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## Journal of Functional Analysis

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# Rigidity theorems of spacelike entire self-shrinking graphs in the pseudo-Euclidean space



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#### ARTICLE INFO

Article history: Received 10 April 2020 Accepted 1 July 2021 Available online 16 July 2021 Communicated by Natasa Sesum

MSC: 53C40 53C24

Keywords: Pseudo-distance Entire graph Self-shrinker Rigidity

#### ABSTRACT

In this paper, we firstly establish a new volume growth estimate for spacelike entire graphs in the pseudo-Euclidean space  $\mathbb{R}_n^{m+n}$ . Then by using this volume growth estimate and the Co-Area formula, we prove various rigidity results for spacelike entire self-shrinking graphs.

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## 1. Introduction

The pseudo-Euclidean space  $\mathbb{R}_n^{m+n}$  of index *n* is the linear space  $\mathbb{R}^{m+n}$  with coordinates  $(x_1, x_2, \ldots, x_{m+n})$  and indefinite metric

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$$ds^{2} = \sum_{i=1}^{m} (dx_{i})^{2} - \sum_{\alpha=m+1}^{m+n} (dx_{\alpha})^{2}.$$

For  $a = (a_1, \ldots, a_{m+n}) \in \mathbb{R}_n^{m+n}$  and  $b = (b_1, \ldots, b_{m+n}) \in \mathbb{R}_n^{m+n}$ , introduce

$$\langle a,b\rangle := \sum_{i=1}^{m} a_i b_i - \sum_{\alpha=m+1}^{m+n} a_\alpha b_\alpha, \quad |a|^2 := \langle a,a\rangle, \quad ||a|| := \sqrt{|\langle a,a\rangle|}.$$

An *m*-dimensional submanifold  $M^m$  in  $\mathbb{R}_n^{m+n}$  is called spacelike if the induced metric on  $M^m$  is a Riemannian metric. The mean curvature flow (MCF) in the pseudo-Euclidean space is a one-parameter family of immersions  $X_t = X(\cdot, t) : M^m \to \mathbb{R}_n^{m+n}$  with the corresponding image  $M_t = X_t(M^m)$  such that

$$\begin{cases} \frac{\partial}{\partial t} X(x,t) = H(x,t), & (x,t) \in M^m \times [0,T); \\ X(x,0) = X(x), & x \in M^m, \end{cases}$$
(1.1)

is satisfied, here H(x,t) is the mean curvature vector of  $M_t$  at X(x,t) in  $\mathbb{R}_n^{m+n}$ . There are many interesting and essential results on the mean curvature flow of spacelike submanifolds in certain Lorentzian manifolds (see e.g. [17-19,22,23,34]).

Let us firstly recall some facts in Euclidean spaces, Chern [12] showed that entire graphs of constant mean curvature (CMC) in  $\mathbb{R}^{m+1}$  are minimal. It is well known that these graphs must be hyperplanes for  $m \leq 7$  (see Bernstein [5] for m = 2, De Giorgi [14] for m = 3, Almgren [3] for m = 4 and Simons [31] for  $m \leq 7$ ) and there are counterexamples for m > 7 (see Bombieri-De Giorgi-Giusti [6]).

In Minkowski space  $\mathbb{R}_1^{m+1}$ , Calabi [7] proposed the Bernstein problem for spacelike maximal hypersurfaces and proved that such hypersurfaces have to be hyperplanes when  $m \leq 4$ . Cheng-Yau [11] solved the problem for all m, in sharp contrast to the situation of the Euclidean space. Later, Ishihara [26] and Jost-Xin [27] generalized the results to higher codimension. The rigidity problem for spacelike submanifolds with parallel mean curvature was studied in [27,33,35].

On the other hand, by the work of Colding-Minicozzi [13] (see also [4]), we know that, in the Euclidean space, minimal submanifolds and self-shrinkers share many geometric properties. Recall that  $M^m$  is said to be a self-shrinker in  $\mathbb{R}_n^{m+n}$  if

$$H = -\frac{1}{2}X^N,\tag{1.2}$$

which is an important class of solutions to (1.1), where  $X^N$  is the normal part of X. So it is natural to consider the rigidity of spacelike self-shrinkers in the pseudo-Euclidean space. Under the global conditions of Lagrangian entire graph or complete with the induced metric, there are plenty of related works, see e.g. [1,8,9,16,25,28]. It should mention that Chen-Qiu [10] proved that only the affine planes are the complete *m*dimensional spacelike self-shrinkers in the pseudo-Euclidean space  $\mathbb{R}_n^{m+n}$ . In this paper, we further study the geometry of the *m*-dimensional spacelike entire self-shrinking graphs in  $\mathbb{R}_n^{m+n}$ . By establishing a new volume growth estimate (see Theorem 3.1) for the spacelike entire graphs and the Co-Area formula (see Federer [20] for Lipschitz functions or Fleming-Rishel [21] for BV functions), we give various growth estimates on the mean curvature and the *w*-function when the spacelike self-shrinking graph is not a linear subspace, these lead to rigidity results if the growth conditions are not satisfied.

**Theorem 1.1.** Let  $X : M^m \to \mathbb{R}_n^{m+n}$  be a spacelike entire self-shrinking graph. Assume that the origin  $o \in M^m$  and  $M^m$  is not a linear subspace. Then the mean curvature satisfies

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^4 e^{-\frac{z}{4}}\right)} \le 4\sqrt{m},\tag{1.3}$$

where  $D_R := M^m \cap \{p \in \mathbb{R}^{m+n}_n : z(p) \leq R^2\}$  and  $z = |X|^2$  is the pseudo-distance.

In Theorem 1.1, we denote  $||H||^2 := -\langle H, H \rangle$ , which is nonnegative. We use the same notation for other timelike quantities.

**Remark 1.2.** Clearly, Theorem 1.1 implies a rigidity result for the spacelike entire selfshrinking graph if

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^4 e^{-\frac{z}{4}}\right)} > 4\sqrt{m}.$$

In particular, by the above Theorem 1.1 and Theorem 3.1 which is stated in section 3, if  $||H||^2 \leq Ce^{\alpha z}$  for some constant C > 0 and  $\alpha < \frac{1}{8}$ , then the spacelike entire self-shrinking graph has to be a linear subspace (see also Theorem 1.1 in [28] or Theorem 2.1 in [29]). Moreover, the growth condition can be weakened as  $||H||^2 \leq Ce^{\alpha z}$  for  $\alpha < \frac{1}{6} + \frac{1}{6\sqrt{m}}$ , see Corollary 5.2 in section 5.

**Theorem 1.3.** Let  $X : M^m \to \mathbb{R}_n^{m+n}$  be a spacelike entire self-shrinking graph. Assume that the origin  $o \in M^m$  and  $M^m$  is not a linear subspace. Then the w-function satisfies

$$\limsup_{R \to \infty} \frac{(\log R)^2}{\int_{D_R} w^2 (\log w)^2 e^{-\frac{z}{4}}} < \infty.$$
(1.4)

Here the definition of w-function is given in Section 2.

**Remark 1.4.** Ding-Wang [15] showed that the spacelike entire self-shrinking graph satisfying  $\lim_{|x|\to\infty} \frac{\log \det(g_{ij}(x))}{|x|} = 0$  is a linear subspace (see Theorem 3 in [15]). By using the above Theorem 1.3, we can improve their result, see the details in Remark 5.5 in section 5.

The article will be organized as follows. In the next section, we shall give some preliminaries. In Section 3, we establish a new volume growth estimate for spacelike entire self-shrinking graphs. Subsequently, in Section 4, we give the proof of Theorem 1.1 and Theorem 1.3. Finally, as applications, various rigidity results for the spacelike selfshrinkers are presented in Section 5.

## 2. Preliminaries

Let  $M^m$  be an *m*-dimensional spacelike submanifold in  $\mathbb{R}_n^{m+n}$ . Let  $\overline{\nabla}$  be the ambient connection on  $\mathbb{R}_n^{m+n}$ , and  $\nabla$  the induced connection on M. If there is no confusion, we also denote the normal connection on the normal bundle NM of M in  $\mathbb{R}_n^{m+n}$  by  $\nabla$ . The second fundamental form B of  $M^m$  in  $\mathbb{R}_n^{m+n}$  is defined by

$$B_{UW} := \left(\overline{\nabla}_U W\right)^N$$

for  $U, W \in \Gamma(TM^m)$ . We use the notation  $(\cdot)^T$  and  $(\cdot)^N$  for the orthogonal projections into the tangent bundle  $TM^m$  and the normal bundle  $NM^m$ , respectively. For  $\nu \in$  $\Gamma(NM^m)$  we define the shape operator  $A^{\nu}: TM^m \to TM^m$  by

$$A^{\nu}(U) := -\left(\overline{\nabla}_U \nu\right)^T$$
.

We have the following

$$\langle A^{\nu}(U), W \rangle = \langle A^{\nu}(W), U \rangle = \langle B_{UW}, \nu \rangle$$

Taking the trace of B gives the mean curvature vector H of  $M^m$  in  $\mathbb{R}_n^{m+n}$  and

$$H := \operatorname{trace}(B) = \sum_{i=1}^{m} B_{e_i e_i},$$

where  $\{e_i\}$  is a local orthonormal frame field of  $TM^m$ . The Gauss equation, Codazzi equation and Ricci equation are (cf. [32])

$$R_{ijkl} = \left\langle B_{e_ie_k}, B_{e_je_l} \right\rangle - \left\langle B_{e_ie_l}, B_{e_je_k} \right\rangle,$$
$$(\nabla_{e_i}B)_{e_je_k} = \left( \nabla_{e_j}B \right)_{e_ie_k},$$
$$R\left(e_i, e_j, \nu, \mu\right) = \left\langle A^{\nu}\left(e_i\right), A^{\mu}\left(e_j\right) \right\rangle - \left\langle A^{\nu}\left(e_j\right), A^{\mu}\left(e_i\right) \right\rangle$$

All spacelike *m*-planes (oriented *m*-subspaces) in  $\mathbb{R}_n^{m+n}$  form the pseudo-Grassmannian manifold  $G_{m,n}^n$ . It is a specific Cartan-Hadamard manifold which is the noncompact dual space of the Grassmannian manifold  $G_{m,n}$ .

Let  $P_1, P_2 \in G_{m,n}^n$  be two spacelike *m*-planes in  $\mathbb{R}_n^{m+n}$ . The angles between  $P_1$  and  $P_2$  are defined by the critical values of angle  $\theta$  between a nonzero vector x in  $P_1$  and its orthogonal projection  $x^*$  in  $P_2$  as x runs through  $P_1$ .

Assume that  $e_1, ..., e_m$  are oriented orthonormal vectors which span  $P_1$  and  $a_1, ..., a_m$  for  $P_2$ . For a nonzero vector in  $P_1$ ,

$$x = \sum_{i} x_i e_i,$$

its orthonormal projection in  $P_2$  is

$$x^* = \sum_i x_i^* a_i$$

Hence for any  $y \in P_2$ , we obtain

$$\langle x - x^*, y \rangle = 0.$$

Let  $W_{ij} := \langle e_i, a_j \rangle$  and  $W := (W_{ij})$ . Then we get

$$x_j^* = \sum_i W_{ij} x_i.$$

A direct computation yields

$$\langle x, x^* \rangle = |x^*|^2 = \sum_{i,j,k} x_i W_{ij} W_{kj} x_k.$$

Since  $WW^T$  is symmetric, so we can choose appropriate orthonormal vectors  $\{e_1, ..., e_m\}$ , such that  $WW^T = \text{diag}\{\mu_1^2, ..., \mu_m^2\}$  with  $\mu_i = \cosh \theta_i \ge 1$ . Hence

$$\langle x, x^* \rangle \ge |x| |x^*|.$$

The angle  $\theta$  between x and  $x^*$  is defined by

$$\cosh \theta = \frac{\langle x, x^* \rangle}{|x||x^*|}.$$

Fix  $P_2 \in G_{m,n}^n$ , which is spanned by a unit *m*-vector  $a_1 \wedge \cdots \wedge a_m$ . For any  $P \in G_{m,n}^n$ , spanned by an *m*-vector  $e_1 \wedge \cdots \wedge e_m$ , define the *w*-function on  $G_{m,n}^n$  as follows

$$w := \langle P, P_2 \rangle = \langle e_1 \wedge \dots \wedge e_m, a_1 \wedge \dots \wedge a_m \rangle = \det W.$$

Then, up to multiplying by -1, the w-function given by the spacelike m-plane P satisfies  $w \ge 1$  when restricted on  $M^m$ . Now we have

$$w = \prod_{i} \cosh \theta_{i} = \prod_{i} \frac{1}{\sqrt{1 - \lambda_{i}^{2}}}, \quad \lambda_{i} = \tanh \theta_{i}.$$

Choose timelike vectors  $a_{m+\alpha}$  such that  $\{a_i, a_{m+\alpha} | i = 1, ..., m; \alpha = 1, ..., n\}$  is an oriented orthonormal Lorentzian basis of  $\mathbb{R}_n^{m+n}$ . Then

$$\{e_i = \cosh \theta_i a_i + \sinh \theta_i a_{m+i} | i = 1, \dots, m\}$$

is an oriented tangent orthonormal basis of  $M^m$ , here  $\theta_i = 0$  for  $i > \min\{m, n\}$ .

## 3. Volume growth estimate

We derive the following volume growth estimate for the spacelike entire graphs in pseudo-Euclidean space  $\mathbb{R}_n^{m+n}$ .

**Theorem 3.1.** Let  $X : M^m \longrightarrow \mathbb{R}_n^{m+n}$  be an m-dimensional spacelike entire graph. Let  $z = \langle X, X \rangle$ . Assume that the origin  $o \in M^m$ , then

$$\limsup_{R \to \infty} R^{-2m} \int_{\{z \le R^2\} \cap M^m} w < \infty.$$
(3.1)

Consequently, for any  $\alpha > 0$ ,

$$\int_{M^m} w e^{-\alpha z} < \infty.$$

**Proof.** Since  $M^m$  is an entire graph,  $M^m$  can be written as  $\{X = (x, u(x)) | x \in \mathbb{R}^m, u = (u^1, u^2, ..., u^n)\}$ . By using the singular value decomposition (see [30]), by an action of  $SO(m) \times SO(n)$  we can choose a new Lorentzian coordinates  $\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$  on  $\mathbb{R}_n^{m+n}$  such that at a considered point to be calculated,

$$du\left(\frac{\partial}{\partial x_i}\right) = \lambda_i \frac{\partial}{\partial x_{m+i}}.$$

Here  $\lambda_i = 0$  for  $i > \min\{m, n\}$ . For simplicity, we denote  $E_i = \frac{\partial}{\partial x_i}, E_{m+\alpha} = \frac{\partial}{\partial x_{m+\alpha}},$  $(i = 1, ..., m, \text{ and } \alpha = 1, ..., n)$ . Since  $M^m$  is spacelike, we have  $|\lambda_i| < 1$ . Let

$$e_i := \frac{1}{\sqrt{1 - \lambda_i^2}} (E_i + \lambda_i E_{m+i}).$$

Define the w-function as

$$w = \langle e_1 \wedge \dots \wedge e_m, E_1 \wedge \dots \wedge E_m \rangle.$$

Then we derive

$$w = \prod_{i} \frac{1}{\sqrt{1 - \lambda_i^2}} = \frac{1}{\sqrt{\det(g_{ij})}},$$
(3.2)

where  $g_{ij} = \delta_{ij} - \sum_{\alpha} u_i^{\alpha} u_j^{\alpha}$  is the induced metric on  $M^m$ . Moreover,  $z = |x|_{\mathbb{R}^m}^2 - |u(x)|_{\mathbb{R}^n}^2$ . Since M is spacelike and u(0) = 0 we have |u(x)| < |x| (for  $|x| \neq 0$ ), thus for given  $\varepsilon > 0$ , there exists a constant  $\delta > 0$ , such that for each point  $(x, u(x)) \in M$  with  $|x| = \varepsilon$ , we get  $|u(x)| \le \delta < \varepsilon$ . Without loss of generality, assume  $x \neq 0$ , let  $\bar{x} = \frac{\varepsilon}{|x|}x$  and  $(\bar{x}, u(\bar{x})) \in M$ , then  $|\bar{x}| = \varepsilon$  and  $|u(\bar{x})| \le \delta$ . Again since M is spacelike, we obtain

$$|u(x) - u(\bar{x})| \le |x - \bar{x}|.$$

It follows that

$$\begin{aligned} |u(x)|^{2} &\leq |x|^{2} + |\bar{x}|^{2} - |u(\bar{x})|^{2} + 2\langle u(x), u(\bar{x}) \rangle - 2\langle x, \bar{x} \rangle \\ &\leq |x|^{2} + \varepsilon^{2} - |u(\bar{x})|^{2} + 2|u(x)||u(\bar{x})| - 2\varepsilon|x| \\ &\leq |x|^{2} - 2(\varepsilon - \delta)|x| + \varepsilon^{2}. \end{aligned}$$

This implies that

$$|u(x)| \le |x| + \varepsilon$$

and

$$|x| \le \frac{z+\varepsilon^2}{2(\varepsilon-\delta)} =: C_1(z+1), \tag{3.3}$$

where  $C_1$  is a positive constant depending only on  $\varepsilon$  and  $\delta$ . Direct computation gives us

$$|X^{T}|^{2} = \sum_{i} \langle X, e_{i} \rangle^{2} = \sum_{i} \frac{1}{1 - \lambda_{i}^{2}} \left( \langle X, E_{i} \rangle + \lambda_{i} \langle X, E_{m+i} \rangle \right)^{2}$$
$$\leq \sum_{i} \frac{2}{1 - \lambda_{i}^{2}} \left( \langle X, E_{i} \rangle^{2} + \langle X, E_{m+i} \rangle^{2} \right)$$
$$\leq 2w^{2} (|x|^{2} + |u(x)|^{2}).$$

Therefore we obtain

$$|X^{T}| \le C_2 w(z+1), \tag{3.4}$$

where  $C_2$  is a positive constant depending only on  $\varepsilon$  and  $\delta$ .

Since M is entire, by (3.3), z is proper. Hence, for every R > 0,

$$\int_{\{z \le R^2\} \cap M^m} w = \int_{\{x \in \mathbb{R}^m : |x|_{\mathbb{R}^m}^2 - |u(x)|_{\mathbb{R}^n}^2 \le R^2\}} w\sqrt{\det g} dx \le \int_{\{|x|_{\mathbb{R}^m} \le C_1(R^2 + 1)\}} dx$$
$$= C_1^m \left(R^2 + 1\right)^m \int_{\{|x|_{\mathbb{R}^m} \le 1\}} dx,$$

which gives the desired estimate (3.1).

Since  $|\nabla \sqrt{z}| = \frac{|X^T|}{\sqrt{z}} \ge 1$  whenever  $X \neq o$ , by the Co-Area formula and integration by parts, we obtain

$$\begin{split} \int_{M^m} w e^{-\alpha z} &= \int_0^\infty \left( \int_{\{\sqrt{z}=R\}\cap M^m} \frac{1}{|\nabla\sqrt{z}|} w e^{-\alpha z} \right) dR \\ &= \int_0^\infty e^{-\alpha R^2} d \int_{\{\sqrt{z} \le R\}\cap M^m} w \\ &= 2\alpha \int_0^\infty R e^{-\alpha R^2} \left( \int_{\{\sqrt{z} \le R\}\cap M^m} w \right) dR \\ &\leq 2\alpha \int_0^1 R e^{-\alpha R^2} \left( \int_{\{\sqrt{z} \le R\}\cap M^m} w \right) dR + C \int_1^\infty R e^{-\alpha R^2} R^{2m} dR \\ &< \infty. \quad \Box \end{split}$$

## 4. Proof of Theorem 1.1 and Theorem 1.3

Let  $V := -\frac{1}{2}X^T$  and  $\Delta_V := \Delta + \langle V, \nabla \cdot \rangle$  be the drift-Laplacian.

**Proof of Theorem 1.1.** Let  $\{e_1, ..., e_m\}$  be a local tangent orthonormal frame field on  $M^m$  such that  $\nabla_{e_i} e_j = 0$  at a considered point to be calculated. From the self-shrinker equation (1.2), we obtain

$$\nabla_{e_j} H = -\frac{1}{2} \nabla_{e_j} \left( X - \langle X, e_k \rangle e_k \right)^N = \frac{1}{2} \langle X, e_k \rangle B_{jk}$$
(4.1)

and

$$\nabla_{e_i} \nabla_{e_j} H = \frac{1}{2} B_{ij} - \langle H, B_{ik} \rangle B_{jk} + \frac{1}{2} \langle X, e_k \rangle \nabla_{e_i} B_{jk}.$$

Then using the Codazzi equation, we derive

$$\begin{split} \Delta_V |H|^2 =& \Delta |H|^2 + \left\langle V, \nabla |H|^2 \right\rangle \\ =& 2 \langle \nabla_{e_i} \nabla_{e_i} H, H \rangle + 2 |\nabla H|^2 + \left\langle V, \nabla |H|^2 \right\rangle \\ =& |H|^2 - 2 \langle H, B_{ik} \rangle^2 + \frac{1}{2} \nabla_{X^T} |H|^2 + 2 |\nabla H|^2 - \frac{1}{2} \left\langle X^T, \nabla |H|^2 \right\rangle \\ =& |H|^2 - 2 |A^H|^2 + 2 |\nabla H|^2. \end{split}$$

It follows that

$$\Delta_V \|H\|^2 = \|H\|^2 + 2|A^H|^2 + 2\|\nabla H\|^2 \ge 2|A^H|^2, \qquad (4.2)$$

where  $||H||^2$  is the absolute value of the square of the mean curvature vector H. By (4.1), we get

By (4.1), we get

$$\begin{aligned} \nabla_{X^T} \|H\|^2 &= -2 \left\langle \nabla_{X^T} H, H \right\rangle \\ &= -2 \left\langle X, e_j \right\rangle \left\langle \nabla_{e_j} H, H \right\rangle \\ &= -2 \left\langle X, e_j \right\rangle \left\langle \frac{1}{2} \left\langle X, e_k \right\rangle B_{jk}, H \right\rangle \\ &= - \left\langle B(X^T, X^T), H \right\rangle \\ &= - \left\langle A^H(X^T), X^T \right\rangle. \end{aligned}$$

Note that  $X = X^T + X^N$ , therefore we have

$$z = \langle X, X \rangle = |X^T|^2 + |X^N|^2 = |X^T|^2 - ||X^N||^2,$$

where  $||X^N||^2$  is the absolute value of the square of the timelike vector  $X^N$ . Then by the self-shrinker equation (1.2), we obtain

$$\left|X^{T}\right|^{2} = z + 4\|H\|^{2}.$$

Denote  $B_R := \{p \in \mathbb{R}_n^{m+n} : z(p) \leq R^2\}$  and  $D_R := M^m \cap B_R$ . By (3.3), z is proper, this implies that  $D_R$  is compact in  $M^m$ . Thus a direct computation yields

$$\int_{D_R} \Delta_V \|H\|^2 e^{-\frac{z}{4}} = \int_{D_R} e^{\frac{z}{4}} \operatorname{div} \left(e^{-\frac{z}{4}} \nabla \|H\|^2\right) e^{-\frac{z}{4}}$$
$$= \int_{D_R} \operatorname{div} \left(e^{-\frac{z}{4}} \nabla \|H\|^2\right)$$
$$= \int_{\partial D_R} \left\langle e^{-\frac{z}{4}} \nabla \|H\|^2, \frac{X^T}{|X^T|} \right\rangle$$

$$= \int_{\partial D_R} \frac{1}{|X^T|} \nabla_{X^T} ||H||^2 e^{-\frac{z}{4}}$$
  
=  $- \int_{\partial D_R} \frac{\langle A^H(X^T), X^T \rangle}{|X^T|} e^{-\frac{z}{4}}$   
 $\leq \int_{\partial D_R} |A^H| |X^T| e^{-\frac{z}{4}}$   
 $\leq \left( R \int_{\partial D_R} \frac{|A^H|^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( R^{-1} \int_{\partial D_R} \frac{(z+4||H||^2)^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}}.$ 

.

Namely

$$\int_{D_R} \Delta_V \|H\|^2 e^{-\frac{z}{4}} \le \left( R \int_{\partial D_R} \frac{|A^H|^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( R^{-1} \int_{\partial D_R} \frac{(z+4\|H\|^2)^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}}.$$
 (4.3)

The Cauchy inequality implies

$$\left|A^{H}\right|^{2} = \sum_{i,j} \left\langle B_{ij}, H \right\rangle^{2} \ge \sum_{i} \left\langle B_{ii}, H \right\rangle^{2} \ge \frac{\left(\sum_{i} \left\langle B_{ii}, H \right\rangle\right)^{2}}{m} = \frac{\|H\|^{4}}{m}.$$
 (4.4)

By the assumption that  $M^m$  is not a linear subspace, we can conclude that  $M^m$  is not maximal, i.e.,  $H \neq 0$ . Otherwise, by the proof of Theorem 4.2 in [27], we derive that M is a linear subspace, this yields the contradiction. Then there exists  $R_0 > 0$ , such that for any  $R > R_0$ ,

$$\int_{D_R} |A^H|^2 e^{-\frac{z}{4}} \ge \frac{1}{m} \int_{D_R} ||H||^4 e^{-\frac{z}{4}} > 0.$$

Let

$$F(R) := \int_{D_R} |A^H|^2 e^{-\frac{z}{4}}, \quad G(R) := \int_{D_R} (z+4||H||^2)^2 e^{-\frac{z}{4}}.$$

By the Co-Area formula, we have

$$\begin{split} F(R) &= \int_{0}^{R} \int_{\partial D_{r}} \frac{\left|A^{H}\right|^{2} e^{-\frac{z}{4}}}{\left|\nabla\sqrt{z}\right|} = \int_{0}^{R} \left(re^{-\frac{r^{2}}{4}} \int_{\partial D_{r}} \frac{\left|A^{H}\right|^{2}}{\left|X^{T}\right|}\right) dr, \\ G(R) &= \int_{0}^{R} \int_{\partial D_{r}} \frac{\left(z+4\|H\|^{2}\right)^{2} e^{-\frac{z}{4}}}{\left|\nabla\sqrt{z}\right|} = \int_{0}^{R} \left(re^{-\frac{r^{2}}{4}} \int_{\partial D_{r}} \frac{\left(z+4\|H\|^{2}\right)^{2}}{\left|X^{T}\right|}\right) dr. \end{split}$$

It follows that

$$F'(R) = Re^{-\frac{R^2}{4}} \int_{\partial D_R} \frac{|A^H|^2}{|X^T|}, \quad G'(R) = Re^{-\frac{R^2}{4}} \int_{\partial D_R} \frac{(z+4||H||^2)^2}{|X^T|}.$$

From (4.2) and (4.3), we obtain

$$4F(R)^{2} \le F'(R) \cdot R^{-2}G'(R).$$

Namely,

$$\frac{R^2}{G'(R)} \le \frac{F'(R)}{4F(R)^2} = -\frac{1}{4} \left(\frac{1}{F(R)}\right)', \quad \forall R > R_0$$

Therefore for any fixed r satisfying  $R > r > R_0$ ,

$$\frac{1}{4}(R^2 - r^2)^2 = \left(\int_r^R s ds\right)^2 \le \int_r^R \frac{s^2}{G'(s)} ds \cdot \int_r^R G'(s) ds$$
$$\le -\frac{1}{4} \left(\frac{1}{F(R)} - \frac{1}{F(r)}\right) \cdot \left(G(R) - G(r)\right),$$

which gives

$$(R^2 - r^2)^2 \le \frac{G(R)}{F(r)} \tag{4.5}$$

We claim that

$$\int_{M^m} |A^H|^2 e^{-\frac{z}{4}} = \infty.$$
(4.6)

In fact, let R go to infinity and then r go to infinity,

$$\limsup_{R \to \infty} \frac{R^4}{G(R)} \le \liminf_{r \to \infty} \frac{1}{F(r)} = \frac{1}{\int_{M^m} |A^H|^2 e^{-\frac{z}{4}}} \le \frac{m}{\int_{M^m} \|H\|^4 e^{-\frac{z}{4}}}$$

This implies that

$$\limsup_{R \to \infty} \frac{R^4}{\int_{D_R} 2z^2 e^{-\frac{z}{4}} + \int_{D_R} 32 \|H\|^4 e^{-\frac{z}{4}}} \le \limsup_{R \to \infty} \frac{R^4}{G(R)} \le \frac{m}{\int_{M^m} \|H\|^4 e^{-\frac{z}{4}}}.$$
 (4.7)

By Theorem 3.1,  $\int_{M^m} e^{\alpha z} < \infty$  for any  $\alpha < 0$ , so we can conclude that

$$\int_{D_R} z^2 e^{-\frac{z}{4}} \le \int_{M^m} z^2 e^{-\frac{z}{4}} < \infty.$$
(4.8)

Thus from the inequality (4.7), we get

$$\limsup_{R \to \infty} \frac{R^4}{\int_{D_R} 32 \|H\|^4 e^{-\frac{z}{4}}} \le \frac{m}{\int_{M^m} \|H\|^4 e^{-\frac{z}{4}}}.$$
(4.9)

If  $\int_{M^m} \|H\|^4 e^{-\frac{z}{4}} < \infty$ , then by (4.9), we conclude that

$$\infty > \frac{m}{\int_{M^m} \|H\|^4 e^{-\frac{z}{4}}} \ge \limsup_{R \to \infty} \frac{R^4}{\int_{D_R} 32 \|H\|^4 e^{-\frac{z}{4}}} \ge \limsup_{R \to \infty} \frac{R^4}{\int_{M^m} 32 \|H\|^4 e^{-\frac{z}{4}}} = \infty.$$

This yields the contradiction. Hence

$$\int_{M^m} \|H\|^4 e^{-\frac{z}{4}} = \infty.$$
(4.10)

Then (4.6) follows from (4.4).

From (4.2), (4.3) and (4.4), we get

$$2 \int_{D_{R}} |A^{H}|^{2} e^{-\frac{z}{4}} \leq \int_{D_{R}} \Delta_{V} ||H||^{2} e^{-\frac{z}{4}}$$

$$\leq \left( R \int_{\partial D_{R}} \frac{|A^{H}|^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( R^{-1} \int_{\partial D_{R}} \frac{(z + 4\sqrt{m} |A^{H}|)^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}}$$

$$\leq 4\sqrt{m} \int_{\partial D_{R}} \frac{\left(\frac{z}{4\sqrt{m}} + |A^{H}|\right)^{2}}{|X^{T}|} e^{-\frac{z}{4}}.$$
(4.11)

 $\operatorname{Set}$ 

$$\hat{F}(R) = \int_{D_R} \left( \left| A^H \right| + \frac{z}{4\sqrt{m}} \right)^2 e^{-\frac{z}{4}}.$$

 $\forall \varepsilon \in (0, \frac{1}{2})$ , for given positive constant  $\delta < \frac{\varepsilon}{1-\varepsilon}$ ,

$$\hat{F}(R) \leq (1+\delta) \int_{D_R} |A^H|^2 e^{-\frac{z}{4}} + \left(1 + \frac{1}{\delta}\right) \cdot \frac{1}{16m} \int_{D_R} z^2 e^{-\frac{z}{4}}$$

$$\leq (1+\delta) \int_{D_R} |A^H|^2 e^{-\frac{z}{4}} + \left(1 + \frac{1}{\delta}\right) \cdot \frac{1}{16m} \int_{M^m} z^2 e^{-\frac{z}{4}}.$$
(4.12)

Since  $\delta < \frac{\varepsilon}{1-\varepsilon}$ , we get  $\frac{1}{1-\varepsilon} - (1+\delta) > 0$ . By (4.6), we obtain

$$\lim_{R \to \infty} \int_{D_R} |A^H|^2 e^{-\frac{z}{4}} = \infty.$$

Note that  $\int_{M^m} z^2 e^{-\frac{z}{4}} < \infty$  by (4.8). Thus there exists  $R_1 > 0$ , such that when  $R > R_1$ , we have

$$\int_{D_R} |A^H|^2 e^{-\frac{z}{4}} > \frac{1}{\frac{1}{1-\varepsilon} - (1+\delta)} \left(1 + \frac{1}{\delta}\right) \cdot \frac{1}{16m} \int_{M^m} z^2 e^{-\frac{z}{4}}.$$
(4.13)

Combining (4.12) with (4.13), it follows

$$\hat{F}(R) < \frac{1}{1-\varepsilon} \int_{D_R} \left| A^H \right|^2 e^{-\frac{z}{4}}.$$
(4.14)

By the Co-Area formula, we obtain

$$\hat{F}(R) = \int_{D_R} \left( |A^H| + \frac{z}{4\sqrt{m}} \right)^2 e^{-\frac{z}{4}} = \int_0^R \left( r \int_{\partial D_r} \frac{\left( |A^H| + \frac{z}{4\sqrt{m}} \right)^2}{|X^T|} e^{-\frac{z}{4}} \right) dr.$$

This implies that

$$\hat{F}'(R) = R \int_{\partial D_R} \frac{\left( |A^H| + \frac{z}{4\sqrt{m}} \right)^2}{|X^T|} e^{-\frac{z}{4}}.$$
(4.15)

Thus from (4.11), (4.14) and (4.15), we get

$$0 < \frac{1-\varepsilon}{2\sqrt{m}}\hat{F}(R) \le \frac{1}{R}\hat{F}'(R),$$

which implies for some  $R_2 > R_1$ ,

$$\frac{1-\varepsilon}{4\sqrt{m}} \left( R^2 - R_2^2 \right) \le \log \frac{\hat{F}(R)}{\hat{F}(R_2)}, \quad \forall R > R_2.$$

Namely,

$$\int_{D_R} |A^H|^2 e^{-\frac{\varepsilon}{4}} > (1-\varepsilon) \frac{\hat{F}(R_2)}{\exp\left(\frac{1-\varepsilon}{4\sqrt{m}}R_2^2\right)} \cdot \exp\left(\frac{1-\varepsilon}{4\sqrt{m}}R^2\right).$$

Thus for R sufficiently large, we have

$$\int_{D_R} \left| A^H \right|^2 e^{-\frac{z}{4}} \ge \exp\left(\frac{1-2\varepsilon}{4\sqrt{m}}R^2\right).$$
(4.16)

According to (4.5) and (4.10), as the similar reason to derive (4.14), for sufficiently large R, we obtain

$$\left(R^2 - r^2\right)^2 \leq \frac{\int_{D_R} \left(z + 4\|H\|^2\right)^2 e^{-\frac{z}{4}}}{\int_{D_r} |A^H|^2 e^{-\frac{z}{4}}} \leq \frac{16(1+\varepsilon) \int_{D_R} \|H\|^4 e^{-\frac{z}{4}} + C_{\varepsilon}}{\int_{D_r} |A^H|^2 e^{-\frac{z}{4}}} \\ \leq \frac{16(1+2\varepsilon) \int_{D_R} \|H\|^4 e^{-\frac{z}{4}}}{\int_{D_r} |A^H|^2 e^{-\frac{z}{4}}}.$$

Choosing  $r = (1 - \varepsilon)R$ , by (4.16),

$$(2\varepsilon - \varepsilon^2) R^4 \le \frac{16(1+2\varepsilon) \int_{D_R} \|H\|^4 e^{-\frac{z}{4}}}{\int_{D_{(1-\varepsilon)R}} |A^H|^2 e^{-\frac{z}{4}}} \le \frac{16(1+2\varepsilon) \int_{D_R} \|H\|^4 e^{-\frac{z}{4}}}{\exp\left(\frac{1-2\varepsilon}{4\sqrt{m}}(1-\varepsilon)^2 R^2\right)}.$$

Direct computation gives us

$$\frac{1}{R^2} \left( \log(2\varepsilon - \varepsilon^2) + 4\log R \right) + \frac{(1 - 2\varepsilon)(1 - \varepsilon)^2}{4\sqrt{m}} \le \frac{1}{R^2} \log 16(1 + 2\varepsilon) + \frac{1}{R^2} \log \int_{D_R} \|H\|^4 e^{-\frac{\varepsilon}{4}}.$$

Letting  $R \to \infty$  in the above equality, we get

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^4 e^{-\frac{z}{4}}\right)} \le \frac{4\sqrt{m}}{\left(1 - 2\varepsilon\right)\left(1 - \varepsilon\right)^2}.$$

Let  $\varepsilon$  go to zero, we conclude that

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^4 e^{-\frac{z}{4}}\right)} \le 4\sqrt{m}. \quad \Box$$

Following the idea of the proof of Theorem 1.1, we give the

**Proof of Theorem 1.3.** Let  $B_R := \{p \in \mathbb{R}_n^{m+n} : z(p) \leq R^2\}$  and  $D_R := M^m \cap B_R$ . By (3.3), z is proper, thus  $D_R$  is compact in  $M^m$ . Integration by parts gives us

$$\int_{D_R} \Delta_V(\log w) \log w e^{-\frac{z}{4}} = \int_{D_R} \operatorname{div} \left( e^{-\frac{z}{4}} \nabla \log w \right) \log w$$
$$= \int_{D_R} \operatorname{div} \left( e^{-\frac{z}{4}} (\nabla \log w) \log w \right) - \int_{D_R} \langle e^{-\frac{z}{4}} \nabla \log w, \nabla \log w \rangle$$
$$= \int_{\partial D_R} \left\langle e^{-\frac{z}{4}} (\nabla \log w) \log w, \frac{X^T}{|X^T|} \right\rangle - \int_{D_R} |\nabla \log w|^2 e^{-\frac{z}{4}}.$$
(4.17)

By Proposition 3.1 in [28], we get

$$\Delta_V(\log w) \ge \frac{\|B\|^2}{w^2},$$
(4.18)

where  $||B||^2$  is the absolute value of the square of the second fundamental form.

From (4.17) and (4.18), we have

$$\int_{D_R} \frac{\|B\|^2}{w^2} \log w e^{-\frac{z}{4}} + \int_{D_R} |\nabla \log w|^2 e^{-\frac{z}{4}} \le \int_{\partial D_R} \left\langle e^{-\frac{z}{4}} (\nabla \log w) \log w, \frac{X^T}{|X^T|} \right\rangle.$$
(4.19)

Applying the Cauchy-Schwarz inequality to the right hand side of (4.19), and using (3.4), we get

$$\int_{\partial D_{R}} \left\langle e^{-\frac{z}{4}} (\nabla \log w) \log w, \frac{X^{T}}{|X^{T}|} \right\rangle \\
\leq \int_{\partial D_{R}} |\nabla \log w| \log w e^{-\frac{z}{4}} \\
\leq \left( \int_{\partial D_{R}} \frac{R |\nabla \log w|^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( \int_{\partial D_{R}} \frac{R^{-1} (\log w)^{2} |X^{T}|^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \\
\leq \left( \int_{\partial D_{R}} \frac{R |\nabla \log w|^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( \int_{\partial D_{R}} \frac{R^{-1} (\log w)^{2} 2C_{2}^{2} w^{2} (z+1)^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \\
\leq C_{3} \left( \int_{\partial D_{R}} \frac{R |\nabla \log w|^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( \int_{\partial D_{R}} \frac{R^{3} w^{2} (\log w)^{2}}{|X^{T}|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}}.$$
(4.20)

Combining (4.19) with (4.20), we have

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$$\int_{D_R} |\nabla \log w|^2 e^{-\frac{z}{4}} \le C_3 \left( \int_{\partial D_R} \frac{R |\nabla \log w|^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}} \left( \int_{\partial D_R} \frac{R^3 w^2 (\log w)^2}{|X^T|} e^{-\frac{z}{4}} \right)^{\frac{1}{2}}.$$
 (4.21)

Let

$$\widetilde{F}(R) := \int_{D_R} |\nabla \log w|^2 e^{-\frac{z}{4}}, \quad \widetilde{G}(R) := \int_{D_R} w^2 (\log w)^2 e^{-\frac{z}{4}}.$$

The Co-Area formula gives

$$\begin{split} \widetilde{F}(R) &= \int\limits_{0}^{R} \int\limits_{\partial D_{r}} \frac{|\nabla \log w|^{2} e^{-\frac{z}{4}}}{|\nabla \sqrt{z}|} = \int\limits_{0}^{R} \int\limits_{\partial D_{r}} \frac{r|\nabla \log w|^{2} e^{-\frac{z}{4}}}{|X^{T}|} dr, \\ \widetilde{G}(R) &= \int\limits_{0}^{R} \int\limits_{\partial D_{r}} \frac{w^{2} (\log w)^{2} e^{-\frac{z}{4}}}{|\nabla \sqrt{z}|} = \int\limits_{0}^{R} \int\limits_{\partial D_{r}} \frac{rw^{2} (\log w)^{2} e^{-\frac{z}{4}}}{|X^{T}|} dr. \end{split}$$

Then we get

$$\widetilde{F}'(R) = Re^{-\frac{R^2}{4}} \int_{\partial D_R} \frac{|\nabla \log w|^2}{|X^T|}, \quad \widetilde{G}'(R) = Re^{-\frac{R^2}{4}} \int_{\partial D_R} \frac{w^2 (\log w)^2}{|X^T|}.$$

By (4.21),

$$\widetilde{F}(R)^2 \le C_3^2 \widetilde{F}'(R) \cdot R^2 \widetilde{G}'(R).$$

That is

$$\frac{1}{R^2 \widetilde{G}'(R)} \le C_3^2 \frac{\widetilde{F}'(R)}{(\widetilde{F}(R))^2} = -C_3^2 \left(\frac{1}{\widetilde{F}(R)}\right)', \quad \forall R > 1.$$

Hence for any fixed  $r \in (1, R)$ , we derive

$$\left(\log\frac{R}{r}\right)^{2} = \left(\int_{r}^{R} \frac{1}{s} ds\right)^{2}$$
$$\leq \int_{r}^{R} \frac{1}{s^{2} \widetilde{G}'(s)} ds \cdot \int_{r}^{R} \widetilde{G}'(s) ds$$
$$\leq -C_{3}^{2} \left(\frac{1}{\widetilde{F}(R)} - \frac{1}{\widetilde{F}(r)}\right) \cdot (\widetilde{G}(R) - \widetilde{G}(r))$$

$$\leq C_3^2 \frac{\widetilde{G}(R)}{\widetilde{F}(r)}.$$

Let R go to infinity and then r go to infinity,

$$\limsup_{R \to \infty} \frac{(\log R)^2}{\widetilde{G}(R)} \le C_3^2 \liminf_{r \to \infty} \frac{1}{\widetilde{F}(r)} = \frac{C_3^2}{\int_{M^m} |\nabla \log w|^2 e^{-\frac{z}{4}}}$$

Since  $M^m$  is not a linear subspace, by (4.18) we know that w can not be a constant. Hence the right hand side of the above inequality is finite. It follows that

$$\limsup_{R\to\infty} \frac{(\log R)^2}{\int_{D_R} w^2 (\log w)^2 e^{-\frac{z}{4}}} < \infty. \quad \Box$$

#### 5. Rigidity results for spacelike self-shrinkers

In this section, we shall give various rigidity results for spacelike self-shrinkers which can be viewed as the applications of Theorem 1.1 and Theorem 1.3.

By Theorem 1.1, we have

**Corollary 5.1.** Let  $X: M^m \to \mathbb{R}^{m+n}_n$  be an *m*-dimensional spacelike entire self-shrinking graph. Assume that the origin  $o \in M^m$  and  $M^m$  is not a linear subspace. Then the mean curvature satisfies

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^3 w e^{-\frac{z}{4}}\right)} \le 4\sqrt{m}.$$
(5.1)

**Proof.** (3.4) implies that

$$||H||^{4} \le \frac{1}{2} ||H||^{3} |X^{T}| \le C_{2} ||H||^{3} w(z+1).$$
(5.2)

Then the conclusion follows from (1.3) and (5.2).  $\Box$ 

As a consequence of Corollary 5.1, we obtain

**Corollary 5.2.** Let  $X: M^m \to \mathbb{R}_n^{m+n}$  be an m-dimensional spacelike entire self-shrinking graph. Assume that the origin  $o \in M^m$ . If the mean curvature satisfies  $||H||^2 \leq Ce^{\alpha z}$  for any  $\alpha < \frac{1}{6} + \frac{1}{6\sqrt{m}}$ , here C is a positive constant. Then  $M^m$  must be a linear subspace.

**Proof.** Choose  $\alpha_0 < \frac{1}{6}$  such that  $\alpha < \alpha_0 + \frac{1}{6\sqrt{m}}$ . Suppose that  $M^m$  is not a linear subspace. The assumption implies that

$$\int_{D_R} \|H\|^3 w e^{-\frac{z}{4}} \le C^{3/2} e^{\frac{3(\alpha - \alpha_0)}{2}R^2} \int_{D_R} w e^{\left(\frac{3\alpha_0}{2} - \frac{1}{4}\right)z}$$

Since  $\alpha_0 < \frac{1}{6}$ , we get  $\frac{3\alpha_0}{2} - \frac{1}{4} < 0$ . Then by Theorem 3.1,  $\int_{M^m} w e^{\left(\frac{3\alpha_0}{2} - \frac{1}{4}\right)z} < \infty$ . Therefore

$$\limsup_{R \to \infty} \frac{R^2}{\log\left(\int_{D_R} \|H\|^3 w e^{-\frac{z}{4}}\right)} \ge \frac{2}{3(\alpha - \alpha_0)} > 4\sqrt{m}.$$

Comparing the above inequality with (5.1), we conclude that  $M^m$  is a linear subspace.  $\Box$ 

By using Theorem 1.3 and Theorem 3.1, we derive the following rigidity result.

**Corollary 5.3.** Let  $X: M^m \to \mathbb{R}^{m+n}_n$  be an *m*-dimensional spacelike entire self-shrinking graph. Assume that the origin  $o \in M^m$  and the w-function satisfies

$$\limsup_{x \to \infty} \frac{\log w}{z} < \frac{1}{4}.$$

Then  $M^m$  has to be a linear subspace.

**Proof.** Suppose that  $M^m$  is not a linear subspace. Let  $f(R) = \max_{\{z=R\}} w$ , since  $\Delta_V \log w \geq \frac{\|B\|^2}{w^2} > 0$ , then the maximum principle implies that f(R) is nondecreasing in R. If f is bounded by some positive constant, then by using Theorem 3.1 and the assumption, we have

$$\int_{M^m} w^2 (\log w)^2 e^{-\frac{z}{4}} < \infty.$$
 (5.3)

Otherwise,  $\lim_{R\to\infty} f(R) = \infty$ , then for any  $\epsilon > 0$ , we obtain  $\log f(R) \le f(R)^{\frac{\epsilon}{2}}$  when R is large. Therefore by the assumption, we can conclude that

$$f(R) < e^{\frac{1-\epsilon}{4}R}$$

for R large enough. It follows that

$$f(R)(\log f(R))^2 \le f(R)^{1+\epsilon} \le e^{\frac{1-\epsilon^2}{4}R}$$
, for  $R$  large.

Then by Theorem 3.1 again, we can also obtain (5.3). Since  $M^m$  is not a linear subspace, by (4.18), w can not be a constant, in particular,  $w \neq 1$ , that is,  $\int_{M^m} w^2 (\log w)^2 e^{-\frac{x}{4}} \neq 0$ . Therefore, we have

$$\limsup_{R \to \infty} \frac{(\log R)^2}{\int_{D_R} w^2 (\log w)^2 e^{-\frac{z}{4}}} = \infty.$$

This is a contradiction with (1.4).  $\Box$ 

By (3.2), Corollary 5.3 can be rewritten as

**Corollary 5.4.** Let  $M^m := \{(x, u(x)) | x \in \mathbb{R}^m, u = (u^1, u^2, ..., u^n)\}$  be an *m*-dimensional spacelike entire self-shrinking graph in  $\mathbb{R}_n^{m+n}$ . Assume the origin  $o \in M^m$  and the induced metric  $(g_{ij})$  satisfies

$$\liminf_{|x| \to \infty} \frac{\log \det(g_{ij}(x))}{|x|^2 - |u(x)|^2} > -\frac{1}{2},$$

where  $g_{ij}(x) = \delta_{ij} - \sum_{\alpha=1}^{n} u_i^{\alpha}(x) u_j^{\alpha}(x)$ . Then  $M^m$  is a linear subspace.

**Proof.** By (3.2), we have  $w = \frac{1}{\sqrt{\det(g_{ij})}}$ . Clearly,  $z = |x|^2 - |u(x)|^2$ . Thus the conclusion follows from Corollary 5.3.  $\Box$ 

**Remark 5.5.** Ding-Wang [15, Theorem 3] showed that a spacelike entire self-shrinking graph passing through the origin is a linear subspace under the condition that

$$\lim_{|x| \to \infty} \frac{\log \det(g_{ij}(x))}{|x|} = 0.$$
(5.4)

Since  $det(g_{ij}) < 1$ , Ding-Wang's assumption (5.4) implies that for any positive constant  $\varepsilon$ , we have

$$\frac{\log \det(g_{ij})}{|x|} > -\varepsilon, \quad \text{as } |x| \to \infty.$$

By (3.3), the function z is proper, choosing  $\varepsilon = \frac{1}{4C_1}$ , then we obtain

$$\frac{\log \det(g_{ij})}{z} \ge 2C_1 \frac{\log \det(g_{ij})}{|x|} > -2C_1 \varepsilon = -\frac{1}{2}, \quad \text{as } |x| \to \infty.$$

Then by Corollary 5.4, the spacelike entire self-shrinking graph is a linear subspace. Hence the condition in Corollary 5.4 is weaker than the one in [15, Theorem 3].

**Remark 5.6.** If m = 1, then the growth condition is not necessary. In other words, the only entire graphic spacelike self-shrinking curve through the origin in the pseudo-Euclidean space  $\mathbb{R}_n^{1+n}$  has to be a linear subspace. In fact, assume that  $M^1 = \{(t, u^1(t), \ldots, u^n(t)) : t \in \mathbb{R}\}$  is a spacelike self-shrinking curve, then

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$$\frac{u_{tt}^{\alpha}}{1-\sum_{\beta=1}^{n}u_{t}^{\beta}u_{t}^{\beta}} = \frac{1}{2}\left(tu_{t}^{\alpha}-u^{\alpha}\right), \quad \forall t \in \mathbb{R}, \quad \alpha = 1, \dots, n.$$
(5.5)

Since  $M^1$  contains the origin, we have  $u^1(0) = \cdots = u^n(0) = 0$ . Denote by  $u_t^{\alpha}(0) = a^{\alpha}, \alpha = 1, \ldots, n$ , then  $\{u^{\alpha}(t) = a^{\alpha}t, \alpha = 1, \ldots, n\}$  is a solution to (5.5) and  $M^1$  is a linear subspace. By the uniqueness theorem of ODE system, we know that  $M^1$  has to be a linear subspace.

**Remark 5.7.** If one shows that spacelike self-shrinkers are the Type I singularity models of MCFs in pseudo-Euclidean spaces, then by applying the blow up analysis of MCFs and using Corollary 5.2 or 5.3, one might prove that the MCF does not develop any Type I singularity under certain conditions.

Now we give a nontrivial spacelike entire self-shrinking graph which does not contain the origin (cf. [24]).

**Example 5.1.** Consider a  $C^2$  function  $u : \mathbb{R} \to \mathbb{R}$  satisfying

$$\frac{u''}{1-u'^2} = \frac{1}{2}(tu'-u), \quad |u'| < 1.$$
(5.6)

If we find a nontrivial solution u to (5.6), i.e., u is not a linear function, then

$$M^{m} = \left\{ (x_{1}, x_{2}, \dots, x_{m}, u(x_{1}), 0, \dots, 0) \in \mathbb{R}_{n}^{m+n} : (x_{1}, x_{2}, \dots, x_{m}) \in \mathbb{R}^{m} \right\}$$

is a nontrivial entire spacelike self-shrinking graph in  $\mathbb{R}_n^{m+n}$ , i.e., this graph is not an affine plane. According to Chen-Qiu's result ([10]), this entire graphic self-shrinker can not be complete.

Indeed, we consider the following ODE

$$\begin{cases} w'' = \frac{1}{2}(tw' - w)(1 - w'^2), \\ w(0) = a \in (-\infty, 0), \quad w'(0) = b \in (-1, 1). \end{cases}$$

Assume the maximal existence interval is  $(-T_1, T_2)$  with  $T_1, T_2 \in (0, \infty]$  such that  $|w'(t)| < 1, \forall t \in (-T_1, T_2)$ . Set  $\phi = tw' - w$ , then

$$\phi' = \frac{t}{2} \left( 1 - w'^2 \right) \phi$$

Consider a function  $f' = \frac{t}{2} (1 - w'^2)$ , f(0) = 0, then  $f \ge 0$  and  $e^{-f}\phi$  is a constant. In particular  $\phi \ge -a$ . Thus

$$(\tanh^{-1} w')' = \frac{1}{2} (tw' - w) \ge -\frac{a}{2}.$$

For  $t \in [0, T_2)$ , we have

$$w' \ge \tanh\left(-\frac{at}{2} + \tanh^{-1}b\right) = -\tanh\left(\frac{at}{2} - \tanh^{-1}b\right),\tag{5.7}$$

which implies

$$w \ge -\frac{2}{a}\log\cosh\left(-\frac{at}{2} + \tanh^{-1}b\right) + \frac{2}{a}\log\cosh\left(\tanh^{-1}b\right) + a$$
$$\ge -\frac{2}{a}\left(-\frac{at}{2} + \tanh^{-1}b - \log 2\right) + \frac{2}{a}\log\cosh\left(\tanh^{-1}b\right) + a$$
$$\ge t + \frac{2}{a}\left(\log 2 + 2\tanh^{-1}|b|\right) + a.$$

Hence  $\phi \leq -a - \frac{2}{a} \left( \log 2 + 2 \tanh^{-1} |b| \right)$ . Thus

$$(\tanh^{-1} w')' \le -\frac{a}{2} - \frac{1}{a} (\log 2 + 2 \tanh^{-1} |b|),$$

which implies

$$w' \le \tanh\left(\left(-\frac{a}{2} - \frac{1}{a}\left(\log 2 + 2\tanh^{-1}|b|\right)\right)t + \tanh^{-1}b\right).$$
 (5.8)

Then (5.7) and (5.8) implies that  $T_2 = +\infty$ .

For  $t \in (-T_1, 0)$ , a similar argument gives

$$-\tanh\left(\left(\frac{a}{2}+\frac{1}{a}\left(\log 2+2\tanh^{-1}|b|\right)\right)t-\tanh^{-1}b\right)\leq w'\leq \tanh\left(-\frac{at}{2}+\tanh^{-1}b\right),$$

which implies that  $T_1 = -\infty$ .

The above example implies that for  $|t| > -\frac{a}{2}$ ,

$$\frac{\log(1-w'^2)}{t^2-w^2} \ge \frac{\log(1-w'^2)}{-2a|t|-a^2} \ge \frac{2\log\cosh\left(-\left(\frac{a}{2}+\frac{1}{a}\left(\log 2+2\tanh^{-1}|b|\right)\right)|t|+\tanh^{-1}|b|\right)}{2a|t|+a^2},$$

and we get

$$\liminf_{|t| \to \infty} \frac{\log (1 - w'^2)}{t^2 - w^2} \ge -\frac{1}{2} - \frac{(\log 2 + 2 \tanh^{-1} |b|)}{a^2} \to -\frac{1}{2}, \quad \text{as } a \to -\infty.$$

We also have

$$a + \frac{2}{a} \left( \log 2 + 2 \tanh^{-1} |b| \right) \le \liminf_{|t| \to \infty} \frac{\log \left(1 - w'^2\right)}{|t|}$$
$$\le \limsup_{|t| \to \infty} \frac{\log \left(1 - w'^2\right)}{|t|} \le a \to -\infty, \quad \text{as } a \to -\infty.$$

Motivated by Corollary 5.4 and the above example, we would like to propose the following

**Conjecture 1.** Let  $u = (u^1, u^2, ..., u^n)$  be an entire smooth solution to

$$\sum_{i,j=1}^{m} g^{ij}(x) u_{ij}^{\alpha}(x) = \frac{1}{2} \left( \sum_{i=1}^{m} x_i u_i^{\alpha}(x) - u^{\alpha}(x) \right), \quad x \in \mathbb{R}^m, \quad \alpha = 1, \dots, n,$$

where  $g_{ij}(x) = \delta_{ij} - \sum_{\alpha=1}^{n} u_i^{\alpha}(x) u_j^{\alpha}(x)$  and  $(g^{ij}(x))_{1 \le i,j \le m}$  is the inverse matrix of  $(g_{ij}(x))_{1 \le i,j \le m}$ . Assume  $u^1(0) = \cdots = u^m(0) = 0$  and

$$\liminf_{|x| \to \infty} \frac{\log \det(g_{ij}(x))}{|x|^2 - |u(x)|^2} \ge -\frac{1}{2},$$

then  $u^{\alpha}(x)$  are linear functions for each  $\alpha = 1, \ldots, n$ .

**Remark 5.8.** (1) As it was stated in the above conjecture, the optimal situation is expected that the lower bound  $-\frac{1}{2}$  can be achieved.

(2) A complete spacelike submanifold  $M^m$  in  $\mathbb{R}_n^{m+n}$  is an entire graph (cf. [2]), but the converse claim is not always the case. Chen-Qiu [10] showed that any complete spacelike self-shrinker  $M^m$  in  $\mathbb{R}_n^{m+n}$  is an affine plane, thus the entire spacelike self-shrinking graph  $M^m$  in Example 5.1 is not complete.

(3) From the computation in Example 5.1, we can see that Ding-Wang's assumption (5.4) is not satisfied.

#### Acknowledgment

This research is partially supported by the National Natural Science Foundation of China (Nos. 11771339, 11801420, 11971358), Fundamental Research Funds for the Central Universities (No. 2042019kf0198) and the Youth Talent Training Program of Wuhan University. The authors would like to thank Professor Y. L. Xin for his valuable suggestions and constant support. The first author also would like to express his gratitude to Professor Tobias H. Colding for his invitation to MIT and their hospitality. The second author thanks the Max Planck Institute for Mathematics in the Sciences for good working conditions when this work was carried out. The authors thank the referee for valuable suggestions which improved the paper.

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