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Calculus of Variations

Topological degree for Chern–Simons Higgs models on finite graphs

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Abstract

Let (V, E) be a finite connected graph. We are concerned about the Chern–Simons Higgs model

$$\Delta u = \lambda e^u(e^u - 1) + f, \tag{0.1}$$

where Δ is the graph Laplacian, λ is a real number and f is a function on V. When $\lambda > 0$ and $f = 4\pi \sum_{i=1}^{N} \delta_{p_i}$, $N \in \mathbb{N}$, $p_1, \cdots, p_N \in V$, the equation (0.1) was investigated by Huang et al. (Commun Math Phys 377:613–621, 2020) and Hou and Sun (Calc Var 61:139, 2022) via the upper and lower solutions principle. We now consider an arbitrary real number λ and a general function f, whose integral mean is denoted by $\overline{f}$, and prove that when $\lambda\overline{f} < 0$, the equation (0.1) has a solution; when $\lambda\overline{f} > 0$, there exist two critical numbers $\Lambda^* > 0$ and $\Lambda_* < 0$ such that if $\lambda \in (\Lambda^*, +\infty) \cup (-\infty, \Lambda_*)$, then (0.1) has at least two solutions, including one local minimum solution; if $\lambda \in (0, \Lambda^*) \cup (\Lambda_*, 0)$, then (0.1) has no solution; while if $\lambda = \Lambda^*$ or Λ_*, then (0.1) has at least one solution. Our method is calculating the topological degree and using the relation between the degree and the critical group of a related functional. Similar method is also applied to the Chern–Simons Higgs system, and a partial result for the multiple solutions of the system is obtained.

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1 Introduction

The Chern–Simons Higgs model, introduced by Hong, Kim, Pac [19] and Jackiw, Weinberg [27], has always attracted the attention of many mathematicians in the fields of geometry and physics, see for examples [2, 3, 9, 10, 31, 40, 41, 43, 46]. Among many versions, the self-dual Chern–Simons Higgs vortex equation on a flat 2-torus Σ can be written as

$$\Delta u = \frac{4}{k^2} e^u (e^u - 1) + 4\pi \sum_{i=1}^{k_0} m_i \delta_{p_i}, \tag{1}$$

where $k > 0$ is the Chern–Simons constant, $m_i \in \mathbb{N}$, $p_i \in \Sigma$, $i = 1, \cdots, k_0$. The solution of the above equation is called a vertex solution, each p_i is called a vertex point, and m_i stands for the multiplicity of p_i. From the view of physics, the vortex points are closely related to the local maximum point of the magnetic flux in the Chern–Simons Higgs model. Let u_0 be a solution of

$$\begin{cases} \Delta u_0 = -\frac{4\pi N}{|\Sigma|} + 4\pi \sum_{i=1}^{k_0} m_i \delta_{p_i} \\ \int_\Sigma u_0 dv_g = 0, \end{cases}$$

where $N = \sum_{i=1}^{k_0} m_i$. Set $v = u - u_0$. Then (1) can be written in a more favourable form

$$\Delta v = \lambda h e^v (h e^v - 1) + \frac{4\pi N}{|\Sigma|}, \tag{2}$$

where $\lambda = \frac{4}{k^2}$ and $h = e^{u_0}$ is a positive function on Σ. A solution v of (2) is called of finite energy if $v \in W^{1,2}(\Sigma)$, a usual Sobolev space. Indeed, it is known that the corresponding physical energy of the solution v is finite if $u \in W^{1,2}(\Sigma)$. Thus, solutions of finite energy are physically meaningful in (2) and there have been many existence results for $W^{1,2}(\Sigma)$ solutions of (2), see [2, 8, 9, 40, 41, 43–45] and the references therein. By using the principle of upper and lower solutions, Caffarelli and Yang constructed a maximal solution. In addition to the above references, [10, 29] also indicated that the Eq. (2) admits a variational structure.

Different from the theoretical significance on Riemann surfaces, the analysis on graphs is very important for applications, such as image processing, data mining, network and so on. Among lots of directions, partial differential equations arising in geometry or physics are worth studying on graphs. Various equations, including the heat equation [20, 26, 32, 33], the Fokker-Planck and Schrödinger equations [6, 7], have been studied by many mathematicians. In particular, Grigor'yan, Lin and Yang [13–15] studied the existence of solutions for a series of nonlinear elliptic equations on graphs by using the variational methods. In this direction, Zhang, Zhao, Han and Shao [17, 18, 49] obtained nontrivial solutions to nonlinear Schrödinger equations with potential wells. Similar problems on infinite metric graphs were studied by Akduman-Pankov [1]. The Kazdan-Warner equation was extended by Keller-Schwarz [28] to canonically compactifiable graphs. Semi-linear heat equations on locally finite graphs were studied by Ge, Jiang, Lin and Wu [12, 32, 33]. For other related works, we refer the readers to [11, 16, 22, 23, 34–36, 38, 39, 47, 48, 50] and the references therein.

To describe the Chern–Simons Higgs model in the graph setting, we introduce some notations. Let (V, E) be a connected finite graph, where V is the set of vertices and E is the set of edges. Let $\mu : V \to (0, +\infty)$ and $\{w_{xy} : xy \in E\}$ be its measure and weights respectively. The weight w_{xy} is always assumed to be positive and symmetric. The Laplacian of a function $u : V \to \mathbb{R}$ reads as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)),$$

where $y \sim x$ means y is adjacent to x, i.e. $xy \in E$. The gradient of u is defined as

$$\nabla u(x) = \left(\sqrt{\frac{w_{xy_1}}{2\mu(x)}}(u(y_1) - u(x)), \cdots, \sqrt{\frac{w_{xy_{\ell_x}}}{2\mu(x)}}(u(y_{\ell_x}) - u(x)) \right),$$

where $\{y_1, \cdots, y_{\ell_x}\}$ are all distinct points adjacent to x. Clearly, such an ℓ_x is unique and $\nabla u(x) \in \mathbb{R}^{\ell_x}$. The integral of u is given by

$$\int_V u d\mu = \sum_{x \in V} \mu(x) u(x).$$

Now we consider an analog of (2) on a connected finite graph, namely

$$\Delta u = \lambda e^u (e^u - 1) + f \quad \text{in} \quad V, \tag{3}$$

where $\lambda \in \mathbb{R}$, $f : V \to \mathbb{R}$ is a function. It was proved by Huang, Lin and Yau [24] that if $\lambda > 0$ and $f = 4\pi \sum_{i=1}^N \delta_{p_i}$, there exists a critical number $\lambda^* > 0$ such that (3) has a solution when $\lambda > \lambda^*$, while (3) has no solution when $0 < \lambda < \lambda^*$. The critical case $\lambda = \lambda^*$ was solved by Hou and Sun [21], who proved that (3) has also a solution. Such results are essentially based on the method of upper and lower solutions principle. This together with variational method may lead to existence results for other forms of Chern–Simons Higgs models, see Chao and Hou [5]. Recently, a more delicate analysis was employed by Huang, Wang and Yang [25] to get existence of solutions of the Chern–Simons Higgs system.

Topological degree theory is a powerful tool in studying partial differential equations in the Euclidean space or Riemann surfaces, see for example Li [30]. It was first used by Sun and Wang [42] to solve the Kazdan-Warner equation on finite graphs. Very recently, it was also employed by Liu [37] to deal with the mean field equation. Our aim is to use this powerful tool to study the Chern–Simons Higgs model. The first and most important step is to get a priori estimate for solutions, say

Theorem 1 *Let (V, E) be a connected finite graph with symmetric weights, i.e. $w_{xy} = w_{yx}$ for all $xy \in E$. Let $\sigma \in [0, 1]$, λ and f satisfy*

$$\Lambda^{-1} \le |\lambda| \le \Lambda, \quad \Lambda^{-1} \le \left| \int_V f d\mu \right| \le \Lambda, \quad \|f\|_{L^\infty(V)} \le \Lambda \tag{4}$$

for some real number $\Lambda > 0$. If u is a solution of

$$\Delta u = \lambda e^u (e^u - \sigma) + f \quad \text{in} \quad V, \tag{5}$$

then there exists a constant C, depending only on Λ and the graph V, such that $|u(x)| \le C$ for all $x \in V$.

When $\sigma = 1$, the Eq. (5) is exactly (3). In the case $\lambda > 0$ and $f = 4\pi \sum_{i=1}^N \delta_{p_i}$, where $p_1, \cdots, p_N \in V$ and $N \in \mathbb{N}$, let λ^* be the critical number in [24]. Then for any $\lambda_k > \lambda^*$ with $\lambda_k \to \lambda^*$ as $k \to \infty$, there exists a solution u_{λ_k} of (3) with $\lambda = \lambda_k$, $k = 1, 2, \cdots$. It follows from Theorem 1 that (u_{λ_k}) is uniformly bounded in V. Hence up to a subsequence, (u_{λ_k}) uniformly converges to some u^*, which is a solution of (3) with $\lambda = \lambda^*$. This gives another proof of a result of Hou and Sun [21].

Denote $X = L^\infty(V)$ and define a map $F : X \to X$ by

$$F(u) = -\Delta u + \lambda e^u(e^u - 1) + f. \tag{6}$$

The second step is to calculate the topological degree of F by using its homotopic invariance property.

Theorem 2 *Let (V, E) be a connected finite graph with symmetric weights, and $F : X \to X$ be a map defined by (6). Suppose that $\lambda \int_V f d\mu \neq 0$. Then there exists a large number $R_0 > 0$ such that for all $R \geq R_0$,*

$$\deg(F, B_R, 0) = \begin{cases} 1 & \text{if } \lambda > 0,\ \int_V f d\mu < 0 \\ 0 & \text{if } \lambda \int_V f d\mu > 0 \\ -1 & \text{if } \lambda < 0,\ \int_V f d\mu > 0, \end{cases}$$

where $B_R = \{u \in X : \|u\|_{L^\infty(V)} < R\}$ is a ball in X.

As an application of the above topological degree, our existence results for the Chern–Simons Higgs model read as follows:

Theorem 3 *Let (V, E) be a connected finite graph with symmetric weights. Then we have the following:*

(a) *If $\lambda \int_V f d\mu < 0$, then the Eq. (3) has a solution;*
(b) *If $\lambda \int_V f d\mu > 0$, then two subcases are distinguished: (i) $\int_V f d\mu > 0$. There exists a real number $\Lambda^* > 0$ such that when $\lambda > \Lambda^*$, (3) has at least two different solutions; when $0 < \lambda < \Lambda^*$, (3) has no solution; when $\lambda = \Lambda^*$, (3) has at least one solution; (ii) $\int_V f d\mu < 0$. There exists a real number $\Lambda_* < 0$ such that when $\lambda < \Lambda_*$, (3) has at least two different solutions; when $\Lambda_* < \lambda < 0$, (3) has no solution; when $\lambda = \Lambda_*$, (3) has at least one solution.*

We remark that Case (b) (i) includes $\lambda > 0$ and $f = 4\pi \sum_{i=1}^N \delta_{p_i}$ as a special case, which was studied in [5, 21, 24, 25]. In the subcase $\lambda > \Lambda^* > 0$ or $\lambda < \Lambda_* < 0$, we shall construct a local minimum solution, and then use the topological degree to obtain the existence of another solution. Our arguments are essentially different from those in [5, 25, 36]. Note that a solution of (3) is a critical point of the functional $J_\lambda : X \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_V |\nabla u|^2 d\mu + \frac{\lambda}{2}\int_V (e^u - 1)^2 d\mu + \int_V f u d\mu. \tag{7}$$

Here a local minimum solution of (3) means a local minimum critical point of J_λ.

Also we consider the Chern–Simons Higgs system

$$\begin{cases} \Delta u = \lambda e^v(e^u - 1) + f \\ \Delta v = \lambda e^u(e^v - 1) + g, \end{cases} \tag{8}$$

where λ is a real number, and f, g are functions on V. Similar to the single equation, we need also a priori estimate.

Theorem 4 *Let (V, E) be a connected finite graph with symmetric weights. Suppose that $\sigma \in [0, 1]$, λ, η are two positive real numbers, f, g are two functions verifying that $\int_V f d\mu > 0$ and $\int_V g d\mu > 0$. If (u, v) is a solution of the system*

$$\begin{cases} \Delta u = \lambda e^v(e^u - \sigma) + f \\ \Delta v = \eta e^u(e^v - \sigma) + g, \end{cases} \tag{9}$$

then there exists a constant C, depending only on λ, η, f, g and the graph V, such that

$$\|u\|_{L^\infty(V)} + \|v\|_{L^\infty(V)} \le C.$$

To compute the topological degree, we define a map $\mathcal{F} : X \times X \to X \times X$ by

$$\mathcal{F}(u, v) = (-\Delta u + \lambda e^v(e^u - 1) + f, -\Delta v + \eta e^u(e^v - 1) + g). \tag{10}$$

Theorem 5 *Let (V, E) be a connected finite graph with symmetric weights, and $\mathcal{F}$ be a map defined by (10). If $\lambda > 0$, $\eta > 0$, $\int_V f d\mu > 0$ and $\int_V g d\mu > 0$, then there exists a large number $R_0 > 0$ such that for all $R \ge R_0$,*

$$\deg(\mathcal{F}, B_R, (0, 0)) = 0,$$

where $B_R = \{(u, v) \in X \times X : \|u\|_{L^\infty(V)} + \|v\|_{L^\infty(V)} < R\}$ is a ball in $X \times X$.

Define a functional $\mathcal{J}_\lambda : X \times X \to \mathbb{R}$ by

$$\mathcal{J}_\lambda(u, v) = \int_V \nabla u \nabla v d\mu + \lambda \int_V (e^u - 1)(e^v - 1) d\mu + \int_V (f v + g u) d\mu. \tag{11}$$

Note that for all $(\phi, \psi) \in X \times X$,

$$\begin{aligned} \langle \mathcal{J}_\lambda'(u, v), (\phi, \psi) \rangle &= \frac{d}{dt}\Big|_{t=0} \mathcal{J}(u + t\phi, v + t\psi) \\ &= \int_V \left\{ \left(-\Delta v + \lambda e^u(e^v - 1) + g\right)\phi + \left(-\Delta u + \lambda e^v(e^u - 1) + f\right)\psi \right\} d\mu. \end{aligned} \tag{12}$$

Clearly (u, v) is a critical point of $\mathcal{J}_\lambda$ if and only if it is a solution of the system (8). As a consequence of Theorem 5, we have the following

Theorem 6 *Let (V, E) be a connected finite graph with symmetric weights, $\lambda > 0$, $\int_V f d\mu > 0$, $\int_V g d\mu > 0$, and $\mathcal{J}_\lambda$ be a functional defined by (11). If either $\mathcal{J}_\lambda$ has a non-degenerate critical point, or $\mathcal{J}_\lambda$ has a local minimum critical point, then it must have another critical point.*

It should be remarked that Theorem 6 gives another solution of (8) under the condition that $\mathcal{J}_\lambda$ has a non-degenerate or a local minimum critical point beforehand. So it is only a partial result for the problem of multiple solutions of the system (8).

The remaining part of this paper is organized as follows: In Sect. 2, we give a priori estimate for solutions of (3) (Theorem 1); The topological degree of $F : X \to X$ (Theorem 2) was calculated in Sect. 3; In Sect. 4, we prove the existence result (Theorem 3); The priori estimate and existence of solutions of the Chern–Simons Higgs system (Theorems 4, 5 and 6) are discussed in Sect. 5.

2 A priori estimate

In this section, we shall prove Theorem 1. In order to provide readers with a clear understanding of the proof, we demonstrate the entire process from simple cases to complex cases. Precisely the proof will be divided into several lemmas as below.

The first priori estimate is for fixed λ and f.

Lemma 7 *Suppose that u is a solution of (3), where $\lambda \neq 0$ and $\int_V f d\mu \neq 0$. Then there exists a constant C, depending only on λ, f and the graph V, such that $|u(x)| \leq C$ for all $x \in V$.*

Proof If u is a solution of (3), then integration by parts gives

$$0 = \int_V \Delta u d\mu = \lambda \int_V e^u(e^u - 1)d\mu + \int_V f d\mu. \tag{13}$$

Firstly, we show that u has a uniform upper bound. With no loss of generality, we may assume $\max_V u > 0$. For otherwise, u has already upper bound 0. Observing

$$\left|\int_{u<0} e^u(e^u - 1)d\mu\right| \leq \frac{1}{4}|V|,$$

we derive from (13) that

$$\int_{u\geq 0} e^u(e^u - 1)d\mu \leq a := \frac{1}{4}|V| + \frac{1}{|\lambda|}\left|\int_V f d\mu\right|.$$

This together with the fact

$$\int_{u\geq 0} e^u(e^u - 1)d\mu = \sum_{x\in V,\, u(x)\geq 0} \mu(x)e^{u(x)}(e^{u(x)} - 1) \geq \mu_0 e^{\max_V u}(e^{\max_V u} - 1)$$

leads to

$$\max_V u \leq \log \frac{1 + \sqrt{1 + 4a/\mu_0}}{2}, \tag{14}$$

where $\mu_0 = \min_{x\in V} \mu(x) > 0$, since V is finite.

Secondly, we prove that u has also a uniform lower bound. To see this, in view of (3) and (14), we calculate for any $x \in V$,

$$\begin{aligned} |\Delta u(x)| &\leq |\lambda| \left|e^{u(x)}(e^{u(x)} - 1)\right| + |f(x)| \\ &\leq |\lambda|(e^{2u(x)} + e^{u(x)}) + |f(x)| \\ &\leq |\lambda| \left(\frac{(1 + \sqrt{1 + 4a/\mu_0})^2}{4} + \frac{1 + \sqrt{1 + 4a/\mu_0}}{2}\right) + \|f\|_{L^\infty(V)} \\ &=: b. \end{aligned}$$

Hence, there holds

$$\|\Delta u\|_{L^\infty(V)} \leq b. \tag{15}$$

We may assume $V=\{x_1,\cdots,x_\ell\}$, $u(x_1)=\max_V u$, $u(x_\ell)=\min_V u$, and without loss of generality $x_1x_2, x_2x_3, \cdots, x_{\ell-1}x_\ell$ is the shortest path connecting x_1 and x_ℓ. It follows that

$$\begin{aligned} 0\leq u(x_1)-u(x_\ell) &\leq \sum_{j=1}^{\ell-1}|u(x_j)-u(x_{j+1})| \\ &\leq \frac{\sqrt{\ell-1}}{\sqrt{w_0}}\left(\sum_{j=1}^{\ell-1} w_{x_jx_{j+1}}(u(x_j)-u(x_{j+1}))^2\right)^{1/2} \\ &\leq \frac{\sqrt{\ell-1}}{\sqrt{w_0}}\left(\int_V |\nabla u|^2 d\mu\right)^{1/2}, \end{aligned} \tag{16}$$

where $w_0=\min_{x\in V,\, y\sim x} w_{xy}>0$. Denoting $\overline{u}=\frac{1}{|V|}\int_V u d\mu$, we obtain by integration by parts

$$\begin{aligned} \int_V |\nabla u|^2 d\mu &= -\int_V (u-\overline{u})\Delta u d\mu \\ &\leq \left(\int_V (u-\overline{u})^2 d\mu\right)^{1/2}\left(\int_V (\Delta u)^2 d\mu\right)^{1/2} \\ &\leq \left(\frac{1}{\lambda_1}\int_V |\nabla u|^2 d\mu\right)^{1/2}\left(\int_V (\Delta u)^2 d\mu\right)^{1/2}, \end{aligned}$$

which gives

$$\int_V |\nabla u|^2 d\mu \leq \frac{1}{\lambda_1}\int_V (\Delta u)^2 d\mu \leq \frac{1}{\lambda_1}\|\Delta u\|^2_{L^\infty(V)}|V|, \tag{17}$$

where $\lambda_1=\inf_{\overline{v}=0,\int_V v^2 d\mu=1}\int_V |\nabla v|^2 d\mu>0$. Combining (16) and (17), we conclude

$$\max_V u-\min_V u\leq\sqrt{\frac{(\ell-1)|V|}{w_0\lambda_1}}\|\Delta u\|_{L^\infty(V)}. \tag{18}$$

We remark that (18) holds for arbitrary function u, such an inequality was obtained by Sun and Wang [42] by using the equivalence of all norms in a finite dimensional vector space, and here we give an explicit constant instead of C. The power of (18) is evident. In view of (15), we have

$$\max_V u-\min_V u\leq c_0:=b\sqrt{\frac{(\ell-1)|V|}{w_0\lambda_1}}. \tag{19}$$

Coming back to (13), we have

$$\int_V e^u(e^u-1)d\mu=c_1:=-\frac{1}{\lambda}\int_V f d\mu. \tag{20}$$

By the assumptions $\lambda\neq 0$ and $\int_V f d\mu\neq 0$, we know $c_1\neq 0$. Now we *claim* that

$$\max_V u>-A:=\log\min\left\{1,\frac{|c_1|}{4|V|}\right\}. \tag{21}$$

For otherwise, $\max_V u \le -A$, which together with (20) implies

$$\begin{aligned}|c_1| &= \left|\int_V e^u(e^u-1)d\mu\right| \\ &\le \int_V (e^{2u}+e^u)d\mu \\ &\le (e^{2\max_V u}+e^{\max_V u})|V| \\ &\le 2e^{-A}|V| \\ &< \frac{|c_1|}{2}.\end{aligned}$$

This contradicts $c_1 \neq 0$, and thus confirms our claim (21). Inserting (21) into (19), we obtain

$$-A-c_0 \le \min_V u \le \max_V u \le \log\frac{1+\sqrt{1+4a/\mu_0}}{2},$$

as we desired. □

The second priori estimate is for the changing λ and f.

Lemma 8 *Let u be a solution of (3). If λ and f satisfy (4), then there exists a constant C, depending only on Λ and the graph V, such that $|u(x)| \le C$ for all $x \in V$.*

Proof It suffices to modify the argument in the proof of Lemma 7.

Similar to (14), we first have the upper bound estimate

$$\max_V u \le \log\frac{1+\sqrt{1+4a/\mu_0}}{2}, \tag{22}$$

where $\mu_0 = \min_{x\in V}\mu(x)$ and $a = |V|+\Lambda^2$. Next, instead of (19), we have

$$\max_V u - \min_V u \le c_0 = b\sqrt{\frac{(\ell-1)|V|}{w_0\lambda_1}}, \tag{23}$$

where $\lambda_1 = \inf_{\overline{v}=0,\int_V v^2 d\mu=1}\int_V |\nabla v|^2 d\mu$, ℓ denotes the number of all points of V, $w_0 = \min_{x\in V,\, y\sim x} w_{xy}$ and

$$b = \Lambda\left(\frac{(1+\sqrt{1+4a/\mu_0})^2}{4}+\frac{1+\sqrt{1+4a/\mu_0}}{2}+1\right).$$

To proceed, we shall show

$$\max_V u > -A = \log\min\left\{1, \frac{1}{4|V|\Lambda^2}\right\}. \tag{24}$$

Suppose not. We have $\max_V u \le -A$ and

$$\begin{aligned}\frac{1}{\Lambda^2} \le \left|\frac{1}{\lambda}\int_V f d\mu\right| = \left|\int_V e^u(e^u-1)d\mu\right| &\le \int_V (e^{2u}+e^u)d\mu \\ &\le 2e^{-A}|V| \\ &< \frac{1}{2\Lambda^2},\end{aligned}$$

which is impossible. Thus (24) holds. Combining (22), (23) and (24), we get the desired result. □

The third priori estimate is not only for changing λ and f, but also for the changing parameter σ.

Lemma 9 *Let $\sigma \in [0, 1]$, λ and f satisfy (4) for some real number $\Lambda > 0$. If u is a solution of (5), then there exists a constant C, depending only on Λ and the graph V, such that $|u(x)| \le C$ for all $x \in V$.*

Proof If u is a solution of (5), then integration by parts gives

$$0 = \int_V \Delta u d\mu = \lambda \int_V e^u(e^u - \sigma)d\mu + \int_V f d\mu.$$

Similar to (14), keeping in mind $\sigma \in [0, 1]$, we first have the same upper bound estimate as (22), namely

$$\max_V u \le \log \frac{1 + \sqrt{1 + 4a/\mu_0}}{2},$$

where $\mu_0 = \min_{x \in V} \mu(x)$ and $a = |V| + \Lambda^2$. Next, we have the same estimates as (23) and (24), which is independent of the parameter $\sigma \in [0, 1]$. In particular

$$\max_V u > -A = \log \min \left\{ 1, \frac{1}{4|V|\Lambda^2} \right\}.$$

This ends the proof of the lemma, and completes the proof of Theorem 1. □

3 Topological degree

In this section, we shall prove Theorem 2. Precisely we shall compute the topological degree of certain maps related to the Chern–Simons Higgs model.

Proof of Theorem 2 Assume $V = \{x_1, \cdots, x_\ell\}$. Let $X = L^\infty(V)$. We may identify X with the Euclidean space $\mathbb{R}^\ell$. Without causing ambiguity, we define a map $F : X \times [0, 1] \to X$ by

$$F(u, \sigma) = -\Delta u + \lambda e^u(e^u - \sigma) + f, \quad (u, \sigma) \in X \times [0, 1].$$

Obviously, F is a smooth map. For the fixed real number λ and the fixed function f, since $\lambda \overline{f} \ne 0$, there must exist a large number $\Lambda > 0$ such that

$$\Lambda^{-1} \le |\lambda| \le \Lambda, \quad \Lambda^{-1} \le \left| \int_V f d\mu \right| \le \Lambda, \quad \|f\|_{L^\infty(V)} \le \Lambda. \tag{25}$$

Here and in the sequel, $\overline{f}$ denotes the integral mean of a function f. Then it follows from Theorem 1 that there exists a constant $R_0 > 0$, depending only on Λ and the graph V, such that for all $\sigma \in [0, 1]$, all solutions of $F(u, \sigma) = 0$ satisfy $\|u\|_{L^\infty(V)} < R_0$. Denote a ball centered at $0 \in X$ with radius r by $B_r \subset X$, and its boundary by $\partial B_r = \{u \in X : \|u\|_{L^\infty(V)} = r\}$. Thus we conclude

$$0 \notin F(\partial B_R, \sigma), \quad \forall \sigma \in [0, 1], \ \forall R \ge R_0.$$

By the homotopic invariance of the topological degree, we have

$$\deg(F(\cdot,1), B_R, 0) = \deg(F(\cdot,0), B_R, 0), \quad \forall R \geq R_0. \tag{26}$$

Given any $\epsilon > 0$, we define another smooth map $G_\epsilon : X \times [0,1] \to X$ by

$$G_\epsilon(u,t) = -\Delta u + \lambda e^{2u} + (t + (1-t)\epsilon) f, \quad (u,t) \in X \times [0,1].$$

Notice that

$$\left|(t + (1-t)\epsilon) \int_V f d\mu\right| \geq \min\{1, \epsilon\} \left|\int_V f d\mu\right|, \quad \forall t \in [0,1].$$

Applying Theorem 1 again, we find a constant $R_\epsilon > 0$, depending only on ϵ, Λ and the graph V, such that all solutions u of $G_\epsilon(u,t) = 0$ satisfy $\|u\|_{L^\infty(V)} < R_\epsilon$ for all $t \in [0,1]$. This implies

$$0 \notin G_\epsilon(\partial B_{R_\epsilon}, t), \quad \forall t \in [0,1].$$

Hence the homotopic invariance of the topological degree leads to

$$\deg(G_\epsilon(\cdot,1), B_{R_\epsilon}, 0) = \deg(G_\epsilon(\cdot,0), B_{R_\epsilon}, 0). \tag{27}$$

To calculate $\deg(G_\epsilon(\cdot,0), B_{R_\epsilon}, 0)$, we need to understand the solvability of the equation

$$G_\epsilon(u,0) = -\Delta u + \lambda e^{2u} + \epsilon f = 0. \tag{28}$$

Now we *claim* two properties of solutions of (28): (i) If $\lambda \overline{f} < 0$, then there exists an $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, (28) has a unique solution u_ϵ, which satisfies $e^{2u_\epsilon} \leq C\epsilon$, where C is a constant depending only on Λ and the graph V; (ii) If $\lambda \overline{f} > 0$, then (28) has no solution for all $\epsilon > 0$.

To see Claim (i), for any $\epsilon > 0$, we let v_ϵ be the unique solution of the equation

$$\begin{cases} \Delta v = \epsilon f - \epsilon \overline{f} & \text{in} \quad V \\ \overline{v} = 0. \end{cases}$$

Then the solvability of (28) is equivalent to that of the equation

$$\Delta w = \lambda e^{2v_\epsilon} e^{2w} + \epsilon \overline{f}. \tag{29}$$

Note that the existence of solutions to (29), under the assumptions that ϵ is sufficiently small and $\lambda \overline{f} < 0$, follows from ([13], Theorems 2 and 4). Hence there exists some $\epsilon_1 > 0$ such that if $0 < \epsilon < \epsilon_1$, then the Eq. (28) has a solution u_ϵ. Integrating both sides of (28), we have by (25),

$$\int_V e^{2u_\epsilon} d\mu = -\frac{\epsilon}{\lambda} \int_V f d\mu \leq \Lambda^2 \epsilon,$$

which leads to

$$e^{2u_\epsilon(x)} \leq \frac{\Lambda^2}{\mu_0} \epsilon, \quad \forall x \in V, \tag{30}$$

where $\mu_0 = \min_{x \in V} \mu(x)$. We also need to prove the uniqueness of the solution. Let φ be an arbitrary solution of (28), namely it satisfies

$$\Delta \varphi = \lambda e^{2\varphi} + \epsilon f. \tag{31}$$

The same procedure as above gives

$$\int_V e^{2\varphi} d\mu \leq \Lambda^2 \epsilon, \quad e^{2\varphi(x)} \leq \frac{\Lambda^2}{\mu_0}\epsilon \quad \text{for all} \quad x \in V. \tag{32}$$

Subtracting (31) from (28) and integrating by parts, we have

$$0 = \int_V \Delta(u_\epsilon - \varphi) d\mu = \lambda \int_V (e^{2u_\epsilon} - e^{2\varphi}) d\mu,$$

which leads to

$$\min_V (u_\epsilon - \varphi) < 0 < \max_V (u_\epsilon - \varphi).$$

As a consequence, there holds

$$|u_\epsilon - \varphi| \leq \max_V (u_\epsilon - \varphi) - \min_V (u_\epsilon - \varphi). \tag{33}$$

Also we derive from (28), (30), (31), and (32),

$$\begin{aligned} |\Delta(u_\epsilon - \varphi)(x)| &= \left| \lambda \left(e^{2u_\epsilon(x)} - e^{2\varphi(x)} \right) \right| \\ &\leq 2\Lambda \left(e^{2u_\epsilon(x)} + e^{2\varphi(x)} \right) |u_\epsilon(x) - \varphi(x)| \\ &\leq \frac{4\Lambda^3}{\mu_0} \epsilon |u_\epsilon(x) - \varphi(x)|. \end{aligned} \tag{34}$$

Combining (18), (33) and (34), we obtain

$$\max_V (u_\epsilon - \varphi) - \min_V (u_\epsilon - \varphi) \leq \sqrt{\frac{(\ell - 1)|V|}{w_0 \lambda_1}} \frac{4\Lambda^3}{\mu_0} \epsilon \left(\max_V (u_\epsilon - \varphi) - \min_V (u_\epsilon - \varphi) \right). \tag{35}$$

Choose

$$\epsilon_0 = \min \left\{ \epsilon_1, \sqrt{\frac{w_0 \lambda_1}{(\ell - 1)|V|}} \frac{\mu_0}{8\Lambda^3} \right\}.$$

If we take $0 < \epsilon < \epsilon_0$, then (35) implies $\varphi \equiv u_\epsilon$ on V, and thus (28) has a unique solution. Hence (i) holds.

To see Claim (ii), in the case $\lambda \overline{f} > 0$, if (28) has a solution u, then there holds

$$0 = \int_V \Delta u d\mu = \lambda \int_V e^{2u} d\mu + \int_V f d\mu,$$

which is impossible. This confirms (ii), and our claims hold.

Let us continue to prove the theorem. Note that $-\Delta : X \to X$ is a nonnegative definite symmetric operator, its eigenvalues are written as

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\ell-1},$$

where ℓ is the number of all points in V. By Claim (i), in the case $\lambda \overline{f} < 0$, we may choose a sufficiently small $\epsilon > 0$ such that $G_\epsilon(u, 0) = 0$ has a unique solution u_ϵ verifying

$$2|\lambda| e^{2u_\epsilon(x)} < \lambda_1.$$

A straightforward calculation shows

$$DG_\epsilon(u_\epsilon, 0) = -\Delta + 2\lambda e^{2u_\epsilon} \mathrm{I},$$

where we identify the linear operator $-\Delta$ with the $\ell \times \ell$ matrix corresponding to $-\Delta$, and denote the $\ell \times \ell$ diagonal matrix diag$[1, 1, \cdots, 1]$ by I. Clearly
$\deg(G_\epsilon(\cdot, 0), B_{R_\epsilon}, 0) = \operatorname{sgn}\det(DG_\epsilon(u_\epsilon, 0)) = \operatorname{sgn}\left\{2\lambda e^{2u_\epsilon(x)}\Pi_{j=1}^{\ell-1}(\lambda_j + 2\lambda e^{2u_\epsilon(x)})\right\} = \operatorname{sgn}\lambda.$

This together with (26) and (27) leads to

$$\begin{aligned}\deg(F(\cdot, 1), B_{R_\epsilon}, 0) &= \deg(F(\cdot, 0), B_{R_\epsilon}, 0)\\ &= \deg(G_\epsilon(\cdot, 1), B_{R_\epsilon}, 0)\\ &= \deg(G_\epsilon(\cdot, 0), B_{R_\epsilon}, 0)\\ &= \operatorname{sgn}\lambda.\end{aligned}$$

By Claim (ii), in the case $\lambda\overline{f} > 0$, since $G_\epsilon(u, 0) = 0$ has no solution, we obtain

$$\deg(F(\cdot, 1), B_{R_\epsilon}, 0) = \deg(G_\epsilon(\cdot, 0), B_{R_\epsilon}, 0) = 0.$$

Thus the proof of Theorem 2 is completed. □

4 Existence results

In this section, we shall prove Theorem 3 by using the topological degree in Theorem 2.

Proof of Theorem 3 (a).

If $\lambda\overline{f} < 0$, then by Theorem 2, we find some large $R_0 > 1$ such that

$$\deg(F, B_{R_0}, 0) \neq 0.$$

Thus the Kronecker's existence theorem implies (3) has a solution. □

In the remaining part of this section, we always assume $\lambda\overline{f} > 0$. We first prove that (3) has a local minimum solution for large $|\lambda|$, say

Lemma 10 *If $|\lambda|$ is chosen sufficiently large, then the Eq. 3 has a local minimum solution.*

Proof Let us first consider the subcase $\lambda > 0$ and $\overline{f} > 0$. Set

$$L_\lambda u = -\Delta u + \lambda e^u(e^u - 1) + f. \tag{36}$$

For real numbers A and λ, there hold

$$L_\lambda A = \lambda e^A(e^A - 1) + f, \quad L_\lambda \log\frac{1}{2} = -\frac{1}{4}\lambda + f.$$

Clearly, taking sufficiently large $A > 1$ and $\lambda > 1$, we have

$$L_\lambda A > 0, \quad L_\lambda \log\frac{1}{2} < 0. \tag{37}$$

Recall the functional $J_\lambda : X = L^\infty(V) \to \mathbb{R}$ defined by (7). Since $X \cong \mathbb{R}^\ell$, $J_\lambda \in C^2(X, \mathbb{R})$, and $\{u \in X : \log\frac{1}{2} \le u \le A\}$ is a bounded closed subset of X, it is easy to find some $u_\lambda \in X$ satisfying $\log\frac{1}{2} \le u_\lambda(x) \le A$ for all $x \in V$ and

$$J_\lambda(u_\lambda) = \min_{\log\frac{1}{2} \le u \le A} J_\lambda(u). \tag{38}$$

We *claim* that

$$\log\frac{1}{2} < u_\lambda(x) < A \quad \text{for all} \quad x \in V. \tag{39}$$

Suppose not. There must hold $u_\lambda(x_0) = \log\frac{1}{2}$ for some $x_0 \in V$, or $u_\lambda(x_1) = A$ for some $x_1 \in V$. If $u_\lambda(x_0) = \log\frac{1}{2}$, then we take a small $\epsilon > 0$ such that

$$\log\frac{1}{2} \le u_\lambda(x) + t\delta_{x_0}(x) \le A, \quad \forall x \in V, \ \forall t \in (0, \epsilon).$$

On one hand, in view of (37) and (38), we have

$$\begin{aligned}
0 &\le \left.\frac{d}{dt}\right|_{t=0} J_\lambda(u_\lambda + t\delta_{x_0}) \\
&= \int_V \left(-\Delta u_\lambda + \lambda e^{u_\lambda}(e^{u_\lambda} - 1) + f\right)\delta_{x_0} d\mu \\
&= -\Delta u_\lambda(x_0) + \lambda e^{u_\lambda(x_0)}(e^{u_\lambda(x_0)} - 1) + f(x_0) \\
&< -\Delta u_\lambda(x_0).
\end{aligned} \tag{40}$$

On the other hand, since $u_\lambda(x) \ge u_\lambda(x_0)$ for all $x \in V$, we conclude $\Delta u_\lambda(x_0) \ge 0$, which contradicts (40). Hence $u_\lambda(x) > \log\frac{1}{2}$ for all $x \in V$. In the same way, we exclude the possibility of $u_\lambda(x_1) = A$ for some $x_1 \in V$. This confirms our claim (39). Combining (38) and (39), we conclude that u_λ is a local minimum critical point of J_λ, in particular, u_λ is a solution of (3).

Now we consider the subcase $\lambda < 0$ and $\overline{f} < 0$. Let φ be the unique solution of

$$\begin{cases} \Delta\varphi = f - \overline{f} \\ \overline{\varphi} = 0. \end{cases}$$

Using the notation of the operator L_λ given by (36), we have

$$\begin{aligned}
L_\lambda(\varphi - A) &= -\Delta\varphi + \lambda e^{\varphi - A}(e^{\varphi - A} - 1) + f \\
&= \lambda e^{\varphi - A}(e^{\varphi - A} - 1) + \overline{f} \\
&< 0
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
L_\lambda(\log\frac{1}{2}) &= \lambda e^{\log\frac{1}{2}}(e^{\log\frac{1}{2}} - 1) + f \\
&= -\frac{\lambda}{4} + f \\
&> 0,
\end{aligned}$$

provided that $\lambda < 4\min_V f$ and $A > 1$ is chosen sufficiently large. Similar to (38) and (39), there exists some u_λ satisfying $\varphi(x) - A < u_\lambda(x) < \log\frac{1}{2}$ for all $x \in V$ and

$$J_\lambda(u_\lambda) = \min_{\varphi - A \le u \le \log\frac{1}{2}} J_\lambda(u) = \min_{\varphi - A < u < \log\frac{1}{2}} J_\lambda(u).$$

This implies u_λ is a local minimum solution of (3). □

To proceed, we also need the following:

Lemma 11 *If $\lambda_1 > 0$ such that the equation $L_{\lambda_1} u = 0$ has a solution u_{λ_1}, then for any $\lambda > \lambda_1$, we have*

$$L_\lambda\left(u_{\lambda_1} + \log\frac{\lambda_1}{\lambda}\right) < 0.$$

Similarly, if $\lambda_2 < 0$ *such that* $L_{\lambda_2} u_{\lambda_2} = 0$*, then for any* $\lambda < \lambda_2$*, there hods*

$$L_\lambda \left(u_{\lambda_2} + \log \frac{\lambda_2}{\lambda} \right) > 0.$$

Proof If $\lambda > \lambda_1 > 0$, then

$$\begin{aligned} L_\lambda \left(u_{\lambda_1} + \log \frac{\lambda_1}{\lambda} \right) &= -\Delta u_{\lambda_1} + \lambda_1 e^{u_{\lambda_1}} \left(\frac{\lambda_1}{\lambda} e^{u_{\lambda_1}} - 1 \right) + f \\ &< -\Delta u_{\lambda_1} + \lambda_1 e^{u_{\lambda_1}} (e^{u_{\lambda_1}} - 1) + f \\ &= 0. \end{aligned}$$

If $\lambda < \lambda_2 < 0$, then

$$\begin{aligned} L_\lambda \left(u_{\lambda_2} + \log \frac{\lambda_2}{\lambda} \right) &= -\Delta u_{\lambda_2} + \lambda_2 e^{u_{\lambda_2}} \left(\frac{\lambda_2}{\lambda} e^{u_{\lambda_2}} - 1 \right) + f \\ &> -\Delta u_{\lambda_2} + \lambda_2 e^{u_{\lambda_2}} \left(e^{u_{\lambda_2}} - 1 \right) + f \\ &= 0, \end{aligned}$$

as we desired. □

As a consequence, we have

Lemma 12 *Assume* $L_{\lambda_1} u_{\lambda_1} = L_{\lambda_2} u_{\lambda_2} = 0$ *on* V*. If either* $\lambda > \lambda_1 > 0$ *or* $\lambda < \lambda_2 < 0$*, then the equation (3) has a local minimum solution* u_λ*.*

Proof Assume $\lambda > \lambda_1 > 0$. Let $A > 1$ be a sufficiently large constant such that $L_\lambda A > 0$ and $u_{\lambda_1} + \log \frac{\lambda_1}{\lambda} < A$ on V. Then there exists some u_λ such that

$$J_\lambda(u_\lambda) = \min_{u_{\lambda_1} + \log \frac{\lambda_1}{\lambda} \le u \le A} J_\lambda(u).$$

Suppose there is some point $x_0 \in V$ satisfying $u_\lambda(x_0) = u_{\lambda_1}(x_0) + \log \frac{\lambda_1}{\lambda}$. Let $\epsilon > 0$ be so small that for $t \in (0, \epsilon)$, there holds

$$u_{\lambda_1}(x) + \log \frac{\lambda_1}{\lambda} \le u_\lambda(x) + t\delta_{x_0}(x) \le A \quad \text{for all} \quad x \in V.$$

Similarly as we did in the proof of Lemma 10, we have by Lemma 11,

$$\begin{aligned} 0 &\le \left. \frac{d}{dt} \right|_{t=0} J_\lambda(u_\lambda + t\delta_{x_0}) \\ &= -\Delta u_\lambda(x_0) + \lambda e^{u_\lambda(x_0)} (e^{u_\lambda(x_0)} - 1) + f(x_0) \\ &= -\Delta \left(u_\lambda - u_{\lambda_1} \right)(x_0) + L_\lambda \left(u_{\lambda_1} + \log \frac{\lambda_1}{\lambda} \right)(x_0) \\ &< -\Delta \left(u_\lambda - u_{\lambda_1} \right)(x_0). \end{aligned}$$

This contradicts the fact that x_0 is a minimum point of $u_\lambda - u_{\lambda_1} - \log \frac{\lambda_1}{\lambda}$. Hence

$$u_\lambda(x) > u_{\lambda_1}(x) + \log \frac{\lambda_1}{\lambda}, \quad \forall x \in V.$$

In the same way we obtain $u(x) < A$ for all $x \in V$. Therefore u_λ is a local minimum critical point of J_λ.

Assume $\lambda < \lambda_2 < 0$. The constant $A > 1$ is chosen sufficiently large such that $\varphi - A < u_{\lambda_2} + \log \frac{\lambda_2}{\lambda}$ on V, and $\varphi - A$ satisfies (41). Clearly there exists some u_λ such that

$$J_\lambda(u_\lambda) = \min_{\varphi - A \le u \le u_{\lambda_2} + \log \frac{\lambda_2}{\lambda}} J_\lambda(u).$$

If there is some point $x_1 \in V$ satisfying $u_\lambda(x_1) = u_{\lambda_2}(x_1) + \log \frac{\lambda_2}{\lambda}$, then there is a small $\epsilon > 0$ such that for $t \in (0, \epsilon)$, there holds

$$\varphi(x) - A \le u_\lambda(x) - t\delta_{x_1}(x) \le u_{\lambda_2}(x) + \log \frac{\lambda_2}{\lambda} \quad \text{for all} \quad x \in V.$$

Thus we have by Lemma 11,

$$\begin{aligned} 0 &\le \left.\frac{d}{dt}\right|_{t=0} J_\lambda(u_\lambda - t\delta_{x_1}) \\ &= \Delta u_\lambda(x_1) - \lambda e^{u_\lambda(x_1)}(e^{u_\lambda(x_1)} - 1) - f(x_0) \\ &= \Delta \left(u_\lambda - u_{\lambda_2}\right)(x_0) - L_\lambda \left(u_{\lambda_2} + \log \frac{\lambda_2}{\lambda}\right)(x_0) \\ &< \Delta \left(u_\lambda - u_{\lambda_2}\right)(x_0). \end{aligned}$$

This contradicts the fact that x_1 is a maximum point of $u_\lambda - u_{\lambda_2} - \log \frac{\lambda_2}{\lambda}$. Hence

$$u_\lambda(x) < u_{\lambda_2}(x) + \log \frac{\lambda_2}{\lambda}, \quad \forall x \in V.$$

In the same way we obtain $u(x) > \varphi(x) - A$ for all $x \in V$. Therefore u_λ is a local minimum critical point of J_λ. Thus we complete the proof of the lemma. □

We conclude from Lemmas 10 and 12 that the following two critical numbers are well defined.

$$\Lambda^* = \inf \left\{\lambda > 0 : \lambda \overline{f} > 0,\ J_\lambda \text{ has a local minimum critical point}\right\} \tag{42}$$

$$\Lambda_* = \sup \left\{\lambda < 0 : \lambda \overline{f} > 0,\ J_\lambda \text{ has a local minimum critical point}\right\}. \tag{43}$$

Lemma 13 *If $\overline{f} > 0$, then $\Lambda^* \ge 4\overline{f}$; If $\overline{f} < 0$, then $\Lambda_* \le 4\overline{f}$.*

Proof Suppose $\lambda \ne 0$ and u is a solution of $\Delta u = \lambda e^u(e^u - 1) + f$. Integration by parts gives

$$-\frac{\int_V f d\mu}{\lambda} = \int_V e^u(e^u - 1)d\mu \ge -\frac{|V|}{4},$$

since $e^u(e^u - 1) \ge -\frac{1}{4}$. The conclusion follows from (42) and (43) immediately. □

We are now ready to complete the proof of the remaining part of the theorem.

Proof of Theorem 3 (b).

We first consider the solvability of the Eq. (3) *under the assumption* $\lambda \in (0, \Lambda^*] \cup [\Lambda_*, 0)$.

If $\lambda \in (0, \Lambda^*) \cup (\Lambda_*, 0)$, then (3) has no solution. Indeed, suppose there exists a number $\lambda_1 \in (0, \Lambda^*) \cup (\Lambda_*, 0)$ such that (3) has a solution at $\lambda = \lambda_1$. With no loss of generality, we assume $\lambda_1 \in (\Lambda_*, 0)$, then by Lemma 12, (3) has a local minimum solution at any $\lambda \in [\Lambda_*, \lambda_1)$. This contradicts the definition of Λ_*. Hence (3) has no solution for any $\lambda \in (0, \Lambda^*) \cup (\Lambda_*, 0)$.

Note that for any $j \in \mathbb{N}$, there exists a solution u_j of (3) with $\lambda = \Lambda_* - 1/j$. According to Theorem 1, (u_j) is uniformly bounded in V. Thus up to a subsequence, (u_j) uniformly converges to some function u^*, a solution of (3) with $\lambda = \Lambda_*$. In the same way, (3) has also a solution at $\lambda = \Lambda^*$.

We next consider multiple solutions of (3) *under the assumption* $\lambda \in (\Lambda^*, +\infty) \cup (-\infty, \Lambda_*)$.

If $\lambda \in (\Lambda^*, +\infty) \cup (-\infty, \Lambda_*)$, by (42) and (43), we let u_λ be a local minimum critical point of J_λ. With no loss of generality, we may assume u_λ is the unique critical point of J_λ. For otherwise, J_λ has already at least two critical points, and the proof terminates. According to ([4], Chapter 1, page 32), the q-th critical group of J_λ at u_λ is defined by

$$\mathsf{C}_q(J_\lambda, u_\lambda) = \mathsf{H}_q(J_\lambda^c \cap U, \{J_\lambda^c \setminus \{u_\lambda\}\} \cap U, \mathsf{G}), \tag{44}$$

where $J_\lambda(u_\lambda) = c$, $J_\lambda^c = \{u \in X : J_\lambda(u) \leq c\}$, U is a neighborhood of $u_\lambda \in X$, H_q is the singular homology group with the coefficients groups G, say $\mathbb{Z}$, $\mathbb{R}$. By the excision property of H_q, this definition is not dependent on the choice of U. It is easy to calculate

$$\mathsf{C}_q(J_\lambda, u_\lambda) = \delta_{q0}\mathsf{G}. \tag{45}$$

Note that J_λ satisfies the Palais-Smale condition. Indeed, if $J_\lambda(u_j) \to c \in \mathbb{R}$ and $J'(u_j) \to 0$ as $j \to \infty$, then using the method of proving Theorem 1, we obtain (u_j) is uniformly bounded. Since X is pre-compact, then up to a subsequence, (u_j) converges uniformly to some u^*, a critical point J_λ. Thus the Palais-Smale condition follows. Notice also that

$$DJ_\lambda(u) = -\Delta u + \lambda e^u(e^u - 1) + f = F(u),$$

where F is given as in Theorem 2. According to ([4], Chapter 2, Theorem 3.2), in view of (45), we have for sufficiently large $R > 1$,

$$\deg(F, B_R, 0) = \deg(DJ_\lambda, B_R, 0) = \sum_{q=0}^{\infty}(-1)^q \text{rank}\, \mathsf{C}_q(J_\lambda, u_\lambda) = 1.$$

This contradicts $\deg(F, B_R, 0) = 0$ derived from Theorem 2. Therefore the Eq. (3) has at least two different solutions, and the proof of Theorem 3 (b) is finished. □

5 Chern–Simons Higgs System

In this section, we shall calculate the topological degree of the map related to the Chern–Simons Higgs system (8), and then use the degree to obtain partial results for multiplicity of solutions to the system. In particular, Theorems 4, 5 and 6 will be proved. We first derive a priori estimate for solutions of (9), a deformation of (8).

Proof of Theorem 4 Let $\sigma \in [0, 1]$, $\lambda > 0$, $\eta > 0$, $\overline{f} > 0$, $\overline{g} > 0$, and (u, v) be a solution of the system (9). Note that there exist a unique solution φ to the equation

$$\begin{cases} \Delta\varphi = f - \overline{f} \\ \int_V \varphi d\mu = 0 \end{cases}$$

and a unique solution ψ to the equation

$$\begin{cases} \Delta\psi = g - \overline{g} \\ \int_V \psi d\mu = 0. \end{cases}$$

Set $w = u - \varphi$ and $z = v - \psi$. Then we have

$$\begin{cases} \Delta w = \lambda e^{\psi} e^{z}(e^{\varphi} e^{w} - \sigma) + \overline{f} \\ \Delta z = \eta e^{\varphi} e^{w}(e^{\psi} e^{z} - \sigma) + \overline{g}, \end{cases} \tag{46}$$

We *claim* that

$$w(x) < -\min_V \varphi \quad \text{for all} \quad x \in V. \tag{47}$$

Suppose not. There necessarily hold $\max_V w \geq -\min_V \varphi$. Take $x_0 \in V$ satisfying $w(x_0) = \max_V w$. Since $\sigma \in [0, 1]$, $\lambda > 0$, $\overline{f} > 0$ and $\varphi(x_0) + w(x_0) \geq 0$, we have

$$0 \geq \Delta w(x_0) = \lambda e^{\psi(x_0)} e^{z(x_0)}(e^{\varphi(x_0)} e^{w(x_0)} - \sigma) + \overline{f} \geq \overline{f} > 0,$$

which is impossible. Hence our claim (47) follows. Keeping in mind $\eta > 0$ and $\overline{g} > 0$, in the same way as above, we also have

$$z(x) < -\min_V \psi \quad \text{for all} \quad x \in V. \tag{48}$$

Inserting (47) and (48) into (46), we obtain

$$\|\Delta w\|_{L^\infty(V)} + \|\Delta z\|_{L^\infty(V)} \leq C$$

for some constant C, depending only on λ, η, f, g and the graph V. The most important thing here is that the constant C is not dependent on the parameter $\sigma \in [0, 1]$. Coming back to the inequality (18), we immediately conclude

$$\max_V w - \min_V w \leq C \tag{49}$$

and

$$\max_V z - \min_V z \leq C.$$

Observe that integration on both sides of the second equation in (46) leads to

$$\int_V e^{\varphi} e^{w}(e^{\psi} e^{z} - \sigma) d\mu = -\frac{\overline{g}}{\eta}|V|.$$

As a consequence, there holds

$$0 < \frac{\overline{g}}{\eta} \leq e^{\max_V w} e^{\max_V \varphi} \left(e^{\max_V \psi} + 1\right) \leq C e^{\max_V w}.$$

Hence $\max_V w \geq -C$, and in view of (49),

$$\min_V w \geq -C. \tag{50}$$

In the same way, from (49) and the first equation of (46), we derive

$$\min_V z \geq -C. \tag{51}$$

In view of (47), (48), (50) and (51), the proof of the theorem is completed. □

Now we calculate the topological degree of the map defined as in (10).

Proof of Theorem 5 Let $X = L^\infty(V)$. Define a map $\mathcal{F} : X \times X \times [0,1] \to X \times X$ by
$\mathcal{F}(u, v, \sigma) = (-\Delta u + \lambda e^v(e^u - \sigma) + f, -\Delta v + \eta e^u(e^v - \sigma) + g), \quad \forall (u, v, \sigma) \in X \times X \times [0,1].$

Obviously $\mathcal{F} \in C^2(X \times X \times [0,1], X \times X)$. On one hand, by Theorem 4, there exists some $R_0 > 0$ such that for any $R \geq R_0$, we have

$$0 \notin \mathcal{F}(\partial B_R, \sigma), \quad \forall \sigma \in [0,1],$$

and thus the homotopic invariance of the topological degree implies

$$\deg(\mathcal{F}(\cdot, 1), B_R, (0,0)) = \deg(\mathcal{F}(\cdot, 0), B_R, (0,0)). \tag{52}$$

Here we denote $B_R = \{(u, v) \in X \times X : \|u\|_{L^\infty(V)} + \|v\|_{L^\infty(V)} < R\}$ and $\partial B_R = \{(u, v) \in X \times X : \|u\|_{L^\infty(V)} + \|v\|_{L^\infty(V)} = R\}$, as usual.

On the other hand, we calculate $\deg(\mathcal{F}(\cdot, 0), B_R, (0,0))$. Since $\lambda > 0$ and $\overline{f} > 0$, integrating both sides of the first equation of the system

$$\begin{cases} \Delta u = \lambda e^{u+v} + f \\ \Delta v = \eta e^{u+v} + g, \end{cases} \tag{53}$$

we get a contradiction, provided that (53) is solvable. This implies

$$\{(u, v) \in X \times X : \mathcal{F}(u, v, 0) = (0,0)\} = \varnothing.$$

As a consequence, there holds

$$\deg(\mathcal{F}(\cdot, 0), B_R, (0,0)) = 0. \tag{54}$$

Combining (52) and (54), we get the desired result. □

Let $\mathcal{J}_\lambda : X \times X \to \mathbb{R}$ be a functional defined as in (11). Note that the critical point of $\mathcal{J}_\lambda$ is a solution of the Chern–Simons system (8). The following property of $\mathcal{J}_\lambda$ will be not only useful for our subsequent analysis, but also of its own interest.

Lemma 14 *Under the assumptions $\lambda > 0$, $\overline{f} > 0$ and $\overline{g} > 0$, $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Proof Let $c \in \mathbb{R}$ and $\{(u_k, v_k)\}$ be a sequence in $X \times X$ such that $\mathcal{J}_\lambda(u_k, v_k) \to c$ and

$$\mathcal{J}'_\lambda(u_k, v_k) \to (0,0) \quad \text{in} \quad (X \times X)^* \cong \mathbb{R}^\ell \times \mathbb{R}^\ell.$$

This together with (12) gives

$$\begin{cases} -\Delta u_k + \lambda e^{v_k}(e^{u_k} - 1) + f = o_k(1) \\ -\Delta v_k + \lambda e^{u_k}(e^{v_k} - 1) + g = o_k(1), \end{cases} \tag{55}$$

where $o_k(1) \to 0$ uniformly on V as $k \to \infty$. Comparing (55) with the system (8), we have by using the same method as in the proof of Theorem 4,

$$\|u_k\|_{L^\infty(V)} + \|v_k\|_{L^\infty(V)} \leq C$$

for some constant C, provided that $k \geq k_1$ for some large positive integer k_1. Since V is finite, X is pre-compact. Hence, up to a subsequence, $u_k \to u^*$ and $v_k \to v^*$ uniformly in V for some functions u^* and v^*. Obviously $\mathcal{J}'_\lambda(u^*, v^*) = (0,0)$. Thus $\mathcal{J}_\lambda$ satisfies the $(PS)_c$ condition. □

Finally we prove a partial multiple solutions result for the system (8).

Proof of Theorem 6 We distinguish two hypotheses to proceed.

Case 1. *$\mathcal{J}_\lambda$ has a non-degenerate critical point* (u_λ, v_λ).

Since (u_λ, v_λ) is non-degenerate, we have

$$\det D^2\mathcal{J}_\lambda(u_\lambda, v_\lambda) \neq 0.$$

Suppose (u_λ, v_λ) is the unique critical point of $\mathcal{J}_\lambda$. Then we conclude for all $R > \|u_\lambda\|_{L^\infty(V)} + \|v_\lambda\|_{L^\infty(V)}$,

$$\deg(D\mathcal{J}_\lambda, B_R, (0,0)) = \operatorname{sgn} \det D^2\mathcal{J}_\lambda(u_\lambda, v_\lambda) \neq 0. \tag{56}$$

Here and in the sequel, as in the proof of Theorem 5, B_R is a ball centered at $(0, 0)$ with radius R. Notice that $D\mathcal{J}_\lambda(u, v) = \mathcal{F}(u, v)$ for all $(u, v) \in X \times X$, where $\mathcal{F}$ is defined as in (10). By Theorem 5, we have

$$\deg(D\mathcal{J}_\lambda, B_R, (0,0)) = \deg(\mathcal{F}, B_R, (0,0)) = 0,$$

contradicting (56). Hence $\mathcal{J}_\lambda$ must have at least two critical points.

Case 2. *$\mathcal{J}_\lambda$ has a local minimum critical point* $(\varphi_\lambda, \psi_\lambda)$.

Similar to (44), the q-th critical group of $\mathcal{J}_\lambda$ at the critical point $(\varphi_\lambda, \psi_\lambda)$ reads as

$$\mathsf{C}_q(\mathcal{J}_\lambda, (\varphi_\lambda, \psi_\lambda)) = \mathsf{H}_q(\mathcal{J}_\lambda^c \cap \mathscr{U}, \{\mathcal{J}_\lambda^c \setminus \{(\varphi_\lambda, \psi_\lambda)\}\} \cap \mathscr{U}, \mathsf{G}),$$

where $\mathcal{J}_\lambda(\varphi_\lambda, \psi_\lambda) = c$, $\mathcal{J}_\lambda^c = \{(u, v) \in X \times X : \mathcal{J}_\lambda(u, v) \leq c\}$, $\mathscr{U}$ is a neighborhood of $(\varphi_\lambda, \psi_\lambda) \in X \times X$, $\mathsf{G} = \mathbb{Z}$ or $\mathbb{R}$ is the coefficient group of H_q. With no loss of generality, we assume $(\varphi_\lambda, \psi_\lambda)$ is the unique critical point of $\mathcal{J}_\lambda$. Since $(\varphi_\lambda, \psi_\lambda)$ is a local minimum critical point, we easily get

$$\mathsf{C}_q(\mathcal{J}_\lambda, (\varphi_\lambda, \psi_\lambda)) = \delta_{q0}\mathsf{G}.$$

By Lemma 14, $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition. Then applying ([4], Chapter 2, Theorem 3.2) and Theorem 5, we obtain

$$\begin{aligned} 0 = \deg(\mathcal{F}, B_R, (0,0)) &= \deg(D\mathcal{J}_\lambda, B_R, (0,0)) \\ &= \sum_{q=0}^{\infty}(-1)^q \operatorname{rank} \mathsf{C}_q(\mathcal{J}_\lambda, (\varphi_\lambda, \psi_\lambda)) \\ &= 1, \end{aligned}$$

provided that $R > \|\varphi_\lambda\|_{L^\infty(V)} + \|\psi_\lambda\|_{L^\infty(V)}$. This is impossible, and thus $\mathcal{J}_\lambda$ must have another critical point, as we desired. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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