Advances in Mathematics 404 (2022) 108422

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Brouwer degree for Kazdan-Warner equations on a connected finite graph ☆

Linlin Sun, Liuquan Wang *

*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China*

A R T I C L E I N F O

*Article history:*
Received 21 April 2021
Received in revised form 13 April 2022
Accepted 14 April 2022
Available online xxxx
Communicated by YanYan Li

*MSC:*
35R02
35A16

*Keywords:*
Kazdan-Warner equation
Brouwer degree
Existence

A B S T R A C T

We study Kazdan-Warner equations on a connected finite graph via the method of the degree theory. Firstly, we prove that all solutions to the Kazdan-Warner equation with nonzero prescribed function are uniformly bounded and the Brouwer degree is well defined. Secondly, we compute the Brouwer degree case by case. As consequences, we give new proofs of some known existence results for the Kazdan-Warner equation on a connected finite graph.



## 1. Introduction

Let $\Sigma$ be a closed Riemann surface, $h$ and $f$ two smooth functions on $\Sigma$. The Kazdan-Warner equation reads as

☆ This work is partially supported by the National Natural Science Foundation of China (Grant Nos. 11971358, 12171375) and the Hubei Provincial Natural Science Foundation of China (No. 2021CFB400). The authors would like to thank the anonymous referee for his/her careful reading and useful comments.

* Corresponding author.
*E-mail addresses:* sunll@whu.edu.cn (L. Sun), wanglq@whu.edu.cn, mathlqwang@163.com (L. Wang).

https://doi.org/10.1016/j.aim.2022.108422
0001-8708/

$$-\Delta u = he^u - f, \tag{1.1}$$

where $\Delta$ is the Laplace-Beltrami operator. It comes from the prescribed Gaussian curvature problem [5,7,16], and also appears in various contexts such as the abelian Chern-Simons-Higgs models [3,20,21].

The existence of solutions to the Kazdan-Warner equation has been studied in recent decades. Denote by $\mathrm{d}\mu_\Sigma$ the area element of $\Sigma$. If $\int_\Sigma f \mathrm{d}\mu_\Sigma = 0$ and $h \not\equiv 0$, then the Kazdan-Warner equation (1.1) is solvable [16] if and only if $h$ changes sign and

$$\int_\Sigma he^\phi \mathrm{d}\mu_\Sigma < 0,$$

where $\phi$ is the unique solution to

$$-\Delta\phi = \frac{\int_\Sigma f \mathrm{d}\mu_\Sigma}{\int_\Sigma 1 \mathrm{d}\mu_\Sigma} - f, \quad \int_\Sigma \phi \mathrm{d}\mu_\Sigma = 0.$$

If $\int_\Sigma f \mathrm{d}\mu_\Sigma \neq 0$, then the Kazdan-Warner equation (1.1) can be reduced to the following mean field equation

$$-\Delta u = \rho\left(\frac{he^u}{\int_\Sigma he^u \mathrm{d}\mu_\Sigma} - \frac{1}{\int_\Sigma 1 \mathrm{d}\mu_\Sigma}\right) \tag{1.2}$$

where $\rho \in \mathbb{R} \setminus \{0\}$. Many partial existence results of the mean field equation have been obtained for both noncritical and critical cases, see for examples Struwe and Tarantello [22], Ding, Jost, Li and Wang [8], Chen and Lin [6], Djadli [10] and the references therein.

If the prescribed function $h$ is a positive function and $\rho \in \mathbb{R} \setminus 8\pi\mathbb{N}^*$, then every solution $u$ with $\int_\Sigma u \mathrm{d}\mu_\Sigma = 0$ to the mean field equation (1.2) is uniformly bounded. One can define the Leray-Schauder degree for equation (1.2) as follows (cf. [18, p. 422]). Given $\alpha \in (0,1)$, consider

$$X_\alpha = \left\{ u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u \mathrm{d}\mu_\Sigma = 0 \right\},$$

and introduce a compact operator $K_{\rho,h} : X_\alpha \longrightarrow X_\alpha$ by

$$K_{\rho,h}(u) = \rho(-\Delta)^{-1}\left(\frac{he^u}{\int_\Sigma he^u \mathrm{d}\mu_\Sigma} - \frac{1}{\int_\Sigma 1 \mathrm{d}\mu_\Sigma}\right).$$

The Leray-Schahder degree is defined by

$$d_\rho = \lim_{R\to+\infty} \deg\left(\mathrm{Id} - K_{\rho,h}, B_R^{X_\alpha}, 0\right)$$

which is independent of $\alpha$ and $h$. Here $B_R^{X_\alpha}$ stands for the ball with center at the origin and radius $R$ in the Banach space $X_\alpha$ equipped with the $C^{2,\alpha}$-norm. Li [18, p. 422] pointed out that $d_\rho$ should be determined by the Euler number $\chi(\Sigma)$ of $\Sigma$. Chen and Lin [6, Theorem 1.2] proved that

$$d_\rho = \binom{k-\chi(\Sigma)}{k},$$

where $\rho \in (8k\pi, 8(k+1)\pi)$ and $k \in \mathbb{N}$. As a consequence, if the genus of $\Sigma$ is greater than zero, then the mean field equation (1.2) with positive prescribed function $h$ always possesses a solution provided that $\rho$ is not a multiple of $8\pi$.

In this paper, we consider the following Kazdan-Warner equation on a connected finite graph $G = (V, E)$:

$$-\Delta u(x) = h(x)e^{u(x)} - c, \quad x \in V, \tag{1.3}$$

where $\Delta$ is the Laplace operator on $G$ (see (2.1)), $h$ is a real function on $V$ and $c$ is a real number. This equation was studied by several mathematicians (cf. [11–15,17,19]). For example, utilizing the variational method, Grigor'yan, Lin and Yang [13, Theorems 1-3] obtained the following discrete analog of that of Kazdan and Warner [16]:

- when $c = 0$, (1.3) has a solution if and only if $h \equiv 0$ or $h$ changes sign and $\int_V h\mathrm{d}\mu < 0$;
- when $c > 0$, (1.3) is solvable if and only if $\max_V h > 0$;
- when $c < 0$, if (1.3) has a solution, then $\int_V h\mathrm{d}\mu < 0$, and in this case, there exists a constant $c_h \in [-\infty, 0)$ depending on $h$ such that (1.3) has a solution if $c \in (c_h, 0)$, but has no solution for any $c < c_h$.

Grigor'yan, Lin and Yang [13, Theorem 4] pointed out that $c_h = -\infty$ if $\min_V h < \max_V h \le 0$. Ge [11] proved that $c_h > -\infty$ if $h$ changes sign and obtained a solution for $c = c_h$. Recently, Liu and Yang [19] studied the following Kazdan-Warner equation

$$-\Delta u = K_\lambda e^u - \kappa \tag{1.4}$$

where $\int_V \kappa\mathrm{d}\mu < 0$, $K_\lambda = K+\lambda$ and $\min_V K < \max_V K = 0$. They obtained the following discrete analog of that of Ding and Liu [9]: there exists a $\lambda^* \in (0, -\min_V K)$ such that (1.4) has a unique solution if $\lambda \le 0$, at least two distinct solutions if $0 < \lambda < \lambda^*$, at least one solution if $\lambda = \lambda^*$, and no solution if $\lambda > \lambda^*$.

Our aim is to extend the results of Chen and Lin [6] to graphs. We shall prove that every solution to the Kazdan-Warner equation (1.3) is uniformly bounded whenever $h \not\equiv 0$. Consequently, the Brouwer degree $d_{h,c}$ for (1.3) can be well defined. We will give the exact formula for the Brouwer degree $d_{h,c}$. Meanwhile, we will use the degree theory to recover some known existence results.

The remaining part of this paper is briefly organized as follows. In Section 2 we recall some notions on a graph and state our main results. In Section 3 we recall some basic facts regarding functions on a connected finite graph. In Section 4 we study the blow-up behavior for the Kazdan-Warner equation and state a discrete analog of that of Brezis and Merle's result in [2]. We will prove a compactness result for the Kazdan-Warner equation on a connected finite graph. In particular, we give a proof of Theorem 2.1. In Section 5 we compute the Brouwer degree for the Kazdan-Warner equation case by case (Theorem 2.3). In Section 6 we shall give new proofs of several known existence results by using the degree theory (Corollary 2.4 and Corollary 2.6). Hereafter we do not distinguish sequence and subsequence unless necessary. Moreover, we use the capital letter $C$ to denote some uniform constants which are independent of the special solutions and not necessarily the same at each appearance.

## 2. Settings and main results

Throughout this paper, $G = (V, E)$ is assumed to be a connected finite graph with vertex set $V$ and edge set $E$. The edges on the graph are allowed to be weighted. Let $\omega : V \times V \longrightarrow \mathbb{R}$ be a weight function in the sense that $\omega_{xy} = \omega_{yx} \geq 0$ and

$$\omega_{xy} > 0 \iff xy \in E.$$

$G$ is connected means that for every $x, y \in V$ there exist $x_i \in V$ such that $x = x_1, y = x_m$ and

$$\omega_{x_i x_{i+1}} > 0, \quad i = 1, \ldots, m-1.$$

We say that $G$ is finite if the number of vertices is finite. Denote by $V^{\mathbb{R}}$ the set of real functions on $V$. Let $\mu$ be a positive function (vertex measure) on $V$ and define the ($\mu$-)Laplace operator $\Delta$ by

$$\Delta u(x) := \frac{1}{\mu_x} \sum_{y \in V} \omega_{xy} \left(u(y) - u(x)\right), \quad x \in V, \quad u \in V^{\mathbb{R}}. \tag{2.1}$$

For any function $f \in V^{\mathbb{R}}$, an integral of $f$ over $V$ is defined by

$$\int_V f \mathrm{d}\mu := \sum_{x \in V} f(x) \mu_x.$$

We have the following Green formula:

$$\int_V \Delta u v \mathrm{d}\mu = -\int_V \Gamma\left(u, v\right) \mathrm{d}\mu, \tag{2.2}$$

where $\Gamma$ is the associated gradient form:

$$\Gamma(u,v)(x) := \frac{1}{2\mu_x}\sum_{y\in V}\omega_{xy}\left(u(y)-u(x)\right)\left(v(y)-v(x)\right). \tag{2.3}$$

Let

$$|\nabla u|(x) := \sqrt{\Gamma(u,u)(x)}.$$

Denote by $L^P(V)$ the space of all functions $f\in V^{\mathbb{R}}$ with finite norm $\|f\|_{L^p(V)}$ which is defined by

$$\|f\|_{L^p(V)} := \begin{cases} \left(\int_V |f|^p \, \mathrm{d}\mu\right)^{1/p}, & 1\le p<\infty, \\ \operatorname{ess\,sup}_V |f|, & p=\infty. \end{cases}$$

We also consider the Sobolev space $W^{1,p}(V)$ which consists of all functions $f\in V^{\mathbb{R}}$ with finite norm $\|f\|_{W^{1,p}(V)}$ which is defined by

$$\|f\|_{W^{1,p}(V)} := \|f\|_{L^p(V)} + \||\nabla f|\|_{L^p(V)}.$$

For every $h,f\in V^{\mathbb{R}}$, we consider the following functional

$$J_{h,f}(u) = \int\limits_V \left(\frac{1}{2}|\nabla u|^2 + fu - he^u\right)\mathrm{d}\mu, \quad u\in W^{1,2}(V).$$

The critical points of $J_{h,f}$ are exactly the solutions to the following (generalized) Kazdan-Warner equation:

$$-\Delta u(x) = h(x)e^{u(x)} - f(x), \quad x\in V. \tag{2.4}$$

We say that $u$ is stable if

$$\int\limits_V \left(|\nabla\xi|^2 - he^u\xi^2\right)\mathrm{d}\mu \ge 0, \quad \forall \xi\in V^{\mathbb{R}},$$

and $u$ is strictly stable if the equality holds only if $\xi\equiv 0$.

The first main theorem is the following a priori estimate.

**Theorem 2.1.** *Let $G=(V,E)$ be a connected finite graph with weight $\omega$ and measure $\mu$. Assume that $h\in V^{\mathbb{R}}$ and $c\in\mathbb{R}$ satisfy:*

1) *if $c$ is positive, then $h$ is positive somewhere;*
2) *if $c$ equals to zero, then $h$ changes sign and the integral of $h$ over $V$ is negative;*

3) *if $c$ is negative, then $h$ is negative somewhere.*

*Then there exists a constant $C$ depending only on $h, c, G, \omega$ and $\mu$ such that every solution $u$ to the Kazdan-Warner equation* (1.3) *satisfies*

$$\max_V |u(x)| \le C.$$

**Remark 2.2.** From [13] (see also Section 1), if $h \not\equiv 0$, then the conditions 1)−3) mentioned in Theorem 2.1 are necessary conditions to solve the Kazdan-Warner equation (1.3). If $h \equiv 0$, then the solutions of (1.3) are not uniformly bounded since any constant function solves it.

Assume that $h, f \in V^{\mathbb{R}}$ satisfy:

1)' if $\int_V f \mathrm{d}\mu > 0$, then $\max_V h > 0$;
2)' if $\int_V f \mathrm{d}\mu = 0$, then $\int_V h e^{\phi} \mathrm{d}\mu < 0 < \max_V h$;
3)' if $\int_V f \mathrm{d}\mu < 0$, then $\min_V h < 0$,

where $\phi$ is the unique solution to

$$-\Delta\phi = \frac{\int_V f \mathrm{d}\mu}{\int_V 1 \mathrm{d}\mu} - f, \quad \min_V \phi = 0.$$

Consider a map

$$F_{h,f} : L^\infty(V) \longrightarrow L^\infty(V), \quad u \mapsto F_{h,f}(u) := -\Delta u + f - h e^u.$$

We denote by $B_R$ the ball with center at the origin and radius $R$ in $L^\infty(V)$. Notice that if $u$ solves (2.4) then

$$-\Delta(u - \phi) = h e^{\phi} e^{u-\phi} - \frac{\int_V f \mathrm{d}\mu}{\int_V 1 \mathrm{d}\mu}.$$

Applying Theorem 2.1 we conclude that there is no solution on the boundary $\partial B_R$ for $R$ large. Hence, the Brouwer degree

$$\deg(F_{h,f}, B_R, 0)$$

is well defined for $R$ large. According to the homotopic invariance, $\deg(F_{h,f}, B_R, 0)$ is independent of $R$. Let

$$d_{h,f} := \lim_{R\to+\infty} \deg(F_{h,f}, B_R, 0).$$

If $J_{h,f}$ is a Morse function, i.e., every critical point of $J_{h,f}$ is nondegenerate, then

$$\deg\left(F_{h,f}, B_R, 0\right) = \sum_{u \in B_R, F_{h,f}(u)=0} \det\left(DF_{h,f}(u)\right)$$

whenever $\partial B_R \cap F_{h,f}^{-1}(\{0\}) = \emptyset$. For more details about the Brouwer degree and its various properties we refer the reader to Chang [4, Chapter 3].

The second main theorem is the following

**Theorem 2.3.** *Let $G = (V, E)$ be a connected finite graph and $h, c$ as in Theorem 2.1. Then*

$$d_{h,c} = \begin{cases} -1, & c \geq 0; \\ 1, & c < 0 \text{ and } \max_V h \leq 0; \\ 0, & c < 0 \text{ and } \max_V h > 0. \end{cases}$$

The Kronecker existence implies that there exists at least one solution if the Brouwer degree is nonzero. As applications, we obtain several existence results mentioned in the introduction.

**Corollary 2.4** *(cf. [11,13]). Let $G = (V, E)$ be a connected finite graph and $h \not\equiv 0$.*

(1) *If $c > 0$, then (1.3) is solvable if and only if $\max_V h > 0$.*
(2) *If $c = 0$, then (1.3) is solvable if and only if $h$ changes sign and $\int_V h d\mu < 0$.*
(3) *If $c < 0$ and $h \leq 0$, then (1.3) has a unique (strict global minimum) solution.*
(4) *If $c < 0$ and $\int_V h d\mu < 0 < \max_V h$, then there exists a constant $c_h \in (-\infty, 0)$ such that (1.3) has at least two distinct solutions for $c_h < c < 0$, at least a (stable) solution for $c = c_h$, and no solution for $c < c_h$.*

**Remark 2.5.** Checking the proof of [11, Theorem 1.1], one concludes that

$$c_h \geq -\frac{C \max_V |h|}{\max_V h}$$

if $\max_V h > 0$. The multiplicity of solutions to the Kazdan-Warner equation in the negative case can also be obtained by using the minimax method (cf. [19]).

**Corollary 2.6** *(cf. [19]). Let $G = (V, E)$ be a connected finite graph. There exists a constant $\lambda^* \in (0, -\min_V K)$ satisfying:*

(1) *if $\lambda \leq 0$, then (1.4) has a unique (strict global minimum) solution;*
(2) *if $0 < \lambda < \lambda^*$, then (1.4) has at least two distinct solutions;*
(3) *if $\lambda = \lambda^*$, then (1.4) has at least a stable solution, i.e., (1.4) has a solution $u$ satisfying*

$$\int_V \left(|\nabla\xi|^2 - K_\lambda e^u \xi^2\right) \mathrm{d}\mu \geq 0, \quad \forall \xi \in V^{\mathbb{R}};$$

(4) *if* $\lambda > \lambda^*$, *then* (1.4) *has no solution.*

## 3. Preliminaries

In this section, we provide discrete versions of the strong maximum principle, the elliptic estimate, Kato's inequality and the sub- and super-solutions principle.

We begin with the following strong maximum principle.

**Lemma 3.1** *(Strong maximum principle). If* $u$ *is not a constant function, then there exists* $x_1 \in V$ *such that*

$$u(x_1) = \max_V u, \quad \Delta u(x_1) < 0.$$

**Proof.** Choose $x, y \in V$ such that

$$u(x) = \max_V u, \quad u(y) = \min_V u.$$

Since $G$ is connected, there exist $x_i \in V$ such that $x = x_1, y = x_m$ and

$$\omega_{x_i x_{i+1}} > 0, \quad i = 1, \ldots, m-1.$$

Since $u$ is not a constant function, we have

$$u(x) > u(y).$$

Thus there exists some $1 \leq i \leq m-1$ such that $u(x_1) = \cdots = u(x_i) > u(x_{i+1})$. Without loss of generality, we may assume that $u(x_1) > u(x_2)$. Then $u(x_1) = \max_V u$ and

$$\begin{aligned}
\Delta u(x_1) =& \frac{1}{\mu_{x_1}} \sum_{y \in V} \omega_{x_1 y} \left(u(y) - u(x_1)\right) \\
\leq& \frac{1}{\mu_{x_1}} \omega_{x_1 x_2} \left(u(x_2) - u(x_1)\right) \\
<& 0.
\end{aligned}$$

We complete the proof. □

Since $G$ is a finite graph, all of the spaces $L^p(V)$ and $W^{1,p}(V)$ with $1 \leq p \leq \infty$ are exactly $V^{\mathbb{R}}$, a finite dimensional linear space. Note that every two norms on $V^{\mathbb{R}}$ are equivalent. Denote by $\|\cdot\|$ the norm of $V^{\mathbb{R}}$ for convenience. Set

$$V_0^{\mathbb{R}} = \left\{ u \in V^{\mathbb{R}} : \int_V u \mathrm{d}\mu = 0 \right\}.$$

Then all of $\max_V |\Delta u|$, $\max_V u - \min_V u$ and $\max_V |\nabla u|$ are norms of $u$ on $V_0^{\mathbb{R}}$. Consequently, we have the following elliptic estimate.

**Lemma 3.2** *(Elliptic estimate). There is a positive constant $C$ such that for all $u \in V^{\mathbb{R}}$,*

$$\max_V u - \min_V u \leq C \max_V |\Delta u|.$$

For any function $f \in V^{\mathbb{R}}$, we denote by $f^+ = \max\{f, 0\}$ and $f^- = (-f)^+$. For any set $A$ we define the indicator

$$\chi_A(t) := \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

The following inequality is useful.

**Lemma 3.3** *(Kato's inequality).*

$$\Delta u^+ \geq \chi_{\{u>0\}} \Delta u.$$

**Proof.** By definition,

$$\Delta u^+(x) = \frac{1}{\mu_x} \sum_{y \in V} \omega_{xy} \left( u^+(y) - u^+(x) \right).$$

If $u(x) > 0$, then $u^+(x) = u(x)$ and

$$\Delta u^+(x) \geq \frac{1}{\mu_x} \sum_{y \in V} \omega_{xy} \left( u(y) - u(x) \right) = \Delta u(x).$$

If $u(x) \leq 0$, then $u^+(x) = 0$ and

$$\Delta u^+(x) = \frac{1}{\mu_x} \sum_{y \in V} \omega_{xy} u^+(y) \geq 0.$$

We obtain the desired inequality. □

Let $f : V \times \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. We say that $\phi$ is a sub-solution (super-solution) to

$$-\Delta u(x) = f(x, u(x)), \quad \forall x \in V, \tag{3.1}$$

if $\Delta\phi(x) + f(x, \phi(x)) \geq (\leq 0)$ for all $x \in V$. Denote

$$J(u) = \int\limits_V \left(\frac{1}{2}|\nabla u|^2 - F(\cdot, u(\cdot))\right) \mathrm{d}\mu,$$

where $\frac{\partial F}{\partial u} = f$. Then we have the following sub- and super-solutions principle.

**Lemma 3.4** *(sub- and super-solutions principle). Assume $\phi$ and $\psi$ are sub-solution and super-solution to (3.1) respectively with $\phi \leq \psi$. Then any minimizer of $J$ in $\left\{u \in V^{\mathbb{R}} : \phi \leq u \leq \psi\right\}$ solves (3.1).*

**Proof.** Without loss of generality, assume $\phi \not\equiv \psi$. Then

$$-\Delta(\psi - \phi)(x) \geq f(x, \psi(x)) - f(x, \phi(x)), \quad x \in V.$$

We claim that $\psi > \phi$. Let $u$ be a minimizer of $J$ in $\left\{u \in V^{\mathbb{R}} : \phi \leq u \leq \psi\right\}$. If $u(x_0) = \phi(x_0)$ for some $x_0 \in V$, then $u \equiv \phi$. In fact, since $u - \phi \geq 0$ and $\min_V (u - \phi) = 0$, if $u \not\equiv \phi$, applying Lemma 3.1, then there exists some $x_1 \in V$ such that

$$u(x_1) - \phi(x_1) = 0, \quad \Delta(u - \phi)(x_1) > 0. \tag{3.2}$$

On the one hand, since $u$ is a minimizer of $J$, we have

$$\begin{aligned} 0 \leq & \frac{\mathrm{d}}{\mathrm{d}t} J(u + t\delta_{x_1})|_{t=0} \\ = & \int\limits_V (-\Delta u - f(\cdot, u(\cdot)))\, \delta_{x_1} \mathrm{d}\mu \\ = & -\Delta u(x_1) - f(x_1, u(x_1)). \end{aligned} \tag{3.3}$$

Here for the first inequality we used the fact that $\psi(x_1) > \phi(x_1)$, which can be verified directly. On the other hand, by (3.2) and (3.3), we have

$$\begin{aligned} 0 < & \Delta(u - \phi)(x_1) \\ \leq & -f(x_1, u(x_1)) + f(x_1, \phi(x_1)) \\ = & 0, \end{aligned}$$

which is a contradiction. Similarly if $u(x) = \psi(x)$ for some $x \in V$ then $u \equiv \psi$. This together with the previous fact proves the claim.

If $u \equiv \phi$, then (3.3) implies that $\phi$ is also a super-solution and thus $u$ solves (3.1). Similarly, if $u \equiv \psi$ then $u$ is also a solution. If $\phi(x) < u(x) < \psi(x)$ for any $x \in V$, then for every $\eta \in V^{\mathbb{R}}$, since $u$ is a minimizer,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} J(u + t\eta)|_{t=0} = \int_V (-\Delta u - f(\cdot, u(\cdot)))\, \eta \mathrm{d}\mu.$$

Thus $u$ solves (3.1). □

## 4. Blow-up analysis

First we state the following discrete analog of that of Brezis and Merle's result [2].

**Theorem 4.1.** *Let $G = (V, E)$ be a connected finite graph. Let $u_n \in V^{\mathbb{R}}$ be a sequence of solutions to*

$$-\Delta u_n(x) = h_n(x) e^{u_n(x)} - c_n, \quad x \in V$$

*where $h_n \in V^{\mathbb{R}}$ and $c_n \in \mathbb{R}$ satisfy*

$$\lim_{n\to\infty} h_n(x) = h(x), \quad \forall x \in V$$

*and*

$$\lim_{n\to\infty} c_n = c.$$

*Then after passing to a subsequence, we have the following alternatives:*

(1) *either $u_n$ is uniformly bounded, or*
(2) *$u_n$ converges uniformly to $-\infty$, or*
(3) *there exists $x_0 \in V$ such that $u_n(x_0)$ converges to $+\infty$ and $h(x_0) = 0$. Moreover, $u_n$ is uniformly bounded from below in $V$ and above in $\{x \in V : h(x) > 0\}$.*

**Proof.** If $u_n$ is uniformly bounded from above, then $\Delta u_n$ is uniformly bounded. Applying Lemma 3.2,

$$\max_V u_n - \min_V u_n \le C.$$

If $\min_V u_n$ is uniformly bounded from below, then we obtain the first alternative. If $\liminf_{n\to\infty} \min_V u_n = -\infty$, then we obtain the second alternative.

If $\limsup_{n\to\infty} u_n = +\infty$, then without loss of generality we may assume for some $x_0 \in V$,

$$0 < u_n(x_0) = \max_V u_n \to +\infty$$

as $n \to \infty$. According to Lemma 3.3, we get

$$\begin{aligned}-\Delta u_n^- &= -\Delta\left(-u_n\right)^+\\ &\leq -\chi_{\{-u_n>0\}}\Delta\left(-u_n\right)\\ &=\chi_{\{u_n<0\}}\left(c_n - h_n e^{u_n}\right)\\ &\leq c_n^+ + h_n^-.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\Delta u_n^-\right\|_{L^1(V)} &= \int_V \left|\Delta u_n^-\right| \mathrm{d}\mu\\ &= \int_{\left\{\Delta u_n^- \geq 0\right\}} \Delta u_n^- \mathrm{d}\mu - \int_{\left\{\Delta u_n^- < 0\right\}} \Delta u_n^- \mathrm{d}\mu\\ &= -2\int_{\left\{\Delta u_n^- < 0\right\}} \Delta u_n^- \mathrm{d}\mu\\ &\leq 2\int_{\left\{\Delta u_n^- < 0\right\}} \left(c_n^+ + h_n^-\right) \mathrm{d}\mu\\ &\leq C.\end{aligned}$$

Applying Lemma 3.2,

$$\max_V u_n^- = \max_V u_n^- - \min_V u_n^- \leq C.$$

Thus $u_n$ is uniformly bounded from below.

Since $u_n$ is uniformly bounded from below, we have for every $x_1 \in V$

$$\begin{aligned}h_n\left(x_1\right) e^{u_n(x_1)} - c_n &= -\Delta u_n\left(x_1\right)\\ &= \frac{1}{\mu_{x_1}} \sum_{y\in V} \omega_{x_1 y}\left(u_n\left(x_1\right) - u_n(y)\right)\\ &\leq C\left(u_n\left(x_1\right) + 1\right),\end{aligned}$$

which implies

$$h_n\left(x_1\right) \leq C\left(u_n\left(x_1\right) + 1\right) e^{-u_n(x_1)}.$$

Taking $n \to \infty$, we deduce that

$$h\left(x_1\right) \leq 0$$

whenever $\limsup_{n\to\infty} u_n(x_1) = +\infty$. In other words, $u_n$ is uniformly bounded in $\{x \in V : h(x) > 0\}$.

Now we prove that $h(x_0) = 0$. The above argument yields that $h(x_0) \le 0$. It suffices to prove that $h(x_0) \ge 0$. Applying the maximum principle, we have

$$h_n(x_0) e^{u_n(x_0)} - c_n = -\Delta u_n(x_0) \ge 0.$$

Thus

$$h_n(x_0) \ge c_n e^{-u_n(x_0)}.$$

Letting $n \to \infty$, we deduce that

$$h(x_0) \ge 0.$$

The proof is completed. $\square$

Now we can prove the following compactness result.

**Theorem 4.2.** *Let $G = (V, E)$ be a connected finite graph with weight $\omega$ and measure $\mu$. Assume that there exists a positive constant $A$ satisfying:*

(1) $\max_V (|h| + |c|) \le A$;
(2) *if $h(x) > 0$ for some $x \in V$, then $h(x) \ge A^{-1}$;*
(3) *if $c > 0$, then $c \ge A^{-1}$;*
(4) *if $c = 0$, then $\int_V h \mathrm{d}\mu \le -A^{-1}$;*
(5) *if $c < 0$, then $c \le -A^{-1}$ and $\min_V h \le -A^{-1}$.*

*Then there exists a positive constant $C$ depending only on $A$, $G$, $\omega$ and $\mu$ such that every solution to (1.3) satisfies*

$$\max_{x\in V} |u(x)| \le C.$$

**Proof.** We prove the theorem by contradiction. Assume there is a sequence $u_n \in V^{\mathbb{R}}$ of solutions to

$$-\Delta u_n = h_n e^{u_n} - c_n$$

satisfying

$$\lim_{n\to\infty} h_n = h, \quad \lim_{n\to\infty} c_n = c, \quad \lim_{n\to\infty} \|u_n\| = \infty.$$

Here $h_n$ and $c_n$ satisfy the conditions $(1) - (5)$.

If $u_n$ converges uniformly to $-\infty$, then

$$-\Delta\left(u_n - \min_V u_n\right) = h_n e^{u_n} - c_n,$$

which implies that $u_n - \min_V u_n$ is uniformly bounded due to Lemma 3.2. Thus $u_n - \min_V u_n$ converges uniformly to a solution $w$ of the equation

$$-\Delta w = -c, \quad \min_V w = 0.$$

But this then implies that $c = 0$ and $w \equiv 0$. By assumptions (3) and (5), we may assume $c_n = 0$. Then the assumption (4) gives

$$\int_V h_n \mathrm{d}\mu \le -A^{-1}.$$

Taking $n \to \infty$, we obtain

$$\int_V h \mathrm{d}\mu \le -A^{-1}.$$

However,

$$0 = e^{-\min_V u_n} \int_V h_n e^{u_n} \mathrm{d}\mu = \int_V h_n e^{u_n - \min_V u_n} \mathrm{d}\mu \to \int_V h \mathrm{d}\mu \le -A^{-1}$$

as $n \to \infty$, which is a contradiction.

Applying Theorem 4.1, we may assume that $\max_V u_n$ converges to $+\infty$, and $u_n$ is uniformly bounded from below in $V$, and $u_n$ is uniformly bounded in $\Omega := \{x \in V : h(x) > 0\}$, and $\{x \in V : h(x) = 0\} \neq \emptyset$. For $n$ large, the assumption (2) gives

$$\Omega \subseteq \{x \in V : h_n(x) > 0\} \subseteq \left\{x \in V : h_n(x) \ge A^{-1}\right\} \subseteq \left\{x \in V : h(x) \ge A^{-1}\right\} \subseteq \Omega,$$

and therefore all the set inclusions are in fact set equalities. We have

$$\begin{aligned}\int_V c_n \mathrm{d}\mu = & \int_V h_n e^{u_n} \mathrm{d}\mu \\ = & \int_\Omega h_n e^{u_n} \mathrm{d}\mu + \int_{V\setminus\Omega} h_n e^{u_n} \mathrm{d}\mu \\ \le & C - \int_V h_n^- e^{u_n} \mathrm{d}\mu.\end{aligned}$$

This implies

$$\int_V h_n^- e^{u_n} \mathrm{d}\mu \leq C.$$

Hence

$$\|\Delta u_n\|_{L^1(V)} \leq \int_V |h_n| e^{u_n} \mathrm{d}\mu + C \leq C.$$

According to Lemma 3.2, we know that

$$\max_V u_n \leq \min_V u_n + C.$$

By the assumption that $\max_V u_n$ converges to $+\infty$, we deduce that $u_n$ must converge uniformly to $+\infty$. Consequently, $\Omega = \emptyset$, i.e., $h \leq 0$. By assumption (2), we may assume $h_n \leq 0$. Thus

$$\int_V c_n \mathrm{d}\mu = \int_V h_n e^{u_n} \mathrm{d}\mu \leq 0,$$

with the equality if and only if $h_n \equiv 0$. If $c_n = 0$, then $h_n = 0$, contradicting the assumption (4). Hence $c_n < 0$. By (5) we know that

$$\min_V h_n \leq -A^{-1}.$$

Hence

$$-CA \leq \int_V c_n \mathrm{d}\mu = \int_V h_n e^{u_n} \mathrm{d}\mu \leq -CA^{-1} e^{\min_V u_n}.$$

This implies $\min_V u_n \leq C$, which is a contradiction. □

**Example 4.1.** For every positive number $\varepsilon$, we have

$$-\Delta \ln \varepsilon = 0 = \pm \left(e^{\ln \varepsilon} - \varepsilon\right).$$

Thus $u = \ln \epsilon$ is a solution to the equation $-\Delta u = he^u - c$ with $h = \pm 1$ and $c = \pm \epsilon$. Thus the conditions (1), (3) and the first part of condition (5) are necessary since $\lim\limits_{\varepsilon \to 0} \ln \varepsilon = -\infty$ and $\lim\limits_{\varepsilon \to +\infty} \ln \varepsilon = +\infty$.

We also have

$$-\Delta \left(-\ln \varepsilon\right) = \pm \left(\varepsilon e^{-\ln \varepsilon} - 1\right),$$

which implies that the conditions (1), (2) and the second part of condition (5) are necessary.

Assume $h$ changes sign and $\int_V h d\mu < 0$. The following Kazdan-Warner equation is solvable (see [13] or Corollary 6.2)

$$-\Delta u = he^u.$$

Let $u$ be a solution to the above equation. Then

$$-\Delta (u - \ln \varepsilon) = \varepsilon h e^{u - \ln \varepsilon},$$

which implies that the conditions (1) and (4) are necessary.

Now we can prove Theorem 2.1.

**Proof of Theorem 2.1.** For fixed $h$ and $c$, it is easy to check that the conditions $(1) - (5)$ in Theorem 4.2 hold under assumptions of Theorem 2.1. Therefore, by Theorem 2.1 we know that every solution to (1.3) is uniformly bounded by a positive constant depending only on $h, c$, $G$, $\omega$ and $\mu$. □

## 5. Brouwer degree

In this section, we prove Theorem 2.3. We divided it into three cases: $c > 0, c = 0$ and $c < 0$. These cases correspond to Theorem 5.1, Theorem 5.4 and Theorem 5.5, respectively.

Firstly, we compute the Brouwer degree for the positive case: $c > 0$. In other words, we prove the following

**Theorem 5.1.** *For every connected finite graph $G = (V, E)$, function $h$ with $\max_V h > 0$ and $c > 0$, we have $d_{h,c} = -1$.*

**Proof.** Let $u_t \in V^{\mathbb{R}}$ satisfy

$$-\Delta u_t = \left(h^+ - (1-t)h^-\right) e^{u_t} - (1-t)c - t\varepsilon, \quad t \in [0, 1],$$

where $\varepsilon > 0$ is small to be determined. According to Theorem 4.2, $u_t$ is uniformly bounded. By the homotopic invariance of the Brouwer degree, we may assume $h^- \equiv 0$ and $c = \varepsilon > 0$.

To compute the Brouwer degree $d_{h,f}$, we may assume $h$ vanishes nowhere in $V$. In fact, without loss of generality, we may assume $\mu \equiv 1$ and $V = \{1, 2, \ldots, m\}$. In this case $L := -\Delta = (l_{ij})_{m \times m}$ is a symmetric matrix which can be characterized by

$$\begin{cases} l_{ij} \le 0, & \forall i \ne j, \\ \sum_j l_{ij} = 0, & \forall i, \end{cases}$$

and $\dim \ker L = 1$ (see Remark 5.2). Since $L$ is a symmetric diagonally dominant real matrix with nonnegative diagonal entries, we know that $L$ is positive semi-definite. This can also be seen from Green's formula (2.2).

Now the Kazdan-Warner equation (2.4) is equivalent to

$$L \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} h_1 e^{x_1} - f_1 \\ h_2 e^{x_2} - f_2 \\ \vdots \\ h_m e^{x_m} - f_m \end{pmatrix},$$

where $x_i = u(i)$, $h_i = h(i)$, $f_i = f(i)$. We also assume $h_1 \ne 0, \dots, h_r \ne 0, h_{r+1} = \cdots = h_m = 0$ and $1 \le r \le m$. Write

$$L = \begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix},$$

where $P$ is a $r \times r$ matrix. If $r < m$, then $R$ is positive definite (see Remark 5.3 for details). Now (2.4) is equivalent to

$$\left(P - Q^T R^{-1} Q\right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} h_1 e^{x_1} - \tilde{f}_1 \\ h_2 e^{x_2} - \tilde{f}_2 \\ \vdots \\ h_r e^{x_r} - \tilde{f}_r \end{pmatrix},$$

where

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_r \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} - Q^T R^{-1} \begin{pmatrix} f_{r+1} \\ f_{r+2} \\ \vdots \\ f_m \end{pmatrix}.$$

One can check that $R^{-1} = \left(r^{ij}\right)$ satisfies $r^{ij} \ge 0$ (see [1, p. 137]) and $\tilde{L} := P - Q^T R^{-1} Q = \left(\tilde{l}_{ij}\right)$ satisfies

$$\begin{cases} \tilde{l}_{ji} = \tilde{l}_{ij} \le 0, & \forall i \ne j, \\ \sum_j \tilde{l}_{ij} = 0, & \forall i, \end{cases}$$

and $\dim \ker \tilde{L} = 1$. Thus, we can construct a connected finite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with vertex $\tilde{V} = \{1, 2, \dots, r\}$, weight $\tilde{\omega}_{ij} = -\tilde{l}_{ij}$, measure $\tilde{\mu} \equiv 1$ and $\tilde{L} = -\tilde{\Delta}$. Moreover,

$$\sum_{i=1}^{r} \tilde{f}_i = \sum_{i=1}^{m} f_i,$$

and

$$\det\left(L - \operatorname{diag}\left(h_1 e^{x_1}, \ldots, h_m e^{x_m}\right)\right) = \det R \cdot \det\left(\tilde{L} - \operatorname{diag}\left(h_1 e^{x_1}, \ldots, h_r e^{x_r}\right)\right).$$

We conclude that

$$d_{h,f} = d_{h|_{\tilde{V}}, \tilde{f}}.$$

Applying Theorem 4.2 again, by the homotopic invariance of the Brouwer degree, we may assume $h \equiv 1$ and $\mu \equiv 1$.

We consider

$$-\Delta u(x) = e^{u(x)} - \varepsilon, \quad x \in V. \tag{5.1}$$

Notice that

$$\int_V e^u \mathrm{d}\mu = \varepsilon \int_V 1 \mathrm{d}\mu.$$

We obtain

$$e^{\max_V u} \leq C\varepsilon.$$

If $w$ solves

$$-\Delta w = e^w - \varepsilon,$$

then

$$-\Delta(u - w) = e^u - e^w,$$

which implies

$$|\Delta(u - w)| \leq \max\left\{e^u, e^w\right\} |u - w| \leq C\varepsilon \, |u - w|.$$

Applying Lemma 3.2, we have

$$\begin{aligned} \max_V (u - w) - \min_V (u - w) &\leq C\epsilon \max_V |u - w| \\ &\leq C\epsilon \left(\max_V (u - w) - \min_V (u - w) + \left|\min_V (u - w)\right|\right). \end{aligned}$$

Thus for small $\varepsilon > 0$, we have

$$\max_V (u - w) \leq \min_V (u - w) + \frac{1}{2} \left| \min_V (u - w) \right| .$$

If $u \not\equiv w$, then $e^u - e^w \not\equiv 0$. Since

$$\int_V (e^u - e^w) \, \mathrm{d}\mu = 0,$$

we must have

$$\min_V (u - w) < 0 < \max_V (u - w) .$$

Hence

$$0 < \max_V (u - w) \leq \min_V (u - w) - \frac{1}{2} \min_V (u - w) = \frac{1}{2} \min_V (u - w) < 0,$$

which is a contradiction. Therefore, the Kazdan-Warner equation (5.1) has a unique solution $u = \ln \varepsilon$ if $\varepsilon > 0$ is small.

Note that $-\Delta$ is a nonnegative matrix and 0 is an eigenvalue of $-\Delta$ with multiplicity one. We have for small $\varepsilon > 0$,

$$\det (DF_{1,\varepsilon} (\ln \varepsilon)) = \det (-\Delta - \varepsilon \mathrm{Id}) < 0.$$

Consequently, by the homotopy invariance of the Brouwer degree,

$$d_{h,c} = \lim_{\varepsilon \searrow 0} d_{1,\varepsilon} = \operatorname{sgn} \det (DF_{1,\varepsilon} (\ln \varepsilon)) = -1$$

and the proof is finished. □

**Remark 5.2.** The condition $\dim \ker L = 1$ is equivalent to that the graph $G = (V, E)$ is connected, i.e., has only one connected component. This further guarantees that no rows of $L$ can be zero and therefore $l_{ii} > 0$ for each $i$.

**Remark 5.3.** Here we give details explaining why $R$ is positive definite. First, since $L$ is positive semi-definite, we know that $R$ is also positive semi-definite. Therefore, all the eigenvalues of $R$ are nonnegative. If $R$ is not positive, then it has an eigenvalue 0. Let $Y$ be an eigenvector for the eigenvalue 0. For any column vector $X \in \mathbb{R}^{m-r}$, let $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$. We have

$$Z^{\mathrm{T}} L Z = (X^{\mathrm{T}}, Y^{\mathrm{T}}) \begin{pmatrix} P & Q^{\mathrm{T}} \\ Q & R \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = X^{\mathrm{T}} P X + 2 X^{\mathrm{T}} Q^{\mathrm{T}} Y.$$

Since $L$ is positive semi-definite, we have $X^{\mathrm{T}}PX + 2X^{\mathrm{T}}Q^{T}Y \geq 0$ for any $X \in \mathbb{R}^{m-r}$. Therefore, we must have $Q^{\mathrm{T}}Y = 0$. Note that

$$LZ = \begin{pmatrix} PX \\ QX \end{pmatrix}.$$

It turns out that when $Y \neq 0$, $\left(0, Y^{\mathrm{T}}\right)^{\mathrm{T}}$ is an eigenvector of $L$ of the eigenvalue 0. However, by the definition of $L$, we know that $(1, 1, \cdots, 1)^{\mathrm{T}}$ is an eigenvector of $L$ corresponding to the eigenvalue 0. Since we have assumed that $\dim \ker L = 1$, we have

$$\ker L = \mathrm{span}(1, 1, \cdots, 1)^{\mathrm{T}}.$$

This implies that $Y = 0$, which contradicts the assumption that $Y$ is an eigenvector of $R$.

Secondly, we compute the Brouwer degree for the flat case: $c = 0$. We prove the following

**Theorem 5.4.** *For every connected finite graph $G = (V, E)$ and sign changed function $h \in V^{\mathbb{R}}$ with $\int_V h \mathrm{d}\mu < 0$, we have*

$$d_{h,0} = -1.$$

**Proof.** Let $u_t \in V^{\mathbb{R}}$ be a solution to

$$-\Delta u_t = he^{u_t} - t, \quad t \in [0, 1].$$

We claim that there exists a positive constant $C$ such that

$$\max_V |u_t| \leq C, \quad \forall t \in [0, 1]. \tag{5.2}$$

It suffices to prove

$$\limsup_{t \searrow 0} \max_V |u_t| \leq C. \tag{5.3}$$

We prove it by contradiction. Suppose (5.3) is not true. According to Theorem 4.1, after passing to a subsequence, there are two cases:

(1) $u_t$ converges uniformly to $-\infty$, or
(2) $u_t$ is uniformly bounded from below in $V$ and uniformly bounded in $\{x \in V : h(x) > 0\} \neq \emptyset$.

For the first case, arguing similarly as in the beginning of the proof of Theorem 4.2, we know that $u_t - \min_V u_t$ converges uniformly to 0. We conclude that

$$0 > \int_V h \mathrm{d}\mu = \lim_{t \searrow 0} \int_V h e^{u_t - \min_V u_t} \mathrm{d}\mu = \lim_{t \searrow 0} \int_V t e^{-\min_V u_t} \mathrm{d}\mu \geq 0,$$

which is a contradiction.

For the second case, we have

$$\begin{aligned}\int_V |h| \, e^{u_t} \mathrm{d}\mu = & \int_{\{x \in V : h(x) > 0\}} h e^{u_t} \mathrm{d}\mu - \int_{\{x \in V : h(x) < 0\}} h e^{u_t} \mathrm{d}\mu \\ =& 2 \int_{\{x \in V : h(x) > 0\}} h e^{u_t} \mathrm{d}\mu - \int_V h e^{u_t} \mathrm{d}\mu \\ =& 2 \int_{\{x \in V : h(x) > 0\}} h e^{u_t} \mathrm{d}\mu - \int_V t \mathrm{d}\mu \\ \leq & C.\end{aligned}$$

This implies

$$\min_V u_t \leq C.$$

Moreover, it also implies

$$\max_V |\Delta u_t| \leq \max_V |h| e^{u_t} + 1 \leq C.$$

Applying Lemma 3.2, we have

$$\max_V u_t \leq \min_V u_t + C \max_V |\Delta u| \leq \min_V u_t + C \leq C,$$

which is a contradiction.

The a priori estimate (5.2) implies

$$d_{h,0} = \lim_{t \searrow 0} d_{h,t} = -1.$$

Here we used the homotopic invariance of the Brouwer degree and Theorem 5.1. □

Finally, we compute the Brouwer degree for the negative case: $c < 0$. We prove the following

**Theorem 5.5.** *For every connected finite graph $G = (V, E)$, function $h$ with $\min_V h < 0$ and $c < 0$, we have*

$$d_{h,c} = \begin{cases} 1, & \max_V h \le 0, \\ 0, & \max_V h > 0. \end{cases}$$

**Proof.** Since $c < 0$, a necessary condition for the existence to the Kazdan-Warner equation (1.3) is

$$\int_V h \mathrm{d}\mu < 0.$$

This was proved by Grigor'yan, Lin and Yang [13, Theorem 3]. For the sake of completeness, we reproduce its proof here. In fact, if $u$ solves (1.3), then

$$\begin{aligned}\int_V h \mathrm{d}\mu &= \int_V ce^{-u} \mathrm{d}\mu - \int_V e^{-u} \Delta u \mathrm{d}\mu \\ &< - \int_V e^{-u} \Delta u \mathrm{d}\mu \quad \text{(since } c < 0\text{)} \\ &= \frac{1}{2} \sum_{x,y \in V} \omega_{x,y} \left(u(x) - u(y)\right) \left(e^{-u(x)} - e^{-u(y)}\right) \quad \text{(by (2.2))} \\ &\le 0.\end{aligned} \tag{5.4}$$

First we assume $\max_V h \le 0$. Let $u_t \in V^{\mathbb{R}}$ satisfy

$$-\Delta u_t = \left(-t + (1-t)h\right) e^{u_t} - c, \quad t \in [0, 1].$$

According to Theorem 4.1, $u_t$ is uniformly bounded. By the homotopy invariance of the Brouwer degree, we may assume $h \equiv -1$ and $\mu \equiv 1$. We claim that $u \equiv \ln(-c)$ is the unique solution to

$$-\Delta u = -e^u - c.$$

It suffices to prove that $u$ is a constant function. For otherwise, applying Lemma 3.1, we have

$$-e^{\max_V u} - c > 0, \quad -e^{\min_V u} - c < 0$$

which is a contradiction. Thus

$$d_{h,c} = d_{-1,c} = \operatorname{sgn} \det \left( DF_{-1,c} \left( \ln(-c) \right) \right) = \operatorname{sgn} \det \left( -\Delta - c\mathrm{Id} \right) = 1.$$

Second, we assume $\max_V h > 0$. Let $v_t \in V^{\mathbb{R}}$ satisfy

$$-\Delta v_t = (th_\Lambda + (1-t)h)\, e^{v_t} - c, \quad t \in [0,1],$$

where

$$h_\Lambda(x) = \begin{cases} \Lambda, & h(x) > 0, \\ -1, & h(x) \le 0, \end{cases}$$

and $\Lambda > 0$ is large to be determined. According to Theorem 4.1, $v_t$ is uniformly bounded. By the homotopy invariance of the Brouwer degree, without loss of generality, assume $h \equiv h_\Lambda$. Choose $\Lambda$ large such that

$$\int\limits_V h_\Lambda \mathrm{d}\mu = \Lambda \int\limits_{\{x\in V: h(x)>0\}} 1\mathrm{d}\mu - \int\limits_{\{x\in V: h(x)\le 0\}} 1\mathrm{d}\mu > 0.$$

Consequently, there is no solution if $\Lambda$ is large. Thus, according to Theorem 4.1, by the homotopy invariance and Kronecker existence of the Brouwer degree,

$$d_{h,c} = \lim_{\Lambda\to+\infty} d_{h_\Lambda,c} = 0.$$

We finish the proof. □

## 6. Existence results

As consequences of the degree theory, we state and prove several existence results in the literature. Specifically speaking, we give proofs of Corollary 2.4 and Corollary 2.6. While some proofs and techniques are borrowed from the literature, the main new ingredient in our proofs is that we apply the degree theory to analyze the existence of solutions.

**Corollary 6.1** *(Solvability for positive case). If $c > 0$, then (1.3) is solvable if and only if* $\max_V h > 0$.

**Proof.** If there is a solution $u$, since $\int_V he^u \mathrm{d}\mu = \int_V c\mathrm{d}\mu > 0$, we must have $\max_V h > 0$. Under the assumption $\max_V h > 0$, by Theorem 2.3 we have $d_{h,c} = -1$. In particular, (1.3) has at least one solution. □

**Corollary 6.2** *(Solvability for flat case). If $c = 0$ and $h \not\equiv 0$, then (1.3) is solvable if and only if $h$ changes sign and $\int_V h\mathrm{d}\mu < 0$.*

**Proof.** If (1.3) has a solution $u$, then since $\int_V he^u \mathrm{d}\mu = 0$, we know that $h$ must change sign. Moreover, similar to (5.4), we have

$$\int_V h \mathrm{d}\mu = \frac{1}{2} \sum_{x,y \in V} \omega_{x,y}(u(x) - u(y)) \left(e^{-u(x)} - e^{-u(y)}\right) \leq 0, \tag{6.1}$$

and the equality holds if and only if $u(x) = u(y)$ whenever $\omega_{x,y} > 0$. Since $V$ is connected, this can happen only when $u$ is a constant. But then $h \equiv 0$, contradicting the assumption. Hence we have $\int_V h \mathrm{d}\mu < 0$. We remark that the proof here follows the lines of [13, p. 92].

Conversely, under the assumption that $\int_V h \mathrm{d}\mu < 0$ and $h$ changes sign, by Theorem 2.3 we have $d_{h,0} = -1$. As a consequence, (1.3) has at least one solution. □

In the rest of this section, we consider the negative case. First, we have the following

**Lemma 6.3.** *If $c < 0$, then (1.3) has a solution if and only if there is a super-solution to (1.3).*

**Proof.** When $A$ is large enough, the constant function $-A$ satisfies

$$-\Delta(-A) + c - he^{-A} = c - he^{-A} < 0.$$

Thus $-A$ is a sub-solution to (1.3) since $c < 0$. Applying the sub- and super-solutions method (Lemma 3.4), we complete the proof. □

**Corollary 6.4.** *Assume $c < 0$ and $\int_V h \mathrm{d}\mu < 0$.*

(1) *If $h \leq 0$, then (1.3) has a unique (strict global minimum) solution.*
(2) *If $\max_V h > 0$, then there exists a constant $c_h \in (-\infty, 0)$ such that (1.3) has at least two distinct solutions for $c_h < c < 0$, at least a (stable) solution for $c = c_h$ and no solution for $c < c_h$.*

**Proof.** If $h \leq 0$, we have $d_{h,c} = 1$ for every $c < 0$. Consequently, the Kazdan-Warner equation (1.3) is solvable. Since $h \leq 0$, every solution to (1.3) is stable. Applying the strong maximum principle, one concludes that the solution to (1.3) is unique. The fact $d_{h,c} = 1$ then implies that the unique solution to (1.3) is the strict global minimum of $J_{h,c}$.

From now on, we assume $\max_V h > 0$.

Applying the sub- and super-solution method, Grigor'yan, Lin and Yang [13] proved that (1.3) is solvable for every $c \in [c_0, 0)$ where $-c_0 > 0$ is small. In fact, arguing in a way similar to that of Grigor'yan, Lin and Yang [13, p. 10], solve

$$-\Delta v = h - \frac{\int_V h \mathrm{d}\mu}{\int_V 1 \mathrm{d}\mu}, \qquad \int_V v \mathrm{d}\mu = 0.$$

For constants $a > 0, b = \ln a$, we compute

$$
\begin{aligned}
-\Delta(av+b) =& a\left(h-\frac{\int_V h\mathrm{d}\mu}{\int_V 1\mathrm{d}\mu}\right) \\
=& he^{av+b}-ah\left(e^{av}-1\right)-a\frac{\int_V h\mathrm{d}\mu}{\int_V 1\mathrm{d}\mu} \\
\geq& he^{av+b}-a\left(|h|\left|e^{av}-1\right|+\frac{\int_V h\mathrm{d}\mu}{\int_V 1\mathrm{d}\mu}\right).
\end{aligned}
$$

Choose $a$ and $-c$ small to obtain a super-solution $\bar{u}=av+b$. Thus there exists some $c_1<0$ such that (1.3) has a solution $u_{c_1}$ for $c=c_1$. For any $c\in[c_1,0)$, it is easy to see that

$$\Delta u_{c_1}+he^{u_{c_1}}-c=c_1-c\leq 0.$$

This means that $u_{c_1}$ is a super solution for (1.3), and hence (1.3) has a solution by Lemma 6.3. Let

$$c_h=\inf\{c\in\mathbb{R}:(1.3)\text{ has a solution}\}.$$

Then $c_h\in[-\infty,0)$ and (1.3) has a solution if $c\in(c_h,0)$ and no solution if $c<c_h$.

Moreover, (1.3) has a strict local minimum solution for $c\in(c_h,0)$. In fact, following the idea of Liu and Yang [19], let $u_0\in V^{\mathbb{R}}$ satisfy

$$-\Delta u_0=he^{u_0}-c_0>he^{u_0}-c,$$

where $c_0\in(c_h,0)$. Choose $A>0$ large such that $u_0>-A$ and

$$-\Delta(-A)<he^{-A}-c.$$

Choose $u\in V^{\mathbb{R}}$ such that $-A\leq u\leq u_0$ and

$$J_{h,c}(u)=\min_{-A\leq v\leq u_0}J_{h,c}(v).$$

Applying Lemma 3.1, one can prove that $-A<u<u_0$ and conclude that $u$ is a local minimum of $J_{h,c}$ (cf. [11]). Moreover, $u$ is a strict local minimum. In fact, following the lines of [19, p. 10-11], if there exists $0\not\equiv\xi\in V^{\mathbb{R}}$ such that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}J_{h,c}(u+t\xi)|_{t=0}=0$, then

$$-\Delta\xi=he^u\xi.$$

This implies that $\xi$ is not a constant function since $h\not\equiv 0$. Since $u$ is a local minimum of $J_{h,c}$, analyzing the Taylor expansion of $J_{h,c}(u+t\xi)$ at the point $t=0$, we deduce that

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}J_{h,c}(u+t\xi)|_{t=0}=0,\quad \frac{\mathrm{d}^4}{\mathrm{d}t^4}J_{h,c}(u+t\xi)|_{t=0}\geq 0.$$

However,

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}t^4} J_{h,c}(u+t\xi)\big|_{t=0} &= -\int_V he^u\xi^4 \mathrm{d}\mu \\ &= \int_V \xi^3\Delta\xi \mathrm{d}\mu \\ &= -\frac{1}{2}\sum_{x,y}\omega_{xy}\left(\xi(x)-\xi(y)\right)\left(\xi^3(x)-\xi^3(y)\right) \\ &<0\end{aligned}$$

which is a contradiction. In other words, $u$ is strictly stable which implies that $u$ is a strict local minimum.

If $h \le 0$ and $\min_V h < 0$, then we conclude that $c_h = -\infty$ since (1.3) is solvable for every $c < 0$. If $c_h = -\infty$, then Ge [11] proved that $h \le 0$. In fact, if $\max_V h > 0$, then

$$c_h \ge -\frac{C\,\|\Delta h\|}{\max_V h^+}. \tag{6.2}$$

Following the lines of Ge [11], assume $u_c, \xi_c$ satisfies

$$-\Delta u_c = he^{u_c} - c, \quad (\Delta + c)\,\xi_c = h.$$

Notice that for every $x \in V$,

$$\begin{aligned}-e^{-u_c(x)}\Delta u_c(x) &= \frac{1}{\mu_x}\sum_{y\in V}\omega_{xy}\left(u_c(x)-u_c(y)\right)e^{-u_c(x)} \\ &\le \frac{1}{\mu_x}\sum_{y\in V}\omega_{xy}\left(e^{-u_c(y)}-e^{-u_c(x)}\right) \\ &= \Delta e^{-u_c}(x).\end{aligned} \tag{6.3}$$

Here we used the inequality $e^t - 1 \ge t$ ($\forall t \in \mathbb{R}$) wherein the equality holds if and only if $t = 0$. We have

$$(\Delta + c)\,e^{-u_c} \ge -e^{-u_c}\Delta u_c + ce^{-u_c} = h = (\Delta + c)\,\xi_c,$$

and the strict inequality holds at some point since the inequality in (6.3) cannot always be equality as $u_c$ is not a constant function. Let $g = e^{-u_c} - \xi_c$. Then

$$\Delta g \ge -cg. \tag{6.4}$$

If $g$ is a constant, then we immediately deduce that $g < 0$. If $g$ is not a constant, by Lemma 3.1 we may choose some $x_1 \in V$ such that

$$g(x_1) = \max_V g, \quad \Delta g(x_1) < 0.$$

These together with (6.4) imply $g < 0$. In other words, we have proved that

$$\xi_c > e^{-u_c}.$$

Hence when $-c$ is large enough,

$$0 > c\xi_c = \left(1 + c^{-1}\Delta\right)^{-1} h = h - c^{-1}\Delta h + O\left(c^{-2}\|\Delta h\|\right).$$

We obtain the desired estimate (6.2). As a consequence, if $c_h = -\infty$, then $\max_V h \leq 0$.

If $c_h > -\infty$, then $\max_V h > 0$ and $d_{h,c} = 0$. We have already proved that there exists a strict local minimum solution for every $c_h < c < 0$. Hence, there must be another solution for $c_h < c < 0$.

If $c_h > -\infty$, then we want to prove that there exists a stable solution for $c = c_h$. Let $u_c$ be a strict local minimum solution for each $c_h < c < 0$. According to Theorem 4.1, we know that $u_c$ is uniformly bounded. Thus, letting $c \searrow c_h$, after passing to a subsequence, we obtain a solution to (1.3) for $c = c_h$. Since $u_c$ is stable, we conclude that the limit is also stable.

Up to now, we obtain a stable solution for each $c \in [c_h, 0)$. Moreover, (1.3) has a strict local minimum solution for every $c \in (c_h, 0)$. Since the Brouwer degree $d_{h,c} = 0$ under the assumption $\max_V h > 0$, we conclude that (1.3) has at least two distinct solutions for every $c \in (c_h, 0)$. □

Combining Corollaries 6.1, 6.2 and 6.4, we complete the proof of Corollary 2.4.

Finally, we provide a new proof of Corollary 2.6 different from that in [19].

**Proof of Corollary 2.6.** Without loss of generality, assume $\kappa \equiv -1$. Recall the condition $\min_V K < \max_V K = 0$ for (1.4). Since $K_\lambda \leq 0$ for $\lambda \leq 0$, according to Corollary 2.4, we conclude that (1.4) has only one (strict global minimum) solution when $\lambda \leq 0$. In particular, when $\lambda = 0$, there is a strict global minimum of $J_{K,\kappa}$.

Let $\psi$ be the unique solution to

$$-\Delta\psi = Ke^{\psi} - \kappa + 1.$$

Then

$$-\Delta\psi + \kappa - K_\lambda e^{\psi} = 1 - \lambda e^{\psi}.$$

Thus for small $\lambda$, we obtain a super-solution $\psi$ to (1.4). Applying the sub- and super-solutions method, we conclude that (1.4) is solvable for small $\lambda$. Define

$$\lambda^* = \sup\{\lambda \in \mathbb{R} : (1.4) \text{ has a solution}\}.$$

If (1.4) has a solution, then

$$\int_V K_\lambda e^u \mathrm{d}\mu = \int_V \kappa \mathrm{d}\mu < 0.$$

This implies $\min_V K_\lambda = \min_V K + \lambda < 0$, i.e., $\lambda < -\min_V K$. Hence we have $\lambda^* \leq -\min_V K$.

Applying the sub- and super-solutions principle, if (1.4) has a solution when $\lambda = \lambda_0$, then (1.4) has a solution for every $\lambda < \lambda_0$. One can check that there exists a strict local minimum $u_\lambda$ of $J_{K_\lambda,\kappa}$ for $\lambda < \lambda^*$. Since the Brouwer degree $d_{K_\lambda,\kappa} = 0$ for $0 < \lambda < \lambda^*$, we conclude that there exists another solution to (1.4). By definition, (1.4) has no solution for any $\lambda > \lambda^*$.

Consider the sequence $\{u_\lambda\}_{0<\lambda<\lambda^*}$. We prove that $u_\lambda$ is uniformly bounded to complete the proof. For otherwise, according to Theorem 4.1, since $\int_V \kappa \mathrm{d}\mu < 0$, we may assume $\max_V u_\lambda$ converges to $+\infty$, and $u_\lambda$ is uniformly bounded from below in $V$, and $u_\lambda$ is uniformly bounded in $\{x \in V : K_{\lambda^*}(x) > 0\} \neq \emptyset$. By definition $K_\lambda = K + \lambda$. Thus, for $\lambda^* - \lambda$ small, we conclude that $u_\lambda$ is uniformly bounded in $\{x \in V : K_\lambda(x) > 0\}$. Therefore,

$$\begin{aligned}
\int_V \kappa \mathrm{d}\mu &= \int_V K_\lambda e^{u_\lambda} \mathrm{d}\mu \\
&= \int_{\{x\in V: K_\lambda(x)>0\}} K_\lambda e^{u_\lambda} \mathrm{d}\mu + \int_{\{x\in V: K_\lambda(x)\leq 0\}} K_\lambda e^{u_\lambda} \mathrm{d}\mu \\
&= \int_{\{x\in V: K_{\lambda^*}(x)>0\}} K_\lambda e^{u_\lambda} \mathrm{d}\mu + \int_{\{x\in V: K_\lambda(x)\leq 0\}} K_\lambda e^{u_\lambda} \mathrm{d}\mu \\
&\leq C - \int_V K_\lambda^- e^{u_\lambda} \mathrm{d}\mu.
\end{aligned}$$

This implies $\int_V K_\lambda^- e^{u_\lambda} \mathrm{d}\mu \leq C$. The second line of the above equation also implies

$$\int_V K_\lambda^+ e^{u_\lambda} \mathrm{d}\mu = \int_V \kappa \mathrm{d}\mu + \int_V K_\lambda^- \mathrm{d}\mu.$$

Hence

$$\int_V K_\lambda^+ e^{u_\lambda} \mathrm{d}\mu < \int_V K_\lambda^- e^{u_\lambda} \mathrm{d}\mu \leq C.$$

Thus

$$\|\Delta u_\lambda\|_{L^1(V)} \leq C.$$

We obtain

$$\max_V u_\lambda \leq \min_V u_\lambda + C \leq C$$

which is a contradiction. □

## References


[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, vol. 9, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994, Revised reprint of the 1979 original.

[2] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Commun. Partial Differ. Equ. 16 (8–9) (1991) 1223–1253.

[3] L.A. Caffarelli, Y.S. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem, Commun. Math. Phys. 168 (2) (1995) 321–336.

[4] K.-C. Chang, Methods in Nonlinear Analysis, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

[5] S.-Y.A. Chang, P.C. Yang, Prescribing Gaussian curvature on $S^2$, Acta Math. 159 (3–4) (1987) 215–259.

[6] C.-C. Chen, C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, Commun. Pure Appl. Math. 56 (12) (2003) 1667–1727.

[7] W.X. Chen, W.Y. Ding, Scalar curvatures on $S^2$, Trans. Am. Math. Soc. 303 (1) (1987) 365–382.

[8] W. Ding, J. Jost, J. Li, G. Wang, Existence results for mean field equations, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 16 (5) (1999) 653–666.

[9] W.Y. Ding, J.Q. Liu, A note on the problem of prescribing Gaussian curvature on surfaces, Trans. Am. Math. Soc. 347 (3) (1995) 1059–1066.

[10] Z. Djadli, Existence result for the mean field problem on Riemann surfaces of all genuses, Commun. Contemp. Math. 10 (2) (2008) 205–220.

[11] H. Ge, Kazdan-Warner equation on graph in the negative case, J. Math. Anal. Appl. 453 (2) (2017) 1022–1027.

[12] H. Ge, W. Jiang, Kazdan-Warner equation on infinite graphs, J. Korean Math. Soc. 55 (5) (2018) 1091–1101.

[13] A. Grigor'yan, Y. Lin, Y. Yang, Kazdan-Warner equation on graph, Calc. Var. Partial Differ. Equ. 55 (4) (2016) 92.

[14] A. Huang, Y. Lin, S.-T. Yau, Existence of solutions to mean field equations on graphs, Commun. Math. Phys. 377 (1) (2020) 613–621.

[15] H.-Y. Huang, J. Wang, W. Yang, Mean field equation and relativistic Abelian Chern-Simons model on finite graphs, J. Funct. Anal. 281 (10) (2021) 109218.

[16] J.L. Kazdan, F.W. Warner, Curvature functions for compact 2-manifolds, Ann. Math. (2) 99 (1974) 14–47.

[17] M. Keller, M. Schwarz, The Kazdan-Warner equation on canonically compactifiable graphs, Calc. Var. Partial Differ. Equ. 57 (2) (2018) 70.

[18] Y.Y. Li, Harnack type inequality: the method of moving planes, Commun. Math. Phys. 200 (2) (1999) 421–444.

[19] S. Liu, Y. Yang, Multiple solutions of Kazdan-Warner equation on graphs in the negative case, Calc. Var. Partial Differ. Equ. 59 (5) (2020) 164.

[20] M. Nolasco, Nontopological $N$-vortex condensates for the self-dual Chern-Simons theory, Commun. Pure Appl. Math. 56 (12) (2003) 1752–1780.

[21] T. Ricciardi, G. Tarantello, Vortices in the Maxwell-Chern-Simons theory, Commun. Pure Appl. Math. 53 (7) (2000) 811–851.

[22] M. Struwe, G. Tarantello, On multivortex solutions in Chern-Simons gauge theory, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1) (1998) 109–121.