)&=\frac{h^*(x)}{h(p_\varepsilon)}e^{G_\varepsilon^*(x)-G_\varepsilon^*(0)}-1\\ &=\frac{1}{h(p_\varepsilon)}e^{2\phi(x)+\ln h(x)+G^*(x)-G^*(p_\varepsilon)}-1\\ &=\langle b_\varepsilon,x\rangle+\langle B_\varepsilon x,x\rangle+O(1)|x|^{2+\beta},\end{aligned}$$

where b_ε and B_ε are the gradient and Hessian of $H(x,0)$ at $x=0$. By Lemma 2.4, we have $|b_\varepsilon|=O\left(e^{-\lambda/2}\right)$.

Let z and z_ε satisfy

$$\begin{cases}x=e^{-\frac{\lambda_\varepsilon}{2}}\left(\frac{h(p_\varepsilon)(8\pi-\varepsilon)}{8}\right)^{-\frac{1}{2}}z,\\ y_\varepsilon=e^{-\frac{\lambda_\varepsilon}{2}}\left(\frac{h(p_\varepsilon)(8\pi-\varepsilon)}{8}\right)^{-\frac{1}{2}}z_\varepsilon.\end{cases}$$

Then we get

$$\left|\int_{B_{\delta_0}(p_\varepsilon)} e^{v_\varepsilon}\langle b_\varepsilon, x\rangle dx\right| \le Ce^{-\lambda_\varepsilon}\int_{|z|\le R_0}\left(1+|z-z_\varepsilon|^2\right)^{-2}|z|dz = O\left(e^{-\lambda_\varepsilon}\right),$$

$$\int_{B_{\delta_0}(p_\varepsilon)} e^{v_\varepsilon}|x|^{2+\beta}dx \le Ce^{-\frac{2+\beta}{2}\lambda_\varepsilon}\int_{|z|\le R_0}\left(1+|z|^2\right)^{-2}|z|^{2+\beta}dz = O\left(e^{-\lambda_\varepsilon}\right)$$

and

$$\begin{aligned}&\int_{B_{\delta_0}(p_\varepsilon)} e^{v_\varepsilon}(x_\alpha - p_{\varepsilon,\alpha})(x_\beta - p_{\varepsilon,\beta})dx\\ &= \left((8\pi-\varepsilon)\frac{h(p_\varepsilon)}{8}\right)^{-2} e^{-\lambda_\varepsilon}\int_{|z|\le R_0}\left(1+|z-z_\varepsilon|^2\right)^{-2} z_\alpha z_\beta dz\\ &= \left((8\pi-\varepsilon)\frac{h(p_\varepsilon)}{8}\right)^{-2} e^{-\lambda_\varepsilon}\pi\left[\delta_{\alpha\beta}\ln R_\varepsilon + O\left(e^{-\frac{\lambda_\varepsilon}{2}}\right)\right],\end{aligned}$$

where x_α stands for the α-th coordinate of x and $1\le\alpha,\beta\le 2$. Putting those estimates above together, we have

$$\int_{B_{\delta_0}(p_\varepsilon)}(8\pi-\varepsilon)h(p_\varepsilon)e^{v_\varepsilon}H(x,0)dx = \frac{32\pi}{(8\pi-\varepsilon)h(p_\varepsilon)}\left(B_\varepsilon^{11}+B_\varepsilon^{22}\right)e^{-\lambda_\varepsilon}\lambda_\varepsilon + O(1)e^{-\lambda_\varepsilon}.$$

Note that $\Delta_0 G_\varepsilon^*(0) = \rho_\varepsilon = (8\pi-\varepsilon) + O\left(e^{-\lambda_\varepsilon}\right)$ and $-\Delta_0\phi(0) = \kappa(p_\varepsilon)$. By Lemma 2.4, we know

$$\begin{aligned}B_\varepsilon^{11}+B_\varepsilon^{22} &= \frac{1}{2}\Delta_0 H(0,0)\\ &= \frac{1}{2}(\Delta\ln h(p_\varepsilon) + 8\pi - \varepsilon - 2\kappa(p_\varepsilon)) + O\left(e^{-\lambda_\varepsilon}\right).\end{aligned}$$

For the remainder terms, we use Lemma 2.5 to get

$$\int_{B_{\delta_0}(p_\varepsilon)} e^{v_\varepsilon}H(x,0)\eta_\varepsilon(x)dx = O\left(e^{-\lambda_\varepsilon}\right)$$

$$\int_{B_{\delta_0})(p_\varepsilon)} e^{v_\varepsilon}\eta_\varepsilon^2(x)dx = O\left(e^{-\lambda_\varepsilon} + e^{-\tau\lambda_\varepsilon}|8\pi-\rho_\varepsilon|\right).$$

Therefore,

$$\rho_\varepsilon - 8\pi = \frac{16\pi}{(8\pi-\varepsilon)h(p_\varepsilon)}\left[\Delta\ln h(p_\varepsilon) + 8\pi - 2\kappa(p_\varepsilon)\right]\lambda_\varepsilon e^{-\lambda_\varepsilon} + O\left(e^{-\lambda_\varepsilon}\right)$$

and this completes the proof. □

4 Proof of Theorem 1.3

Proof On one hand, checking the proof in [23, Theorem 1.2] step by step, we have

$$\begin{aligned}\inf_{u\in H^1(\Sigma)} J_{8\pi}(u) = \lim_{\varepsilon\to 0} J_{8\pi}(u_\varepsilon) &\geq -1-\ln\pi-\left(\ln h(p_0)+\frac{1}{2}A(p_0)\right)\\ &\geq -1-\ln\pi-\max_{p\in\Sigma}\left(\ln h^+(p)+\frac{1}{2}A(p)\right).\end{aligned}\tag{4.1}$$

We sketch the proof here. Without loss of generality, up to a conformal change of the metric, we may assume that the metric is the Euclidean metric around p_0 and we also assume p_0 is the origin $o \in \mathbb{B} \subset \Sigma$. Choose $p_\varepsilon \to p_0$ such that

$$\lambda_\varepsilon = u_\varepsilon(p_\varepsilon) = \max_\Sigma u_\varepsilon \to +\infty.$$

Set $r_\varepsilon = e^{-\lambda_\varepsilon/2}$ and

$$\tilde{u}_\varepsilon = u_\varepsilon(p_\varepsilon + r_\varepsilon x) + 2\ln r_\varepsilon, \quad |x| < r_\varepsilon^{-1}(1-|p_\varepsilon|).$$

Then $\tilde{u}_\varepsilon$ converges to w in $C^\infty_{loc}\left(\mathbb{R}^2\right)$ where

$$w(x) = -2\ln\left(1+\pi h(p_0)|x|^2\right).$$

We denote by $o_\varepsilon(1)$ (resp. $o_R(1), o_\delta(1)$) the terms which tents to zero as $\varepsilon \to 0$ (resp. $R \to \infty$, $\delta \to 0$). Moreover, $o_\varepsilon(1)$ may depend on R, δ, while $o_R(1)$ may depend on δ. We have

$$\frac{1}{16\pi}\int_{\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)} |\nabla u_\varepsilon|^2 = \frac{1}{16\pi}\int_{\mathbf{B}_R}|\nabla \tilde{u}_\varepsilon|^2 = \ln\left(\pi h(p_0)R^2\right) - 1 + o_\varepsilon(1) + o_R(1).$$

According to Lemma 2.1, a direct calculation yields

$$\frac{1}{16\pi}\int_{\Sigma\setminus \mathrm{B}_\delta(p_\varepsilon)} |\nabla u_\varepsilon|^2 = -2\ln\delta + \frac{1}{2}A(p_0) + o_\varepsilon(1) + o_\delta(1).$$

Under polar coordinates (r,θ), set

$$u_\varepsilon^*(r) = \frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon\left(p_\varepsilon + re^{\sqrt{-1}\theta}\right)\mathrm{d}\theta.$$

Then

$$\begin{aligned}u_\varepsilon^*(\delta) &= \int_\Sigma u_\varepsilon - 4\ln\delta + A(p_0) + o_\varepsilon(1) + o_\delta(1),\\ u_\varepsilon^*(r_\varepsilon R) &= -2\ln r_\varepsilon - 2\ln\left(\pi h(p_0)R^2\right) + o_\varepsilon(1) + o_R(1).\end{aligned}$$

Solve

$$\begin{cases}-\Delta\xi_\varepsilon = 0, & \text{in } \mathbf{B}_\delta(p_\varepsilon)\setminus\mathbf{B}_{r_\varepsilon R}(p_\varepsilon),\\ \xi_\varepsilon = u_\varepsilon^*, & \text{on } \partial\left(\mathbf{B}_\delta(p_\varepsilon)\setminus\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)\right).\end{cases}$$

We have

$$\frac{1}{16\pi}\int_{\mathbf{B}_\delta(p_\varepsilon)\backslash\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)}|\nabla u_\varepsilon|^2 \geq \frac{1}{16\pi}\int_{\mathbf{B}_\delta(p_\varepsilon)\backslash\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)}\left|\nabla u_\varepsilon^*\right|^2$$
$$\geq \frac{1}{16\pi}\int_{\mathbf{B}_\delta(p_\varepsilon)\backslash\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)}|\nabla \xi_\varepsilon|^2 = \frac{\left(u_\varepsilon^*(\delta)-u_\varepsilon^*(r_\varepsilon R)\right)^2}{8\left(\ln\delta-\ln\left(r_\varepsilon R\right)\right)}.$$

Thus

$$\frac{1}{16\pi}\int_{\mathbf{B}_\delta(p_\varepsilon)\backslash\mathbf{B}_{r_\varepsilon R}(p_\varepsilon)}|\nabla u_\varepsilon|^2 \geq \frac{\left(u_\varepsilon^*(\delta)-u_\varepsilon^*(r_\varepsilon R)\right)^2}{-8\ln r_\varepsilon}\left(1+\frac{\ln(R/\delta)}{-\ln r_\varepsilon}\right)$$
$$=\frac{\left(\tau_\varepsilon+\int\limits_\Sigma u_\varepsilon-2\ln r_\varepsilon\right)^2}{-8\ln r_\varepsilon}+\frac{1}{8}\left(2+\frac{\tau_\varepsilon}{\ln r_\varepsilon}+\frac{\int_\Sigma u_\varepsilon}{\ln r_\varepsilon}\right)^2\ln(R/\delta)$$
$$-\int_\Sigma u_\varepsilon-4\ln(R/\delta)-A(p_0)-2\ln(\pi h(p_0))$$
$$+o_R(1)+o_\delta(1),$$

where

$$\tau_\varepsilon=u_\varepsilon^*(\delta)-u_\varepsilon^*\left(r_\varepsilon R\right)-\int\limits_\Sigma u_\varepsilon+2\ln r_\varepsilon$$
$$=4\ln(R/\delta)+A(p_0)+2\ln(\pi h(p_0))+o_\varepsilon(1)+o_\delta(1)+o_R(1).$$

Hence, we get

$$C\geq J_{8\pi}\left(u_\varepsilon\right)$$
$$\geq -1-\ln\pi-\ln h(p_0)-\frac{1}{2}A(p_0)$$
$$+\frac{\left(\tau_\varepsilon+\int_\Sigma u_\varepsilon-2\ln r_\varepsilon\right)^2}{-8\ln r_\varepsilon}+\frac{1}{8}\left(\left(2+\frac{\tau_\varepsilon}{\ln r_\varepsilon}+\frac{\int_\Sigma u_\varepsilon}{\ln r_\varepsilon}\right)^2-16\right)\ln(R/\delta)$$
$$+o_\varepsilon(1)+o_R(1)+o_\delta(1)$$

which implies

$$\int\limits_\Sigma u_\varepsilon=-\lambda_\varepsilon+O\left(\sqrt{\lambda_\varepsilon}\right)$$

and we obtain (4.1).

On the other hand, checking the proof in [9, Theorem 1.2] step by step, for each p with $h(p)>0$, there exists a sequence $\phi_\varepsilon\in H^1(\Sigma)$ such that

$$J_{8\pi}\left(\phi_\varepsilon\right)=-1-\ln\pi-\left(\ln h(p)+\frac{1}{2}A(p)\right)$$
$$-\frac{1}{4}\left(\Delta\ln h(p)+8\pi-2\kappa(p)+\left|\nabla\left(\ln h+\frac{1}{2}A\right)(p)\right|^2\right)\varepsilon\ln\varepsilon^{-1}$$
$$+o\left(\varepsilon\ln\varepsilon^{-1}\right).$$

Here we used the fact that the Green function G is symmetric. These test functions ϕ_ε can be constructed as following: without loss of generality, assume $p=0$ and

$$8\pi G(x,0) = -2\ln|x| + A(p) + b_1x_1 + b_2x_2 + \beta(x),$$

and take

$$\phi_\varepsilon(x) = \begin{cases} -2\ln\left(|x|^2+\varepsilon\right) + b_1x_1 + b_2x_2 + \ln\varepsilon, & |x| < \alpha_\varepsilon\sqrt{\varepsilon}, \\ 8\pi G(x,0) - \eta\left(\alpha_\varepsilon\sqrt{\varepsilon}\,|x|\right)\beta(x) + C_\varepsilon + \ln\varepsilon, & \alpha_\varepsilon\sqrt{\varepsilon} \le |x| < 2\alpha_\varepsilon\sqrt{\varepsilon}, \\ 8\pi G(x,0) + C_\varepsilon + \ln\varepsilon, & |x| \ge 2\alpha_\varepsilon\sqrt{\varepsilon}, \end{cases}$$

where η is a cutoff function supported in $[0,2]$ and $\eta = 1$ on $[0,1]$ and the positive constants α_ε and C_ε are chosen carefully. The assumption h is positive in [9] is used only to ensure that

$$\lim_{\varepsilon\searrow 0}\int_\Sigma he^{\phi_\varepsilon} > 0.$$

If p is a critical point (e.g., a maximum point) of the function $2\ln h^+ + A$, then

$$J_{8\pi}(\phi_\varepsilon) = -1 - \ln\pi - \left(\ln h(p) + \frac{1}{2}A(p)\right) - \frac{1}{4}\left(\Delta\ln h(p) + 8\pi - 2\kappa(p)\right)\varepsilon\ln\varepsilon^{-1} + o\left(\varepsilon\ln\varepsilon^{-1}\right).$$

This gives

$$\inf_{u\in H^1(\Sigma)} J_{8\pi}(u) = -1 - \ln\pi - \max_{p\in\Sigma}\left(\ln h^+(p) + \frac{1}{2}A(p)\right) = -1 - \ln\pi - \left(\ln h(p_0) + \frac{1}{2}A(p_0)\right).$$

In particular, the blow-up point p_0 must be a maximum point of the function $\ln h^+ + A$. □

Remark 4.1 One can write down the $o_\varepsilon(1)$ as follows. By Lemma 2.3 and (1.4), direct computations give us

$$\begin{aligned}\frac{1}{16\pi}\int_{\Sigma\setminus B_\delta(p_\varepsilon)} |\nabla u_\varepsilon|^2 &= \left(1 - \frac{\varepsilon}{4\pi} + \frac{\varepsilon^2}{64\pi^2} + O\left(e^{-\lambda_\varepsilon}\right)\right) \\ &\quad \left(-2\ln\delta + \frac{1}{2}A(p_\varepsilon) + O\left(e^{-\lambda_\varepsilon}\right) + o_\delta(1)\right) + O\left(e^{-\lambda_\varepsilon}\right) \\ &= -2\ln\delta + \frac{1}{2}A(p_\varepsilon) - \frac{\varepsilon}{4\pi}\left(-2\ln\delta + \frac{1}{2}A(p_\varepsilon) + O\left(e^{-\lambda_\varepsilon}\right) + o_\delta(1)\right) \\ &\quad + O\left(\varepsilon^2\right) + O\left(e^{-\lambda_\varepsilon}\right) + o_\delta(1).\end{aligned}$$

From the proof of Theorem 1.1, we also get the following

$$\int_{B_\delta(p_\varepsilon)} |\nabla\eta_\varepsilon|^2 = O\left(\varepsilon^2\delta\right) + O\left(e^{-\lambda_\varepsilon}\right),$$

$$\frac{1}{16\pi}\int_{B_{r_\varepsilon R}(p_\varepsilon)} |\nabla v_\varepsilon|^2 = \ln\left(\pi h(p_0)R^2\right) - 1 + o_R(1),$$

$$\int\limits_{B_\delta(p_\varepsilon)} \left|\nabla G^*\right|^2 = O\left(\delta^2\right)$$

and

$$\frac{1}{16\pi} \int\limits_{B_{r_\varepsilon R}(p_\varepsilon)} \left|\nabla G^*\right|^2 = O\left(r_\varepsilon^2\right) = O(e^{-\lambda_\varepsilon}).$$

These imply that

$$\frac{1}{16\pi} \int\limits_{B_{r_\varepsilon R}(p_\varepsilon)} |\nabla u_\varepsilon|^2 = \ln\left(\pi h(p_0)R^2\right) - 1 + O\left(\varepsilon^2 e^{-\frac{\lambda_\varepsilon}{2}}\right) + O\left(e^{-\lambda_\varepsilon}\right) + o_R(1).$$

On the neck, $o_\varepsilon(1)$ are the convergent rates in Lemma 2.1 and $\tilde{u}_\varepsilon \to w$.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed in this study.

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