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Calculus of Variations

Sinh-Gordon equations on finite graphs

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Abstract
In this paper, we focus on the sinh-Gordon equations on graphs. We introduce a uniform a priori estimate to define the topological degree for this equation with nonzero prescribed functions on finite, connected, and symmetric graphs. Furthermore, we calculate this topological degree case by case and show several existence results. In particular, we prove that the classical sinh-Gordon equation with nonzero prescribed function is always solvable on such graphs.

Mathematics Subject Classification 35R02 · 35A16

1 Introduction

The classical elliptic sinh-Gordon equation on a Riemann surface Σ reads as follows,

$$-\Delta_{\Sigma} u = \sinh u$$

which plays a very important role in the study of the construction of constant mean curvature surfaces initiated by Wente [26]. We refer the reader to [18] for more details. The sinh-Gordon equation has many applications in geometric analysis, for example in the work on the Wente torus [1].

Beyond its theoretical importance on Riemann surfaces, graph analysis is crucial for various applications, including image processing and data mining. Among the numerous avenues of research, studying partial differential equations from geometry or physics on graphs is particularly promising, as highlighted in [5–12, 14–17, 20, 22–24, 27] and related works.

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In this paper, we examine the sinh-Gordon equations, a family of nonlinear equations, on a finite, connected, and symmetric graph $G = (V, E)$:

$$-\Delta u = h_+ e^u + h_- e^{-u} - c \tag{1.1}$$

where h_+ and h_- are two real prescribed functions defined on the graph and c is a real number. This family includes notable equations such as the Kazdan-Warner equation:

$$-\Delta u = he^u - c \tag{1.2}$$

and the classical sinh-Gordon equation:

$$-\Delta u = h \sinh u - c, \tag{1.3}$$

where h is a real prescribed function defined on the graph.

The Kazdan-Warner equation (1.2) on graphs has been explored by several mathematicians. For instance, Grigor'yan, Lin, and Yang [7, Theorems 1-3] completely solved the solvability problem of the Kazdan-Warner equation on a finite, connected, and symmetric graph by utilizing the variational method. That is, they derived a discrete analog of the results by Kazdan and Warner [19]:

- When $c = 0$, a solution to (1.2) exists if and only if $h \equiv 0$ or h changes sign and $\int_V h \, d\mu < 0$.
- If $c > 0$, then (1.2) is solvable precisely when $\max_V h > 0$.
- For $c < 0$, solvability to (1.2) requires $\int_V h \, d\mu < 0$, and there exists a constant $c_h \in [-\infty, 0)$ depending on h such that (1.2) is solvable for $c \in (c_h, 0)$, but unsolvable for any $c < c_h$.

The degree theory has proven to be an effective approach in studying partial differential equations in Euclidean spaces or Riemannian manifolds, see for example [3]. Drawing from the foundational work of Chen and Lin [4] on mean field equations over closed Riemann surfaces, the author, in collaboration with Wang [25], innovatively employed the degree theory to resolve the Kazdan-Warner equation on finite graphs. This groundbreaking method was later broadened by Li, the author, and Yang [21] to encompass Chern-Simons Higgs models on finite graphs. Very recently, Hou and Qiao [13] also used the degree theory to extend the result obtained by [21].

The author and Wang [25] demonstrated that every solution to the Kazdan-Warner equation (1.2) on a finite, connected, and symmetric graph is uniformly bounded when the prescribed function $h \neq 0$, i.e., h is a nonzero function. Consequently, one can define the topological degree $D_{h,c}$ for (1.2) as follows

$$D_{h,c} = \lim_{R \longrightarrow \infty} \deg \left(-\Delta - he^{(\cdot)} + c, B_R^{L^\infty(V)}, 0 \right).$$

Furthermore, under the assumption $h \neq 0$, they computed the degree case by case and found (see Appendix A for details):

$$D_{h,c} = \begin{cases} -1, & c > 0, \ \max_V h > 0; \\ -1, & c = 0, \ \bar{h} < 0 < \max_V h; \\ 1, & c < 0, \ \min_V h < \max_V h \leq 0; \\ 0, & else. \end{cases}$$

Zhu [28] examined the mean field equation associated with equilibrium turbulence and the Toda system on finite, connected, and symmetric graphs. Specifically, he obtained that the equation

$$-\Delta u = c_1\left(\frac{h_1 e^u}{\int_V h_1 e^u \, \mathrm{d}\mu} - \psi\right) - c_2\left(\frac{h_2 e^{-u}}{\int_V h_2 e^{-u} \, \mathrm{d}\mu} - \psi\right)$$

admits at least one solution provided that the prescribed functions h_1 and h_2 are nonnegative, and the constants c_1 and c_2 are positive, and the function ψ satisfies $\int_V \psi \, \mathrm{d}\mu = 1$.

The aim of this paper is to employ topological methods to investigate the sinh-Gordon equation (1.1) on graphs. We will demonstrate that every solution to the sinh-Gordon equation (1.1) on a finite, connected, and symmetric graph is uniformly bounded, provided $h_+ \neq 0$ and $h_- \neq 0$. This ensures that the topological degree $d_{h_+,h_-,c}$ for (1.1) can be defined explicitly. Furthermore, we will derive the exact formula for the topological degree $d_{h_+,h_-,c}$. Consequently, we will apply the degree theory to establish existence criteria for (1.1).

The remaining part of this paper is briefly organized as follows. In section 2, we introduce key concepts related to graphs and present our principal results. In section 3, we review fundamental properties of functions on finite, connected, and symmetric graphs. In section 4, we obtain a uniform a priori estimate, which gives a proof of the first main result Theorem 2.1. The computation of the topological degree for the sinh-Gordon equation is given in section 5, offering a proof of the second main result Theorem 2.2. This leads to the existence result when the degree is non-zero, see Corollary 2.3. Finally, in section 6, we discuss the existence result for the sinh-Gordon equation, which yields a proof of the third main result Theorem 2.4.

Throughout the paper, we only consider finite, connected and symmetric graphs. Moreover, we do not distinguish between sequences and subsequences unless necessary. Additionally, the capital letter C denotes universal constants that are independent of specific solutions and may vary in different situations.

2 Main results

Let $G = (V, E)$ be a finite, connected, undirected graph with vertex set V and edge set E. We say G is **symmetric** if there exists a symmetric weight function $\omega : V \times V \to \mathbb{R}_{\geq 0}$ satisfying:

$$\omega_{xy} = \omega_{yx} \geq 0, \quad \text{and} \quad \omega_{xy} > 0 \quad \text{if and only if} \quad xy \in E.$$

This induces a weighted graph (G, ω) where edge existence and weights are compatible. In other words, $\omega_{xy} > 0$ if and only if x and y are adjacent vertices. For example, the complete graph K_n with uniform weighting $\omega_{xy} = c$ $(c > 0)$ is symmetric. The graph $G = (V, E)$ is said to be connected if, for any two vertices $x, y \in V$ there exist vertices $x_i \in V$ with $x = x_1, y = x_m$ such that

$$\omega_{x_i x_{i+1}} > 0, \quad i = 1, \ldots, m-1.$$

The finiteness of $G = (V, E)$ stems from the finite number of its vertices. We introduce a positive function (vertex measure) μ on V and define the (μ-)Laplace operator Δ as

$$\Delta u(x) := \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \left(u(y) - u(x)\right), \quad x \in V, \quad u \in L^\infty(V). \tag{2.1}$$

For any function $f \in L^\infty(V)$, the integral of f over V is defined by

$$\int_V f \, \mathrm{d}\mu := \sum_{x \in V} f(x) \mu_x.$$

The Green formula follows as:

$$\int_V \Delta u \cdot v \,\mathrm{d}\mu = -\int_V \Gamma\left(u, v\right) \,\mathrm{d}\mu,$$

where Γ, the associated gradient form, is defined by

$$\Gamma(u, v)(x) := \frac{1}{2\mu_x} \sum_{y\in V} \omega_{xy}\left(u(y) - u(x)\right)\left(v(y) - v(x)\right).$$

Lastly, we denote the square of the gradient of u at a vertex x by

$$|\nabla u(x)|^2 = \Gamma(u, u)(x).$$

Firstly, we prove a uniform a priori estimate for sinh-Gordon equations.

Theorem 2.1 *Every solution to the sinh-Gordon equation (1.1) with nonzero prescribed functions on a finite, connected, and symmetric graph is uniformly bounded.*

Consider the map $F_{h_+,h_-,c} : L^\infty(V) \longrightarrow L^\infty(V)$ defined by

$$F_{h_+,h_-,c}(u) = -\Delta u - h_+e^u - h_-e^{-u} + c.$$

Applying the Theorem 2.1 we conclude that there is no solution on the boundary $\partial B_R^{L^\infty(V)}$ for R large. Hence, the Brouwer degree

$$\deg\left(F_{h_+,h_-,c}, B_R^{L^\infty(V)}, 0\right)$$

is well defined for R large. According to the homotopy invariance, $\deg\left(F_{h_+,h_-,c}, B_R^{L^\infty(V)}, 0\right)$ is independent of R when $R \to +\infty$. In particular, we define the topological degree $d_{h_+,h_-,c}$ for the sinh-Gordon equation (1.1) on the finite, connected, and symmetric graph $G = (V, E)$ as follows

$$d_{h_+,h_-,c} = \lim_{R\to\infty} \deg\left(F_{h_+,h_-,c}, B_R^{L^\infty(V)}, 0\right).$$

Notice that every solutions to the equation (1.1) are critical points of the following functional

$$L^\infty(V) \ni u \mapsto J_{h_+,h_-,c}(u) = \int_V \left(\frac{1}{2}|\nabla u|^2 - h_+e^u + h_-e^{-u} + cu\right) \mathrm{d}\mu.$$

Thus, if $J_{h_+,h_-,c}$ is a Morse function, i.e., every critical point of $J_{h_+,h_-,c}$ is nondegenerate, then

$$\deg\left(F_{h_+,h_-,c}, B_R^{L^\infty(V)}, 0\right) = \sum_{u\in B_R^{L^\infty(V)}, F_{h_+,h_-,c}(u)=0} \operatorname{sgn}\det\left(\mathrm{d}F_{h_+,h_-,c}(u)\right) \tag{2.2}$$

provided $\partial B_R^{L^\infty(V)} \cap F^{-1}_{h_+,h_-,c}(\{0\}) = \emptyset$. For more details about the Brouwer degree and its various properties we refer the reader to Chang [2, Chapter 3].

Secondly, we give the exact expression of the topological degree for sinh-Gordon equations.

Theorem 2.2 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If $h_+ \neq 0$ and $h_- \neq 0$, then*

$$d_{h_+,h_-,c} = \begin{cases} (-1)^{\#V_0}, & V_0 := \{x \in V : h_+(x) > 0\} = \{x \in V : h_-(x) < 0\}; \\ 0, & else. \end{cases}$$

The Kronecker existence theorem suggests that at least one solution exists if the topological degree is nonzero. From this, we can derive the following existence result:

Corollary 2.3 *Consider a finite, connected, and symmetric graph $G = (V, E)$. If $h_+ \neq 0$, $h_- \neq 0$ and*

$$\{x \in V : h_+(x) > 0\} = \{x \in V : h_-(x) < 0\},$$

then the equation (1.1) has at least one solution. Specifically, the classical sinh-Gordon equation (1.3) with nonzero prescribed function admits a solution.

Finally, utilizing the sub-super solution principle Lemma 6.1 and the Theorem 2.2, we arrive at the following existence result:

Theorem 2.4 *Consider a finite, connected, and symmetric graph $G = (V, E)$. Assume $\max_V h_+ > 0,\ h_- \geq 0$. If $\int_V h_+ \,\mathrm{d}\mu < 0$ and*

$$c^*_{h_+,h_-} := \inf_{u \in L^\infty(V)} \max_V \left(\Delta u + h_+ e^u + h_- e^{-u}\right) < 0,$$

then the sinh-Gordon equation (1.1) has no solutions when $c < c^*_{h_+,h_-}$, has at least one solution for $c = c^*_{h_+,h_-}$, and has at least two solutions if $c^*_{h_+,h_-} < c < 0$.

3 Preliminaries

In this section, we present discrete versions of several key mathematical concepts, including the strong maximum principle, Kato's inequality, Hanack's inequality, and the elliptic estimate. As mentioned above, the graph $G = (V, E)$ is always assumed to be finite, connected, and symmetric.

We begin with the following strong maximum principle:

Lemma 3.1 (Strong maximum principle, cf. [25]) *If u is not a constant function, then there exists $x_1 \in V$ such that*

$$u(x_1) = \max_V u, \quad \Delta u(x_1) < 0.$$

Next, we introduce the following Kato's inequality:

Lemma 3.2 (Kato's inequality [25])

$$\Delta u^+ \geq \chi_{\{u>0\}} \Delta u.$$

The following Hanack inequality is also useful:

Lemma 3.3 (Hanack's inequality) *Denote by $L = \max_V u - u$. For every $x \sim y$, we have the following Hanack inequality*

$$L(y) \leq \frac{\mu_x}{\omega_{xy}} \left(\frac{\sum_{z\sim x} \omega_{zx}}{\mu_x} L(x) - \Delta u(x) \right). \tag{3.1}$$

Moreover, if $\max_V u + \min_V u \geq 0$, then we have for every $x \in V$

$$-\Delta u(x) \leq \frac{\sum_{z\sim x} \omega_{zx}}{\mu_x} \left(2u(x) + L(x)\right). \tag{3.2}$$

Proof On the one hand, notice that for every $x, z \in V$,

$$u(x) - u(z) \geq u(x) - \max_V u = -L(x).$$

By the definition of the Laplacian (2.1), we have for each $y \sim x$

$$\begin{aligned}
-\Delta u(x) =& \frac{1}{\mu_x} \sum_{z \sim x} \omega_{xz} (u(x) - u(z)) \\
=& \frac{\omega_{xy}}{\mu_x} (u(x) - u(y)) + \frac{1}{\mu_x} \sum_{z \sim x, z \neq y} \omega_{xz} (u(x) - u(z)) \\
\geq& \frac{\omega_{xy}}{\mu_x} (u(x) - u(y)) - \frac{1}{\mu_x} \sum_{z \sim x, z \neq y} \omega_{xz} L(x) \\
=& \frac{\omega_{xy}}{\mu_x} (L(y) - L(x)) - \frac{1}{\mu_x} \sum_{z \sim x, z \neq y} \omega_{xz} L(x) \\
=& \frac{\omega_{xy}}{\mu_x} L(y) - \frac{1}{\mu_x} \sum_{z \sim x} \omega_{xz} L(x).
\end{aligned}$$

We obtain the Hanack inequality (3.1).

On the other hand, if $\max_V u + \min_V u \geq 0$, then we have for every $x, z \in V$

$$u(x) - u(z) \leq u(x) - \min_V u \leq u(x) + \max_V u = 2u(x) + L(x). \tag{3.3}$$

Inserting the above estimate (3.3) into (2.1), we get

$$-\Delta u(x) = \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} (u(x) - u(y)) \leq \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} (2u(x) + L(x)).$$

We obtain the estimate (3.2) and complete the proof. □

Lastly, we present the following refined elliptic estimate:

Lemma 3.4 (Refined elliptic estimate [25]) *For all $u \in L^\infty(V)$,*

$$\max_V u - \min_V u \leq \frac{A^{\#V-2} - 1}{A - 1} B \cdot \max_V \Delta u,$$

where

$$A = \max_x \max_{y \sim x} \frac{\sum_{z \sim x} \omega_{zx}}{\omega_{yx}}, \quad B = \max_x \max_{y \sim x} \frac{\mu_x}{\omega_{yx}}.$$

Proof Assume

$$\max_V u = u(\bar{x}), \quad \min_V u = u(\underline{x}).$$

Choose a shortest path $x_0 x_1 \ldots x_m$ connecting $x_0 = \bar{x}$ and $x_m = \underline{x}$. We consider the function $L(x) = \max_V u - u(x)$. According to the Lemma 3.3, we get

$$L(x_{i+1}) \leq A \cdot L(x_i) + B \cdot \max_V (-\Delta u), \quad i = 0, 1, 2, \ldots, m-1. \tag{3.4}$$

Since $L(x_0) = 0$, we obtain by induction

$$L(x_i) \leq \left(1 + A + \cdots + A^{i-1}\right) B \cdot \max_V (-\Delta u), \quad i = 1, 2, \ldots, m. \tag{3.5}$$

To see this, since $L(x_1) \leq B \cdot \max_V (-\Delta u)$ according to (3.4), we may assume for some $1 \leq i \leq m-1$

$$L(x_i) \leq \left(1 + A + \cdots + A^{i-1}\right) B \cdot \max_V (-\Delta u)\,.$$

Applying the estimate (3.4), we obtain

$$\begin{aligned} L(x_{i+1}) &\leq A \cdot \left(\left(1 + A + \cdots + A^{i-1}\right) B \cdot \max_V (-\Delta u)\right) B \cdot \max_V (-\Delta u) \\ &= \left(1 + A + \cdots + A^{i}\right) B \cdot \max_V (-\Delta u)\,. \end{aligned}$$

We obtain the estimate (3.5). In particular

$$\max_V u - \min_V u = L(x_m) \leq \left(1 + A + \cdots + A^{m-1}\right) B \cdot \max_V (-\Delta u)\,.$$

Thus

$$\max_V u - \min_V u \leq \left(1 + A + \cdots + A^{\#V-2}\right) B \cdot \max_V (-\Delta u)\,.$$

Replacing u by $-u$, we obtain

$$\max_V u - \min_V u \leq \left(1 + A + \cdots + A^{\#V-2}\right) B \cdot \max_V \Delta u = \frac{A^{\#V-2} - 1}{A - 1} B \cdot \max_V \Delta u.$$

We complete the proof. □

4 A priori estimates

In this section, we give a priori estimate for the sinh-Gordon type equation (1.1). That is, we will give a proof of the Theorem 2.1.

To elucidate the core concept of the proof, we begin by examining the following example.

Example 4.1 Let $G = (V, E)$ be a graph with only two vertices. The sinh-Gordon equation (1.1) becomes

$$\begin{cases} \frac{\omega_{x_1x_2}}{\mu_{x_1}} (u(x_1) - u(x_2)) = h_+(x_1)e^{u(x_1)} + h_-(x_1)e^{-u(x_1)} - c, \\ \frac{\omega_{x_1x_2}}{\mu_{x_2}} (u(x_2) - u(x_1)) = h_+(x_2)e^{u(x_2)} + h_-(x_2)e^{-u(x_2)} - c. \end{cases} \tag{4.1}$$

We assume

$$|h_+(x_1)| + |h_+(x_2)| > 0, \quad |h_-(x_1)| + |h_-(x_2)| > 0. \tag{4.2}$$

Assume u solves the equation (4.1). Without loss of generality, assume

$$u(x_1) \geq |u(x_2)|\,.$$

If $h_+(x_1) > 0$, then

$$h_+(x_1)e^{u(x_1)} + h_-(x_1)e^{-u(x_1)} - c = \frac{\omega_{x_1x_2}}{\mu_{x_1}} (u(x_1) - u(x_2)) \leq \frac{2\omega_{x_1x_2}}{\mu_{x_1}} u(x_1)$$

which implies

$$u(x_1) \leq C.$$

If $h_+(x_1) \le 0$, then

$$\frac{\omega_{x_1x_2}}{\mu_{x_1}}(u(x_1)-u(x_2)) = h_+(x_1)e^{u(x_1)} + h_-(x_1)e^{-u(x_1)} - c \le C$$

which implies

$$0 \le u(x_1) - u(x_2) \le C.$$

If $u(x_2) < 0$, then $u(x_1) \le C$. If $u(x_2) \ge 0$, then

$$\begin{aligned} |h_+(x_1)| + |h_+(x_2)| =& \left| \left(\frac{\omega_{x_1x_2}}{\mu_{x_1}}(u(x_1)-u(x_2)) - h_-(x_1)e^{-u(x_1)} + c \right) e^{-u(x_1)} \right| \\ &+ \left| \left(\frac{\omega_{x_1x_2}}{\mu_{x_2}}(u(x_2)-u(x_1)) - h_-(x_2)e^{-u(x_2)} + c \right) e^{-u(x_2)} \right| \\ \le& Ce^{-u(x_1)}, \end{aligned}$$

which implies

$$u(x_1) \le C.$$

That is, we obtain a uniform a priori estimate for the sinh-Gordon equation (4.1) under the assumption (4.2).

Now, drawing inspiration from the Example 4.1 provided above, we can present a comprehensive proof of the Theorem 2.1.

Proof of the Theorem 2.1 To proceed with the proof by contradiction, we will assume that there exists a sequence of real functions $\{u_n\}$ on a finite, connected, and symmetric graph $G = (V, E)$ satisfying

$$-\Delta u_n = h_+ e^{u_n} + h_- e^{u_n} - c, \quad \text{in } V, \tag{4.3}$$

and

$$\max_V u_n = \max_V |u_n| \to \infty, \tag{4.4}$$

as $n \to \infty$. Our goal is to show that this leads to a contradiction, thereby proving the original statement. According to the a priori estimate stated by Sun and Wang [25, Theorem 2.1], we may assume $h_+ \ne 0$ and $h_- \ne 0$.

We consider the functions $L_n : V \longrightarrow \mathbb{R}$ defined by

$$L_n = \max_V u_n - u_n. \tag{4.5}$$

It follows from (4.4) that

$$\max_V u_n + \min_V u_n \ge 0.$$

According to the Lemma 3.3, we obtain

$$-\Delta u_n \le C\left(u_n^+ + L_n\right), \tag{4.6}$$

and

$$L_n(y) \le C\left((-\Delta u_n(x))^+ + L_n(x)\right), \quad \forall y \sim x. \tag{4.7}$$

Insert (4.6) into the equation (4.3) to obtain

$$h_+e^{u_n} + h_-e^{-u_n} \le C\left(u_n^+ + L_n + 1\right). \tag{4.8}$$

If $L_n(x) \le C$, then by the definition (4.5) and the assumption (4.4), we get $u_n(x) \to +\infty$ as $n \to \infty$. The estimate (4.8) then gives $h_+(x) \le 0$ which implies from (4.3) that

$$-\Delta u_n(x) \le h_-(x)e^{-u_n(x)} - c \le C. \tag{4.9}$$

It follows from (4.9) and (4.7) that

$$L_n(y) \le C, \quad \forall y \sim x.$$

In other words, if $L_n(x)$ is uniformly bounded from above, then $L_n(y)$ is also uniformly bounded from above for every neighbors y of x. That is

$$L_n(x) \le C \quad \Longrightarrow \quad L_n(y) \le C, \quad \forall y \sim x. \tag{4.10}$$

Since the graph is finite, we may assume that for some points $\bar{x}, \underline{x} \in V$

$$u_n(\bar{x}) = \max_V u_n, \quad u_n(\underline{x}) = \min_V u_n, \quad \forall n.$$

Since the graph is connected, we can choose a shortest path $x_0x_1\cdots x_m$ connected $x_0 = \bar{x}$ and $x_m = \underline{x}$. We have

$$L_n(x_0) = L_n(\bar{x}) = 0, \quad \forall n.$$

Applying the estimate (4.10), since $m \le \#V - 1$, we obtain by induction that

$$L_n(x) \le C, \quad \forall x \in V.$$

In particular

$$\max_V u_n - \min_V u_n = L_n(x_m) \le C. \tag{4.11}$$

Together with the above estimate (4.11) and the assumption (4.4), we conclude that

$$\lim_{n\to\infty} u_n(x) = +\infty, \quad \forall x \in V. \tag{4.12}$$

Moreover, by the definition of the Laplacian (2.1), the estimate (4.11) also implies

$$|\Delta u_n| \le C\left(\max_V u_n - \min_V u_n\right) \le C. \tag{4.13}$$

Combine (4.3), (4.11), (4.12) and (4.13),

$$|h_+| = e^{\max_V u_n - u_n}\left|c - \Delta u_n - h_-e^{-u_n}\right| e^{-\max_V u_n} \le Ce^{-\max_V u_n}$$

which implies that $h_+ \equiv 0$. This is a contradiction and we complete the proof. □

5 Topological degrees

In this section, we calculate the topological degree on a case-by-case basis.

We start with a uniform a priori estimate for the sinh-Gordon equation (1.1) that involves variable coefficients.

Theorem 5.1 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. Assume for some positive constant K, the following conditions hold:*

(H1) $K^{-1} \leq \max_V |h_\pm| \leq K,\ |c| \leq K$, *and*
(H2) $h_\pm^2 \geq \pm K^{-1} h_\pm$.

Then there is a uniform positive constant C depending only on the graph such that every solution u to the sinh-Gordon equation (1.1) satisfies

$$\max_V |u| \leq CK. \tag{5.1}$$

Proof Assume u is a solution to the sinh-Gordon equation (1.1), i.e.,

$$-\Delta u = h_+ e^u + h_- e^{-u} - c.$$

Without loss of generality, assume

$$\max_V u + \min_V u \geq 0.$$

We consider the function $L = \max_V u - u$. Under the assumption $(H1)$, according to (3.3), we obtain

$$-\Delta u \leq C\left(u^+ + L\right), \tag{5.2}$$

and

$$L(y) \leq C\left((-\Delta u)^+(x) + L(x)\right), \quad \forall x \sim y. \tag{5.3}$$

Inserting the above estimate (5.2) into the sinh-Gordon equation (1.1), under the assumption $(H1)$, we have

$$h_+ e^u + h_- e^{-u} = -\Delta u + c \leq C\left(u^+ + L + K\right). \tag{5.4}$$

If $h_+(x) > 0$ and $u(x) > 0$, then under the assumption $(H2)$ the above inequality (5.4) gives

$$h_+(x) e^{u(x)} \leq C e^{u(x)/2} + C(L(x) + K)$$

which implies

$$e^{u(x)/2} \leq \frac{C + \sqrt{C^2 + 4h_+(x)C(L(x)+K)}}{2h_+(x)}.$$

We obtain

$$h_+(x) e^{u(x)} \leq \frac{C^2 + 2h_+(x)C(L(x)+K)}{h_+(x)}.$$

Thus

$$h_+(x) e^{u(x)} \leq C\left(L(x) + K\right). \tag{5.5}$$

Notice that the above estimate (5.5) also holds when $h_+(x) \leq 0$ or $u(x) \leq 0$. Inserting (5.5) into (1.1), we get

$$-\Delta u(x) = h_+(x) e^{u(x)} + h_-(x) e^{-u(x)} - c \leq C\left(L(x) + K\right) \tag{5.6}$$

provided $u(x) \geq 0$. It follows from (5.3) and (5.6) that

$$u(x) \geq 0 \quad \Longrightarrow \quad L(y) \leq C\left((-\Delta u)^+(x) + L(x)\right) \leq C_0\left(L(x) + K\right), \quad \forall x \sim y. \tag{5.7}$$

Here the constant C_0 depends only on the graph.

Now we prove the a priori estimate (5.1). We may assume

$$\max_V |u| = \max_V u > \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K. \tag{5.8}$$

Since the graph is finite, connected, and symmetric, we can choose a shortest path $x_0 x_1 \cdots x_m$ such that

$$u(x_0) = \max_V u, \quad u(x_m) = \min_V u.$$

Then $m \leq \#V - 1$ and

$$L(x_0) = \max_V u - u(x_0) = 0.$$

Since $u(x_0) \geq 0$, it follows from (5.7) that

$$L(x_1) \leq C_0 K.$$

If $L(x_i) \leq \left(C_0 + C_0^2 + \cdots + C_0^i\right) K$ for some $1 \leq i < m$, then

$$u(x_i) = \max_V u - L(x_i) > \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K - \left(C_0 + C_0^2 + \cdots + C_0^i\right) K \geq 0,$$

which implies from (5.7) that

$$L(x_{i+1}) \leq C_0 \left(\left(C_0 + C^2 + \cdots + C_0^i\right) K + K\right) = \left(C_0 + C_0^2 + \cdots + C_0^{i+1}\right) K.$$

By using the method of induction, we conclude that

$$L(x_i) \leq \left(C_0 + C_0^2 + \cdots + C_0^i\right) K, \quad 1 \leq i \leq m.$$

In particular

$$\begin{aligned} \max_V u - \min_V u = L(x_m) &\leq \left(C_0 + C_0^2 + \cdots + C_0^m\right) K \\ &\leq \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K < \max_V u. \end{aligned}$$

We obtain $\min_V u > 0$. That is, under the assumption (5.8), we have

$$\max_V u - \min_V u \leq \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K, \quad \min_V u \geq 0. \tag{5.9}$$

By the definition of the Laplacian (2.1),

$$\begin{aligned} |-\Delta u(x)| = \left| \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \left(u(x) - u(y)\right) \right| &\leq \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \left(\max_V u - \min_V u\right) \\ &\leq C \left(\max_V u - \min_V u\right). \end{aligned} \tag{5.10}$$

Inserting the estimate (5.9) into the above estimate (5.10), we obtain

$$|\Delta u| \leq C \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K. \tag{5.11}$$

Recall from the sinh-Gordon equation (1.1) that

$$h_+ = e^{-u} \left(c - \Delta u - h_- e^{-u}\right).$$

Consequently, the assumption $(H1)$ together with the estimate (5.11) implies

$$K^{-1} \leq \max_V |h_+| \leq \left(|c| + \max_V |h_-| + \max_V |\Delta u|\right) e^{-\min_V u}$$
$$\leq C\left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K e^{-\min_V u}.$$

We obtain

$$\min_V u \leq \ln\left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) + \ln C + \ln K. \tag{5.12}$$

Therefore it follows from (5.9) and (5.12) that

$$\max_V |u| = \max_V u = \max_V u - \min_V u + \min_V u$$
$$\leq \left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right) K + \ln\left(C_0 + C_0^2 + \cdots + C_0^{\#V-1}\right)$$
$$+ \ln C + \ln K.$$

In particular, we obtain the desired uniform a priori estimate (5.1) and complete the proof. □

5.1 $h_+ \leq 0, h_- \geq 0$.

Theorem 5.2 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If $h_+ \leq 0$, $h_- \geq 0$ and $\max_V (|h_+| + |h_-|) > 0$, then*

$$d_{h_+,h_-,c} = \begin{cases} 1, & \min_V h_+ < 0,\ \max_V h_- > 0; \\ 1, & c < 0,\ h_- \equiv 0; \\ 1, & c > 0,\ h_+ \equiv 0; \\ 0, & else. \end{cases}$$

Proof By Sun and Wang's result [25, Theorem 2.3], without loss of generality, we may assume $\min_V h_+ = c_+ < 0$ and $\max_V h_- = c_- > 0$. In this case, we will prove

$$d_{h_+,h_-,c} = 1.$$

Consider the following deformations

$$h_{+,t} = (1-t)h_+ - t, \quad h_{-,t} = (1-t)h_- + t, \quad c_t = (1-t)c, \quad t \in [0,1].$$

Assume u_t is a solution to

$$-\Delta u_t = h_{+,t} e^{u_t} + h_{-,t} e^{-u_t} - c_t, \quad t \in [0,1].$$

One can check that for every $t \in [0,1]$,

- the assumption $(H1)$ holds, i.e.,

$$0 < \min\left\{\max_V |h_\pm|, 1\right\} \leq \max_V \left|h_{\pm,t}\right| \leq \max\left\{\max_V |h_\pm|, 1\right\}, \quad |c_t| \leq |c|,$$

- and the assumptions $(H2)$ holds since

$$\max_V h_{+,t} \leq 0, \quad \min_V h_{-,t} \geq 0.$$

According to the Theorem 5.1, we conclude that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. By the homotopy invariance of the topological degree, we know that

$$d_{h_+,h_-,c} = d_{h_{+,0},h_{-,0},c_0} = d_{h_{+,1},h_{-,1},c_1} = d_{-1,1,0}.$$

Notice that the equation

$$-\Delta u = -e^u + e^{-u} = -2\sinh u \tag{5.13}$$

has a unique constant solution $u_c \equiv 0$. To see this, it suffices to prove that the solution to the equation (5.13) is unique. If u and w solve the equation (5.13), then

$$-\Delta(u-w) = -2\sinh u + 2\sinh w.$$

If $u \neq w$, then $u - w$ is not a constant function since the function $t \mapsto \sinh t$ is a strictly increasing function. We obtain by applying the strong maximum principle Lemma 3.1

$$0 < -2\sinh u(x_0) + 2\sinh w(x_0)$$

where $x_0 \in V$ satisfies

$$u(x_0) - w(x_0) = \max_V (u - w).$$

This implies $u < w$. Similar argument yields $u > w$ which is a contradiction. Hence, by homotopy invariance, we can compute the topological degree $d_{h_+,h_-,c}$ as follows

$$d_{h_+,h_-,c} = d_{-1,1,0} = \operatorname{sgn}\det(-\Delta + 2\cdot \mathrm{Id}) = 1.$$

□

5.2 $\max_V h_+ > 0$, $h_- \geq 0$ or $h_+ \leq 0$, $\max_V h_- < 0$.

Theorem 5.3 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If $\max_V h_+ > 0$ and $h_- \geq 0$, then*

$$d_{h_+,h_-,c} = \begin{cases} -1, & h_- \equiv 0,\ c > 0; \\ -1, & h_- \equiv 0,\ \int_V h_+ \,\mathrm{d}\mu < 0,\ c = 0; \\ 0, & else. \end{cases}$$

Proof By Sun and Wang's result [25, Theorem 2.3], without loss of generality, assume $\max_V h_- = c_- > 0$. Set

$$h_{+,t} = h_+^+ - (1-t)h_+^-, \quad c_t = (1-t)c.$$

Assume u_t is a solution to the the following equation

$$-\Delta u_t = h_{+,t}e^{u_t} + h_- e^{u_t} - c_t, \quad t \in [0,1].$$

One can check that

$$0 < \min\left\{\max_V h_+^+, 1\right\} \leq \max_V \left|h_{+,t}\right| \leq \max\left\{\max_V |h_+|, 1\right\}, \quad |c_t| \leq |c|,$$
$$h_{+,t}(x) > 0 \implies h_+(x) > 0 \implies h_{+,t}(x) = h_+(x) > 0,$$
$$\min_V h_- \geq 0.$$

Consequently, According to the Theorem 5.1, we conclude that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. By the homotopy invariance of the topological degree, we know that

$$d_{h_+,h_-,c} = d_{h_{+,0},h_-,c_0} = d_{h_{+,1},h_-,c_1} = d_{h_+^+,h_-,0}.$$

One can check that $d_{h_+^+,h_-,0} = 0$ since the equation

$$-\Delta u = h_+^+ e^u + h_- e^{-u},$$

has no solution. Hence

$$d_{h_+,h_-,c} = d_{h_+^+,h_-,0} = 0.$$

□

Similarly, we obtain

Theorem 5.4 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If $h_+ \leq 0$ and $\min_V h_- < 0$, then*

$$d_{h_+,h_-,c} = \begin{cases} -1, & h_+ \equiv 0,\ c < 0; \\ -1, & h_+ \equiv 0,\ \int_V h_- \mathrm{d}\mu > 0,\ c = 0; \\ 0, & else. \end{cases}$$

5.3 $\max_V h_+ > 0$, $\min_V h_- < 0$.

This situation is more complicated. To obtain a deeper understanding of the potential variations in the topological degree at this time, we first examine the following example.

Example 5.1 Let $G = (V, E)$ be a graph with only two vertices $\{x_1, x_2\}$ and one edge x_1x_2. Without loss of generality, assume $\mu_{x_1} = \mu_{x_2} = \omega_{x_1x_2} = 1$.

Case 1. $h_+ = (1, 0)$, $h_- = (-1, 0)$.

The sinh-Gordon equation (4.1) is equivalent to

$$\begin{cases} x - y = e^x - e^{-x} - c, \\ y - x = -c, \end{cases}$$

which is equivalent to

$$c = e^x - e^{-x} - x + y = x - y.$$

One can check that it has a unique solution $(x_c, y_c) = \left(\ln\left(c + \sqrt{c^2+1}\right), \ln\left(c + \sqrt{c^2+1}\right) - c \right)$. The degree is

$$\operatorname{sgn}\det\begin{pmatrix} 1 - e^{x_c} - e^{-x_c} & -1 \\ -1 & 1 \end{pmatrix} = \operatorname{sgn}\left(-e^{x_c} - e^{-x_c}\right) = -1.$$

Case 2. $h_+ = (1, 0)$, $h_- = (0, -1)$.

The sinh-Gordon equation (4.1) is equivalent to

$$\begin{cases} x - y = e^x - c, \\ y - x = -e^{-y} - c. \end{cases} \tag{5.14}$$

which is equivalent to

$$c = e^x - x + y = x - y - e^{-y}.$$

It (x, y) solves the above equation (5.14), then we have

$$f(x) := 2x - e^x = x + y - c = 2y + e^{-y} = -f(-y).$$

However, a direct computation implies $f \leq 2\ln 2 - 2 < 0$. As a consequence, the above equation can not have any solution. In particular, the topological degree must be zero.

Case 3. $h_+ = (1, 1), h_- = (-1, 0)$.
The sinh-Gordon equation (4.1) is equivalent to

$$\begin{cases} x - y = e^x - e^{-x} - c, \\ y - x = e^y - c, \end{cases} \tag{5.15}$$

which is equivalent to

$$c = e^x - e^{-x} - x + y = e^y - y + x.$$

If (x, y) solves the equation (5.15), then $x > y$ and $c > 0$. Thus

$$\ln\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}\right) < x = y - e^y + c \leq c - 1,$$

$$c - 1 - e^{c-1} + e^{1-c} \leq x - e^x + e^{-x} + c = y < \ln c.$$

Hence, to compute the degree, we may assume $c = 0$. However, when $c = 0$, this equation has no solution. Consequently, the degree must be zero.

Case 4. $h_+ = (1, 1), h_- = (-1, -1)$.
The sinh-Gordon equation (4.1) is equivalent to

$$\begin{cases} x - y = e^x - e^{-x} - c, \\ y - x = e^y - e^{-y} - c, \end{cases}$$

which is equivalent to

$$c = e^x - e^{-x} - x + y = e^y - e^{-y} - y + x.$$

One can check that it has exactly one solution

$$(x_c, y_c) = \left(\ln\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}\right), \ln\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}\right)\right).$$

To compute the degree, we may assume $c \neq 0$. Consequently, the degree is

$$\operatorname{sgn}\det\begin{pmatrix} 1 - e^x - e^{-x} & -1 \\ -1 & 1 - e^y - e^{-y} \end{pmatrix}\Bigg|_{x=y=\ln\left(\frac{c}{2}+\sqrt{\frac{c^2}{4}+1}\right)} = 1.$$

To ensure readers have a clear knowledge of the proof, we will illustrate the process step by step, moving from simple to more complicated cases. The proof is organized into several theorems, as outlined below.

We begin with the following special and simple case.

Theorem 5.5 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If* $\min_V h_+ > 0$, $\max_V h_- < 0$*, then*

$$d_{h_+,h_-,c} = (-1)^{\#V}.$$

Proof Let Λ be the largest eigenvalue of $-\Delta$. We consider the following deformations

$$h_{+,t} = (1-t)h_+ + t\Lambda, \quad h_{-,t} = (1-t)h_- - t\Lambda, \quad c_t = (1-t)c, \quad t \in [0, 1].$$

Assume u_t solves

$$-\Delta u_t = h_{+,t}e^{u_t} + h_{-,t}e^{u_t} - c_t, \quad t \in [0, 1].$$

One can check that

$$0 < \min\left\{\max_V |h_\pm|, \Lambda\right\} \le \max_V |h_{\pm,t}| \le \max\left\{\max_V |h_\pm|, \Lambda\right\}, \quad |c_t| \le |c|,$$

$$\min_V h_{+,t} \ge \min\left\{\min_V h_+, \Lambda\right\} > 0, \quad \max_V h_{-,t} \le \max\left\{\max_V h_-, -\Lambda\right\} < 0.$$

According to the Theorem 5.1, we conclude that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. By the homotopy invariance of the topological degree, we know that

$$d_{h_+,h_-,c} = d_{h_{+,0},h_{-,0},c_0} = d_{h_{+,1},h_{-,1},c_1} = d_{\Lambda,-\Lambda,0}.$$

We consider the following functional

$$L^\infty(V) \ni u \mapsto J_\Lambda(u) = \frac{1}{2}\int_V |\nabla u|^2 \,\mathrm{d}\mu - 2\Lambda \int_V \cosh u \,\mathrm{d}\mu.$$

One can check that the Euler-Lagrangian equations for this functional are

$$-\Delta u = 2\Lambda \sinh u = \Lambda e^u - \Lambda e^{-u}.$$

In other words, u solves (1.1) if and only if u is a critical point of the functional J_Λ.

For every $u, \xi \in L^\infty(V)$ and $t \in \mathbb{R}$, a direct computation gives

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} J_\Lambda(u + t\xi) = \int_V |\nabla \xi|^2 \,\mathrm{d}\mu - 2\Lambda \int_V \cosh u \cdot \xi^2 \,\mathrm{d}\mu.$$

Since $\cosh u = \frac{e^u + e^{-u}}{2} \ge 1$, we obtain

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} J_\Lambda(u + t\xi) \le \Lambda \int_V \xi^2 \,\mathrm{d}\mu - 2\Lambda \int_V \xi^2 \,\mathrm{d}\mu = -\Lambda \int_V \xi^2 \,\mathrm{d}\mu.$$

In particular, J_Λ is a strictly concave function in the finite dimensional space $L^\infty(V)$. On the other hand,

$$\begin{aligned} J_\Lambda(u) \le & C\left(\max_V u - \min_V u\right)^2 - C^{-1}\Lambda\left(\cosh \max_V u + \cosh \min_V u\right) \\ \le & -C^{-1} \cosh \max_V |u| \to -\infty, \end{aligned}$$

as $\max_V |u| \to \infty$. Consequently, J_Λ has exactly one critical point u_Λ which is the global maximum point of the functional J_Λ, i.e., the sinh-Gordon equation (1.1) admits a unique solution u_Λ. Notice that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_\Lambda(u + t\xi) = \int_V (-\Delta u - 2\Lambda \sinh u)\,\xi \,\mathrm{d}\mu = \int_V F_{\Lambda,-\Lambda,0}(u)\xi \,\mathrm{d}\mu,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} J_\Lambda(u+t\xi) = \int_V (-\Delta\xi - 2\Lambda\cosh u\cdot\xi)\,\xi\,\mathrm{d}\mu = \int_V \frac{\partial F_{\Lambda,-\Lambda,0}(u)}{\partial u}\xi^2\,\mathrm{d}\mu.$$

Since J_Λ is strictly concave, we know that all of the eigenvalues of $\frac{\partial F_{\Lambda,-\Lambda,0}(u)}{\partial u}$ are negative. Hence, by the definition of the topological degree (2.2), we get

$$d_{h_+,h_-,c} = d_{\Lambda,-\Lambda,0} = \operatorname{sgn}\det(-\Delta - 2\Lambda\cosh u_\Lambda \mathrm{Id}) = (-1)^{\#V}.$$

□

Next, we consider a slightly more general case.

Theorem 5.6 *Let $G=(V,E)$ be a finite, connected, and symmetric graph. If* $\max_V h_+ > 0$, $\min_V h_- < 0$ *and*

$$V_0 := \{x\in V : h_+(x) > 0\} = \{x \in V : h_-(x) < 0\},$$

then

$$d_{h_+,h_-,c} = (-1)^{\#V_0}.$$

Proof We consider the following deformations

$$h_{+,t}(x) = \begin{cases}(1-t)h_+(x) + t\Lambda, & h_+(x) > 0,\\ (1-t)h_+(x), & h_+(x)\le 0,\end{cases}$$

$$h_{-,t}(x) = \begin{cases}(1-t)h_-(x) - t\Lambda, & h_-(x) < 0,\\ (1-t)h_-(x), & h_-(x)\ge 0,\end{cases}\qquad c_t = (1-t)c,$$

where $t\in[0,1]$ and Λ is a positive number to be determined. Assume u_t solves

$$-\Delta_t u_t = h_{+,t}e^{u_t} + h_{-,t}e^{-u_t} - c_t,\quad t\in[0,1].$$

Notice that for all $t\in[0,1]$

- $$0 < \min\left\{\max_V h_+^+, \max_V \left|h_-^-\right|, \Lambda\right\} \le \max_V \left|h_{\pm,t}\right| \le \max\left\{\max_V |h_\pm|, \Lambda\right\},\quad |c_t|\le|c|,$$
- $$h_{+,t}(x) > 0 \implies h_+(x) > 0 \implies h_{+,t}(x) \ge \min\{h_+(x),\Lambda\} > 0,$$
- $$h_{-,t}(x) < 0 \implies h_-(x) < 0 \implies h_{-,t}(x) \le \max\{h_-(x),-\Lambda\} < 0.$$

Consequently, according to the Theorem 5.1, we conclude that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. By the homotopy invariance of the topological degree, we know that

$$d_{h_+,h_-,c} = d_{h_{+,0},h_{-,0},c_0} = d_{h_{+,1},h_{-,1},c_1} = d_{\Lambda\chi_{V_0},-\Lambda\chi_{V_0},0}.$$

It is well known that the following boundary value problem has a unique solution

$$\begin{cases}\Delta u = 0, & \text{in } V\setminus V_0,\\ u = \phi, & \text{in } V_0,\end{cases}$$

for every function $\phi \in L^{\infty}(V_0)$. We then obtain a linear map $\phi \mapsto P\phi = u$. We define the linear operator $L : L^{\infty}(V_0) \longrightarrow L^{\infty}(V_0)$ as follows

$$\phi \mapsto L\phi := (\Delta(P\phi))\,|_{V_0}$$

Then $u \in L^{\infty}(V)$ solves

$$-\Delta u = 2\Lambda \sinh u \cdot \chi_{V_0}, \quad \text{in } V, \tag{5.16}$$

if and only if $\phi = u|_{V_0}$ solves

$$-L\phi = 2\Lambda \sinh \phi, \quad \text{in } V_0, \tag{5.17}$$

if and only if ϕ is a critical point of the following functional

$$\tilde{J}_{\Lambda}(\phi) = -\int_{V_0} \left(\frac{1}{2}\phi L\phi + 2\Lambda \cosh \phi\right) \mathrm{d}\mu, \quad \forall \phi \in L^{\infty}(V_0).$$

One can check that for every functions $\phi, \eta \in L^{\infty}(V_0)$, we have

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \tilde{J}_{\Lambda}(\phi + t\eta) = -\int_{V_0} \left(\eta L\eta + 2\Lambda \cosh \phi \cdot \eta^2\right) \mathrm{d}\mu.$$

Let Λ_0 be the largest eigenvalue of the operator $-L$. One can choose $\Lambda = \Lambda_0$. A similar argument as in the proof of Theorem 5.5, we obtain that the equation (5.17) has exactly one solution. Consequently, the sinh-Gordon equation (5.16) has exactly one solution. A direct computation then yields

$$d_{h_+,h_-,c} = (-1)^{\#V_0}.$$

□

We now turn our attention to the last case, which stands apart from the previous two and presents a more subtle complexity. That is, we consider the case that $\max_V h_+ > 0$, $\min_V h_- < 0$ and

$$\{x \in V : h_+(x) > 0\} \neq \{x \in V : h_-(x) < 0\}.$$

We have the following

Theorem 5.7 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If* $\max_V h_+ > 0$, $\min_V h_- < 0$ *and*

$$V_+ := \{x \in V : h_+(x) > 0\} \neq \{x \in V : h_-(x) < 0\} =: V_-,$$

then

$$d_{h_+,h_-,c} = 0.$$

Proof Without loss of generality, assume

$$V_- \setminus V_+ \neq \emptyset.$$

Let $\Lambda > 1$ be a positive constant to be determined. We consider the following equation

$$-\Delta u = h_+ e^u + h_- e^{-u} - \Lambda. \tag{5.18}$$

Assume u_Λ solves (5.18). Applying the Theorem 5.1, there exists a positive constant C which depends only on the graph $G = (V, E)$ and the prescribed functions $h_\pm$ such that

$$\max_V |u_\Lambda| \leq C\Lambda. \tag{5.19}$$

On the one hand, since the graph is finite, one can choose $x_\Lambda \in V$ such that

$$u_\Lambda(x_\Lambda) = \max_V u_\Lambda.$$

If $h_+(x_\Lambda) \leq 0$, then applying the maximum principle Lemma 3.1, we get

$$0 \leq -\Delta u_\Lambda(x_\Lambda) = h_+(x_\Lambda)e^{u_\Lambda(x_\Lambda)} + h_-(x_\Lambda)e^{-u_\Lambda(x_\Lambda)} - \Lambda \leq \max_V h_- e^{-u_\Lambda(x_\Lambda)} - \Lambda.$$

We must have $\max_V h_- > 0$ and conclude that

$$\max_V u_\Lambda = u_\Lambda(x_\Lambda) \leq \ln \max_V h_- - \ln \Lambda. \tag{5.20}$$

If $h_+(x_\Lambda) > 0$, then the estimate (5.19) gives

$$\max_V |\Delta u_\Lambda| \leq C\left(\max_V u_\Lambda - \min_V u_\Lambda\right) \leq C\Lambda$$

and hence

$$h_+(x_\Lambda)e^{u_\Lambda(x_\Lambda)} + h_-(x_\Lambda)e^{-u_\Lambda(x_\Lambda)} - \Lambda = -\Delta u_\Lambda(x_\Lambda) \leq C\Lambda. \tag{5.21}$$

The above estimate (5.21) implies

$$\max_V u_\Lambda = u_\Lambda\,(x_\Lambda) \leq C + \ln \Lambda. \tag{5.22}$$

Therefore, we obtain a upper bound from (5.20) and (5.22) that

$$\max_V u_\Lambda \leq C + \ln \Lambda. \tag{5.23}$$

One the other hand, choose $x_0 \in V_- \setminus V_+$, i.e.,

$$h_+(x_0) \leq 0, \quad h_-(x_0) < 0.$$

By the definition,

$$-\Delta u_\Lambda(x_0) = \frac{1}{\mu_{x_0}} \sum_{y \sim x_0} \omega_{x_0 y}\,(u_\Lambda(x_0) - u_\Lambda(y)) \geq C\left(u_\Lambda(x_0) - \max_V u_\Lambda\right).$$

Inserting the above estimate into the equation (5.18), we get

$$C\left(u_\Lambda(x_0) - \max_V u_\Lambda\right) \leq h_+(x_0)e^{u_\Lambda(x_0)} + h_-(x_0)e^{-u_\Lambda(x_0)} - \Lambda \leq h_-(x_0)e^{-u_\Lambda(x_0)} - \Lambda.$$

Hence

$$\max_V u_\Lambda \geq u_\Lambda(x_0) - \frac{h_-(x_0)}{C}e^{-u_\Lambda(x_0)} + \frac{\Lambda}{C} \geq C^{-1}\Lambda + 1 + \ln \frac{-h_-(x_0)}{C}.$$

Therefore, we obtain a lower bound of $\max_V u_\Lambda$ as follows

$$\max_V u_\Lambda \geq C^{-1}\Lambda - C. \tag{5.24}$$

It follows from (5.23) and (5.24) that

$$C^{-1}\Lambda - C \le C + \ln \Lambda$$

which implies

$$\Lambda \le C.$$

In other words, there exist a constant Λ_0 such that the equation (5.18) has no solution for any $\Lambda > \Lambda_0$. In particular,

$$d_{h_+,h_-,\Lambda} = 0, \quad \forall \Lambda > \Lambda_0.$$

Finally, we consider the deformation

$$c_t = (1-t)c + t(\Lambda_0 + 1), \quad t \in [0,1].$$

Assume u_t solves

$$-\Delta u_t = h_+ e^{u_t} + h_- e^{-u_t} - c_t.$$

Applying the Theorem 5.1, we know that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. In particular, by the homotopy invariance of the topological degree, we have

$$d_{h_+,h_-,c} = d_{h_+,h_-,c_0} = h_{h_+,h_-,c_1} = d_{h_+,h_-,\Lambda_0+1} = 0.$$

□

Now we can prove the Theorem 2.2.

Proof of the Theorem 2.2 Combine the Theorems 5.2-5.7, we complete the proof. □

6 An existence result

We first recall with discrete version of sub-super solution principle.

Let $f : V \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. We consider the following equation

$$-\Delta u = f(\cdot, u). \tag{6.1}$$

It is easy to check that u solves the above equation (6.1) if and only if u is a critical point of the following functional

$$J_f(u) := \int_V \left(\frac{1}{2} |\nabla u|^2 - F(\cdot, u) \right) \mathrm{d}\mu,$$

where $\frac{\partial F}{\partial u} = f$. We say that $\phi \in L^\infty(V)$ is a subsolution to the equation (6.1) if

$$-\Delta \phi \le f(\cdot, \phi),$$

and we say that ψ is a super-solution to the equation (6.1) if

$$-\Delta \psi \ge f(\cdot, \psi).$$

We have the following

Lemma 6.1 (Sub-super solution principle, cf. [25]) *Assume ϕ and ψ are the sub- and super-solutions to the equation (6.1) respectively with $\phi \le \psi$. Then any minimizer of the functional J_f in $\{u \in L^\infty(V) : \phi \le u \le \psi\}$ solves (6.1).*

As a consequence, we have the following existence result.

Lemma 6.2 *Assume*

$$\limsup_{t\to-\infty} f(\cdot,t) = f_\infty(\cdot).$$

If $\bar{f}_\infty > 0$ *and*

$$\inf_{u\in L^\infty(V)} \max_V (\Delta u + f(\cdot,u)) < 0,$$

then the equation (6.1) *has at least one solution.*

Proof Let ϕ_∞ be the unique solution to the following equation

$$-\Delta\phi_\infty = f_\infty - \bar{f}_\infty, \quad \bar{\phi}_\infty = 0.$$

Here $\bar{f} = \fint_V f \,\mathrm{d}\mu$ is the average of the integral of the function f over V. For every constant A, we have

$$-\Delta(A+\phi_\infty) - f(\cdot, A+\phi_\infty) = f_\infty - \bar{f}_\infty - f(\cdot, A+\phi_\infty).$$

Since $\bar{f}_\infty > 0$, there exists a sequence of numbers $A_n \to -\infty$ such that

$$-\Delta(A_n+\phi_\infty) - f(\cdot, A_n+\phi_\infty) \le -\frac{1}{2}\bar{f}_\infty < 0.$$

In other words, $A_n + \phi_\infty$ are sub-solutions to the equation (6.1). Applying the Lemma 6.1, we conclude that the equation (6.1) is solvable if and only if it has a super-solution.

Since

$$c_f := \inf_{u\in L^\infty(V)} \max_V (\Delta u + f(\cdot,u)) < 0,$$

we can find a function ψ satisfying

$$\max_V (\Delta\psi + f(\cdot,\psi)) < \frac{c_f}{2} < 0.$$

Thus, ψ is a super-solution and we complete the proof. □

As a consequence, we obtain an existence result, i.e., we can give a proof of the Theorem 2.4. More generally, we have the following

Theorem 6.3 *Let* $G = (V, E)$ *be a finite, connected, and symmetric graph. Assume* $\max_V h_+ > 0,\ h_- \ge 0,\ \max_V h_- > 0.$

(1) There exists a constant Λ_0 *such that* (1.1) *has no solution when* $c < \Lambda_0$.
(2) If $c \le 0$*, then a necessary condition to solve* (1.1) *is* $\int_V h_+ \,\mathrm{d}\mu < 0$.
(3) If $\int_V h_+ \,\mathrm{d}\mu < 0$ *and set*

$$c^*_{h_+,h_-} := \inf_{u\in L^\infty(V)} \max_V \left(\Delta u + h_+e^u + h_-e^{-u}\right),$$

then $c^*_{h_+,h_-} \in \mathbb{R}$*. Moreover, if* $c^*_{h_+,h_-} < 0$*, then the sinh-Gordon equation* (1.1) *has no solution if* $c < c^*_{h_+,h_-}$*, and has at least one solution if* $c = c^*_{h_+,h_-}$*, and has at least two solutions if* $c^*_{h_+,h_-} < c < 0$.

Proof Firstly, since

$$\{h_+ > 0\} \neq \emptyset, \quad \{h_- < 0\} = \emptyset,$$

the argument in the proof of the Theorem 5.7 implies that there exists a constant Λ_0 such that the equation (1.1) has no solution when $c < \Lambda_0$.

Secondly, we assume $c \leq 0$. If $c = 0$ and u solves (1.1), then u can not be a constant function and

$$0 > -\int_V e^{-u} \Delta u \,\mathrm{d}\mu = \int_V h_+ \,\mathrm{d}\mu + \int_V h_- e^{-2u} \,\mathrm{d}\mu \geq \int_V h_+ \,\mathrm{d}\mu.$$

Thus $\int_V h_+ \,\mathrm{d}\mu < 0$. If $c < 0$, then a similar argument implies that $\bar{h}_+ < 0$.

Thirdly, if $c < 0$, then for each constant $A < \ln \frac{-c}{\max_V |h_+|}$

$$-\Delta A - h_+ e^A - h_- e^{-A} + c \leq \max_V |h_+| \, e^A + c < 0.$$

That is, the constant function A is a sub-solution whenever $A < \ln \frac{-c}{\max_V |h_+|}$. Notice that

$$\inf_{u \in L^\infty(V)} \max_V \left(\Delta u + h_+ e^u + h_- e^{-u}\right) \geq \inf_{u \in L^\infty(V)} \max_V \left(\Delta u + h_+ e^u\right) =: c_{h_+}.$$

If h_+ changes its sign and $\bar{h}_+ < 0$, then we claim that

$$\inf_{u \in L^\infty(V)} \max_V \left(\Delta u + h_+ e^u\right) \in (-\infty, 0).$$

To see this, on the one hand, let ϕ be the unique solution to

$$-\Delta \phi = h_+ - \bar{h}_+, \quad \min_V \phi = 0.$$

Then

$$0 \leq \phi \leq C \left(\max_V h_+ - \bar{h}_+\right).$$

For any two constants $a > 0$ and $b = \ln a$, we have

$$\begin{aligned}
\Delta (a\phi + b) + h_+ e^{a\phi + b} =& a \left(\bar{h}_+ + h_+ \left(e^{a\phi} - 1\right)\right) \\
\leq& a \left(\bar{h}_+ + \max_V h_+ \left(e^{aC(\max_V h_+ - \bar{h}_+)} - 1\right)\right) \\
=& \frac{1}{C\left(1 - \bar{h}_+ / \max_V h_+\right)} \left[e^{aC(\max_V h_+ - \bar{h}_+)} - \left(1 - \frac{\bar{h}_+}{\max_V h_+}\right)\right] \\
& \cdot aC \left(\max_V h_+ - \min_V h_+\right).
\end{aligned}$$

Hence

$$\begin{aligned}
c_{h_+} \leq& \inf_{a>0} \max_V \left(\Delta (a\phi + b) + h_+ e^{a\phi + b}\right) \\
\leq& \frac{1}{C\left(1 - \bar{h}_+ / \max_V h_+\right)} \inf_{t>0} \left[e^t - \left(1 - \frac{\bar{h}_+}{\max_V h_+}\right)\right] t \\
<& -\frac{1}{2C} \frac{\ln^2\left(1 - \bar{h}_+ / \max_V h_+\right)}{2 + \ln^2\left(1 - \bar{h}_+ / \max_V h_+\right)} < 0.
\end{aligned}$$

On the other hand, if $c_{h_+} < c < 0$, then according to the Lemma 6.2, we conclude that there exists a solution u_c to the equation

$$-\Delta u_c = h_+ e^{u_c} - c.$$

Thus

$$\Delta e^{-u_c} \geq -e^{-u_c} \Delta u_c \geq h_+ - c e^{-u_c}.$$

Applying the maximum principle Lemma 3.1, we obtain

$$e^{-u_c} \leq (\Delta + c)^{-1} h_+.$$

If $c_{h_+} = -\infty$, then we obtain by letting $c \to -\infty$

$$\max_V h_- \leq 0,$$

which is a contradiction.

Finally, we assume $c < 0$, $c^*_{h_+,h_-} < 0$. On the one hand, if the equation (1.1) is solvable, then by the definition, we must have

$$c^*_{h_+,h_-} \leq c.$$

On the other hand, according to the Lemma 6.2, for every $c \in \left(c^*_{h_+,h_-}, 0\right)$, the equation

$$-\Delta u = h_+ e^u + h_- e^{-u} - c$$

has at least one solution u_c. Letting $c \to c^*_{h_+,h_-}$, according to the Theorem 5.1, we obtain a solution $u_{c^*_{h_+,h_-}}$ to the equation

$$-\Delta u = h_+ e^u + h_- e^{-u} - c^*_{h_+,h_-}.$$

Moreover, if $c^*_{h_+,h_-} < c < 0$, we claim that the equation (1.1) has at least two solutions. We can prove this claim as follows.

Let $A < \min_V u_{c^*_{h_+,h_-}}$ be a subsolution to the equation (1.1). Notice that $u_{c^*_{h_+,h_-}}$ is a super-solution to the equation (1.1). Applying the sub-super solution principle Lemma 6.1, we obtain a solution $u_{c,1}$ which minimizes the functional

$$\left\{u : A \leq u \leq u_{c^*_{h_+,h_-}}\right\} \ni u \to J(u) = \int_V \left(\frac{1}{2}|\nabla u|^2 - h_+ e^u + h_- e^{-u} + cu\right) \mathrm{d}\mu.$$

Applying the maximum principle Lemma 3.1, we know that $A < u_{c,1} < u_{c^*_{h_+,h_-}}$. Moreover, for each $\eta \in L^\infty(V)$,

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} J(u_{c,1} + t\eta) \geq 0.$$

We claim that $u_{c,1}$ is a local strict minimizer. For otherwise, there exists a nonzero function ξ such that

$$-\Delta \xi = \left(h_+ e^{u_{c,1}} - h_- e^{-u_{c,1}}\right) \xi,$$

and

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}t^3}\right|_{t=0} J(u_{c,1} + t\xi) = \left.\frac{\mathrm{d}^4}{\mathrm{d}t^4}\right|_{t=0} J(u_{c,1} + t\xi) = 0.$$

We conclude that ξ is not a constant function since $h_- \geq 0$ and $c < 0$. Moreover,

$$0 = \frac{\mathrm{d}^4}{\mathrm{d}t^4}\bigg|_{t=0} J(u_{c,1} + t\xi) = \int_V \left(-h_+ e^{u_{c,1}} + h_- e^{-u_{c,1}}\right) \xi^4 \,\mathrm{d}\mu = \int_V \Delta\xi \cdot \xi^3 \,\mathrm{d}\mu < 0,$$

which is a contradiction. Therefore, $u_{c,1}$ is a local strict minimizer. According to the Theorem 5.7, we know that the topological degree vanishes. As a consequence, we know that there exists another solution to the equation (1.1). □

Appendix A. The topological degree for the Kazdan-Warner equation

In this appendix, we consider the Kazdan-Warner equation (1.2).

We have the following

Theorem Appendix A.1 (Sun-Wang [25]) *Every solution to the Kazdan-Warner equation with nonzero prescribed function on a finite, connected, and symmetric graph is uniform bounded.*

In fact, we can prove the following

Theorem Appendix A.2 *Let $G = (V, E)$ be a finite, connected, and symmetric graph. Assume there exists a positive constant K satisfying the following conditions:*

A) $K^{-1} \leq \max_V |h| \leq K, \quad |c| \leq K$, and
B) $(\max_V h)^2 - K^{-1} \max_V h \geq 0$, and
C) either $|c| \geq K^{-1}$ or $c \geq 0$ and $\bar{h} \leq -K^{-1}$.

There exists a positive constant C depending only on K and the graph such that every solution u to the Kazdan-Warner equation (1.2) satisfies

$$\max_V |u| \leq C.$$

Proof Notice that the condition $B)$ is different from Sun and Wang [25, Theorem 4.2]. We should modify the original proof of Sun and Wang. For convenience of the reader, we sketch the proof here which is slightly different from Sun and Wang's original one.

Assume the statement if false. One can find a sequence of functions u_n solves

$$-\Delta u_n = h_n e^{u_n} - c_n \tag{A.1}$$

and satisfies

$$\lim_{n\to\infty} \max_V |u_n| = \infty, \tag{A.2}$$

where the functions h_n and constants c_n satisfy the assumptions $A) - C)$ and

$$\lim_{n\to\infty} h_n = h, \quad \lim_{n\to\infty} c_n = c.$$

We claim that u_n can not uniformly bounded from above. For otherwise, $\max_V u_n \leq C$. The equation (A.1) implies

$$|\Delta u_n| \leq C.$$

Applying the Lemma 3.4, we get

$$\max_V u_n - \min_V u_n \leq C \max_V |\Delta u_n| \leq C.$$

Hence, the blowup assumption (A.2) yields

$$\lim_{n\to\infty} u_n = -\infty.$$

Moreover, up to a subsequence, we may assume $u_n - \min_V u_n$ converges to w. By the equation (A.1),

$$-\Delta\left(u_n - \min_V u_n\right) = h_n e^{u_n} - c_n,$$

we obtain

$$-\Delta w = c, \quad \min_V w = 0,$$

which implies $c = 0$ and $w \equiv 0$. According to the equation (A.1) again, we get

$$0 = -e^{-\min_V u_n} \int_V \Delta u_n \, \mathrm{d}\mu = \int_V h_n e^{u_n - \min_V u_n} \, \mathrm{d}\mu - c_n e^{-\min_V u_n} |V|.$$

Under the assumption C), we get

$$-K^{-1} \ge \bar{h} = \lim_{n\to\infty} \fint_V h_n e^{u_n - \min_V u_n} \, \mathrm{d}\mu \ge \liminf_{n\to\infty} c_n e^{-\min_V u_n} \ge 0,$$

which is a contradiction.

We may assume

$$0 < \max_V u_n \to +\infty$$

as $n \to \infty$. According to the Kato inequality Lemma 3.2,

$$-\Delta u_n^- \le -\Delta(-u_n)^+ \cdot \chi_{-u_n>0} = \left(c_n - h_n e^{u_n}\right) \chi_{u_n<0} \le C.$$

Applying the Lemma 3.4, we get

$$\max_V u_n^- = \max_V u_n^- - \min_V u_n^- \le C \max_V \left(-\Delta u_n^-\right) \le C,$$

which implies

$$u_n \ge -C.$$

By the definition of the Laplacian (2.1), for each $y \sim x$

$$-\Delta u_n(x) = \frac{1}{\mu_x} \omega_{xy} \left(u_n(x) - u_n(y)\right) + \frac{1}{\mu_x} \sum_{z\sim x, z\neq y} \omega_{xz} \left(u_n(x) - u_n(z)\right),$$

which gives

$$u_n(y) \le C \left(u_n^+(x) + (\Delta u_n)^+(x)\right).$$

Consequently, if $u_n(x) \le C$ for some point x, then

$$|\Delta u_n(x)| = \left|h_n e^{u_n(x)} - c_n\right| \le C$$

and we get

$$u_n(y) \le C, \quad \forall y \sim x.$$

Thus, we obtain by induction

$$\max_V u_n \le C,$$

which is a contradiction to the blowup assumption (A.2). In other words, we may assume, up to a subsequence,

$$\lim_{n\to\infty} u_n(x) = +\infty, \quad \forall x \in V.$$

If $\max_V h_n \ge K^{-1}$, then we may assume $h_n(x_0) = \max_V h_n$. Thus

$$h_n(x_0)e^{u_n(x_0)} - c_n = -\Delta u_n(x_0) \le C\left(u_n^+(x_0) + 1\right)$$

since $u_n \ge -C$. We must have $u_n(x_0) \le C$ which is impossible.

If $\max_V h_n < K^{-1}$, then by the assumption B) we have $\max_V h_n \le 0$ which gives

$$-\Delta u_n \le C$$

and

$$|\Delta u_n| \le C.$$

It follows from (A.1) that

$$h = \lim_{n\to\infty} h_n = \lim_{n\to\infty} e^{-u_n}(c_n - \Delta u_n) = 0,$$

which is a contradiction to the assumption A). □

Based on this a priori estimate as in the Theorem Appendix A.2, one can define the topological degree $D_{h,c}$ for the Kazdan-Warner equation (1.2) as follows:

$$D_{h,c} = \lim_{R\to\infty} \deg\left(K_{h,c}, B_R^{L^\infty(V)}, 0\right)$$

where

$$K_{h,c}(u) = -\Delta u - he^u + c, \quad \forall u \in L^\infty(V).$$

Theorem Appendix A.3 (Sun-Wang [25]) *Let $G = (V, E)$ be a finite, connected, and symmetric graph. If $h \ne 0$, then*

$$D_{h,c} = \begin{cases} -1, & c > 0,\ \max_V h > 0; \\ -1, & c = 0,\ \bar{h} < 0 < \max_V h; \\ 1, & c < 0,\ \min_V h < \max_V h \le 0; \\ 0, & else. \end{cases}$$

Proof For convenience of the reader, we sketch the proof here which is slightly different from Sun and Wang's original one.

One can check that if the Kazdan-Warner equation (1.2) admits a solution, then we have

- if $c > 0$, then $\max_V h > 0$;
- if $c = 0$, then either $h \equiv 0$ or h changes sign and $\bar{h} < 0$;
- if $c < 0$, then $\min_V h < 0$.

Now we compute the degree case by case as follows.

Case 1. If $c > 0$ and $\max_V h > 0$, then

$$D_{h,c} = D_{h^+,c}.$$

In fact, we consider the following deformation

$$h_t = (1-t)h + t, \quad c_t = (1-t)c + t\epsilon, \quad t \in [0,1],$$

where ϵ is a positive number to be determined. If u_t solves

$$-\Delta u_t = h_t e^{u_t} - c_t,$$

then according to the Theorem Appendix A.2 there exists a constant C such that

$$|u_t| \le C, \quad \forall t \in [0,1].$$

By the homotopy invariance of the topological degree, we know that

$$D_{h,c} = D_{1,\epsilon}.$$

Finally, assume u solves

$$-\Delta u = e^u - \epsilon.$$

If $w \ne u$ also solves

$$-\Delta w = e^w - \epsilon,$$

then

$$\int_V e^u \,\mathrm{d}\mu = \int_V e^w \,\mathrm{d}\mu = \epsilon\, |V|\,.$$

We conclude that $\min_V(u-w) < 0 < \max_V(u-w)$. Moreover, we have the following estimate

$$e^u \le C\epsilon, \quad e^w \le C\epsilon.$$

On the other hand, since

$$-\Delta(u-w) = e^u - e^w \le e^u\,(u-w)\,,$$

we conclude that

$$\max_V(u-w) - \min_V(u-w) \le C \max_V \Delta(w-u) \le C\epsilon \max_V(u-w).$$

If $0 < \epsilon < \epsilon_0 = 1/(2C)$, then we conclude that

$$0 < \max_V(u-w) \le 2\min_V(u-w) < 0,$$

which is a contradiction. In other words, if $0 < \epsilon < \epsilon_0$, then $u_\epsilon = \ln \epsilon$ is the unique solution to

$$-\Delta u = e^u - \epsilon.$$

Hence

$$D_{h,c} = \lim_{\epsilon \searrow 0} D_{1,\epsilon} = -1.$$

Case 2. If $c = 0$, $\bar{h} < 0$ and $\max_V h > 0$, then

$$D_{h,0} = -1.$$

In fact, we consider the following deformation

$$c_t = t, \quad t \in [0, 1].$$

Assume u_t satisfies

$$-\Delta u_t = he^{u_t} - c_t, \quad t \in [0, 1].$$

according to the Theorem Appendix A.2, $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. Hence

$$D_{h,0} = D_{h,1} = -1.$$

Case 3. If $c < 0$, $h \le 0$ and $\min_V h < 0$, then

$$D_{h,c} = 1.$$

In fact, we consider the following deformation

$$h_t = (1 - t)h - t, \quad t \in [0, 1].$$

If u_t satisfies

$$-\Delta u_t = h_t e^{u_t} - c,$$

then Theorem Appendix A.2 implies that $\{u_t\}_{t\in[0,1]}$ is uniformly bounded. As a consequence

$$D_{h,c} = D_{-1,c}.$$

Notice that $\ln(-c)$ is the unique solution to the equation

$$-\Delta u = -e^u - c.$$

As a consequence,

$$D_{h,c} = D_{-1,c} = \operatorname{sgn} \det (-\Delta - c\mathrm{Id}) = 1.$$

Case 4. If $c < 0$ and h changes sign, then

$$D_{h,c} = 0.$$

In fact, according to the Theorem Appendix A.2,

$$D_{h,c} = D_{1,c}.$$

Notice that there is no solution to the equation

$$-\Delta u = e^u - c,$$

since $c < 0$. We must have

$$D_{h,c} = D_{1,c} = 0.$$

□

Data Availability Data sharing is not applicable to this article as obviously no datasets were generated or analyzed during the current study.

Declarations

Conflicts of Interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

References

1. Abresch, U.: Constant mean curvature tori in terms of elliptic functions. J. Reine Angew. Math. **374**, 169–192 (1987)
2. Chang, K.C.: Methods in nonlinear analysis. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2005)
3. Chang, S.Y.A., Yang, P.C.: Prescribing Gaussian curvature on S^2. Acta Math. **159**, 215–259 (1987)
4. Chen, C.C., Lin, C.S.: Topological degree for a mean field equation on Riemann surfaces. Comm. Pure Appl. Math. **56**, 1667–1727 (2003)
5. Chow, S.N., Li, W., Zhou, H.: A discrete Schrödinger equation via optimal transport on graphs. J. Funct. Anal. **276**, 2440–2469 (2019)
6. Ge, H., Jiang, W.: Kazdan-Warner equation on infinite graphs. J. Korean Math. Soc. **55**, 1091–1101 (2018)
7. Grigor'yan, A., Lin, Y., Yang, Y.: Kazdan-Warner equation on graph. Calc. Var. Partial Differential Equations **55**, 92 (2016)
8. Grigor'yan, A., Lin, Y., Yang, Y.: Yamabe type equations on graphs. J. Differential Equations **261**, 4924–4943 (2016)
9. Grigor'yan, A., Lin, Y., Yang, Y.: Existence of positive solutions to some nonlinear equations on locally finite graphs. Sci China Math **60**, 1311–1324 (2017)
10. Gu, Q., Huang, X., Sun, Y.: Semi-linear elliptic inequalities on weighted graphs. Calc. Var. Partial Differential Equations **62**, 42 (2023)
11. Han, X., Shao, M., Zhao, L.: Existence and convergence of solutions for nonlinear biharmonic equations on graphs. J. Differential Equations **268**, 3936–3961 (2020)
12. Hao, L., Sun, Y.: Sharp Liouville type results for semilinear elliptic inequalities involving gradient terms on weighted graphs. Discrete Contin. Dyn. Syst. Ser. S **16**, 1484–1516 (2023)
13. Hou, S., Qiao, W.: Solutions to a generalized Chern-Simons Higgs model on finite graphs by topological degree, (2024)
14. Hua, B., Li, R., Wang, L.: A class of semilinear elliptic equations on groups of polynomial growth. J. Differential Equations **363**, 327–349 (2023)
15. Hua, B., Xu, W.: Existence of ground state solutions to some nonlinear Schrödinger equations on lattice graphs. Calc. Var. Partial Differential Equations **62**, 127 (2023)
16. Huang, A., Lin, Y., Yau, S.T.: Existence of solutions to mean field equations on graphs. Comm. Math. Phys. **377**, 613–621 (2020)
17. Huang, H.Y., Wang, J., Yang, W.: Mean field equation and relativistic Abelian Chern-Simons model on finite graphs. J. Funct. Anal. **281**, 109218 (2021)
18. Jost, J., Wang, G., Ye, D., Zhou, C.: The blow up analysis of solutions of the elliptic sinh-Gordon equation. Calc. Var. Partial Differential Equations **31**, 263–276 (2008)
19. Kazdan, J.L., Warner, F.W.: Curvature functions for compact 2-manifolds. Ann. of Math. **2**(99), 14–47 (1974)
20. Li, H., Vrancken, L., Wang, X., Yao, Z.: Hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant sectional curvature. Calc. Var. Partial Differential Equations **63**, 167 (2024)
21. Li, J., Sun, L., Yang, Y.: Topological degree for Chern-Simons Higgs models on finite graphs. Calc. Var. Partial Differential Equations **63**, 81 (2024)
22. Lin, Y., Wu, Y.: The existence and nonexistence of global solutions for a semilinear heat equation on graphs. Calc. Var. Partial Differential Equations **56**, 102 (2017)
23. Lin, Y., Yang, Y.: A heat flow for the mean field equation on a finite graph. Calc. Var. Partial Differential Equations **60**, 206 (2021)
24. Liu, S., Yang, Y.: Multiple solutions of Kazdan-Warner equation on graphs in the negative case. Calc. Var. Partial Differential Equations **59**, 164 (2020)

25. Sun, L., Wang, L.: Brouwer degree for Kazdan-Warner equations on a connected finite graph. Adv. Math. **404**, 108422 (2022)
26. Wente, H.C.: Counterexample to a conjecture of H. Hopf. Pacific J. Math. **121**, 193–243 (1986)
27. Zhang, N., Zhao, L.: Convergence of ground state solutions for nonlinear Schrödinger equations on graphs. Sci China Math **61**, 1481–1494 (2018)
28. Zhu, X.: Mean field equations for the equilibrium turbulence and Toda systems on connected finite graphs. J. Partial Differ. Equ. **35**, 199–207 (2022)