SCIENCE CHINA
Mathematics

• ARTICLES •

January 2023 Vol. 66 No. 1: 115–142
https://doi.org/10.1007/s11425-021-1962-5

The boundary value problem for the mean field equation on a compact Riemann surface

Jiayu Li[1], Linlin Sun[2] & Yunyan Yang[3,*]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei* 230026*, China;*
[2] *School of Mathematics and Statistics, Wuhan University, Wuhan* 430072*, China;*
[3] *School of Mathematics, Renmin University of China, Beijing* 100872*, China*

Email: jiayuli@ustc.edu.cn, sunll@whu.edu.cn, yunyanyang@ruc.edu.cn

Received December 21, 2021; accepted April 4, 2022; published online July 28, 2022

Abstract Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$, Δ_g be the Laplace-Beltrami operator, and h be a positive smooth function. Using a min-max scheme introduced by Djadli and Malchiodi (2008) and Djadli (2008), we prove that if Σ is non-contractible, then for any $\rho \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}^*$, the mean field equation

$$\begin{cases} \Delta_g u = \rho \dfrac{he^u}{\int_\Sigma he^u dv_g} & \text{in } \Sigma, \\ u = 0 & \text{on } \partial\Sigma \end{cases}$$

has a solution. This generalizes earlier existence results of Ding et al. (Ann Inst H Poincaré Anal Non Linéaire, 1999) and Chen and Lin (2003) in the Euclidean domain. Also we consider the corresponding Neumann boundary value problem. If h is a positive smooth function, then for any $\rho \in (4k\pi, 4(k+1)\pi)$ with $k \in \mathbb{N}^*$, the mean field equation

$$\begin{cases} \Delta_g u = \rho \left(\dfrac{he^u}{\int_\Sigma he^u dv_g} - \dfrac{1}{|\Sigma|} \right) & \text{in } \Sigma, \\ \partial u / \partial \boldsymbol{v} = 0 & \text{on } \partial\Sigma \end{cases}$$

has a solution, where $\boldsymbol{v}$ denotes the unit normal outward vector on $\partial\Sigma$. Note that in this case we do not require the surface to be non-contractible.

Keywords mean field equation, topological method, min-max scheme

MSC(2020) 49J35, 58J05, 93B24

Citation: Li J Y, Sun L L, Yang Y Y. The boundary value problem for the mean field equation on a compact Riemann surface. Sci China Math, 2023, 66: 115–142, https://doi.org/10.1007/s11425-021-1962-5

1 Introduction

As a basic problem of mathematical physics, the mean field equation has attracted the interest of many mathematicians for at least half a century. In addition to the prescribed Gaussian curvature problem [4, 7, 8, 10, 22], it also arises in Onsager's vortex model for turbulent Euler flows [27, p. 256] and in Chern-Simons-Higgs models [6, 14, 16, 29, 33, 35].

* Corresponding author

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. It was proved by Ding et al. [15] that if the complement of Ω contains a bounded region, and $h : \overline{\Omega} \to \mathbb{R}$ is a positive function, then the mean field equation

$$\begin{cases} -\Delta_{\mathbb{R}^2} u = \rho \dfrac{he^u}{\int_\Omega he^u dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has a solution for all $\rho \in (8\pi, 16\pi)$, where $\Delta_{\mathbb{R}^2} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the standard Laplacian operator in $\mathbb{R}^2$. The proof is based on a compactness result of Li and Shafrir [24], the monotonicity technique used by Struwe [28] in dealing with harmonic maps, and a general min-max theorem [34, Theorem 2.8].

It was pointed out by Li [23] that the Leray-Schauder degree for the mean field equation should depend only on the topology of the domain and $k \in \mathbb{N}$ satisfying $\rho \in (8k\pi, 8(k+1)\pi)$. To illustrate this point, he calculated the simplest case $\rho < 8\pi$. Later, by computing the topological degree, Chen and Lin [9] improved Ding-Jost-Li-Wang's result to the following: if Ω is not simply connected, and h is positive on $\overline{\Omega}$, then (1.1) has a solution for all $\rho \in (8k\pi, 8(k+1)\pi)$. Also they were able to compute the topological degree for the mean field equation on the compact Riemann surface (Σ, g) without boundary, namely,

$$\Delta_g u = \rho \left(\frac{he^u}{\int_\Sigma he^u dv_g} - \frac{1}{|\Sigma|} \right) \quad \text{in } \Sigma, \tag{1.2}$$

where Δ_g denotes the Laplace-Beltrami operator, and $|\Sigma|$ stands for the area of Σ with respect to the metric g. Precisely, the degree-counting formula for (1.2) is given by $\binom{k-\chi(\Sigma)}{k}$ for $\rho \in (8k\pi, 8(k+1)\pi)$. As a consequence, if the Euler characteristic $\chi(\Sigma) \leqslant 0$, then (1.2) has a solution.

Note that solutions of (1.2) are critical points of the functional

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 dv_g - \rho \log \int_\Sigma he^u dv_g + \frac{\rho}{|\Sigma|} \int_\Sigma u dv_g, \quad u \in W^{1,2}(\Sigma).$$

A direct method of variation leads to that J_ρ has critical points for $\rho < 8\pi$. When $\rho = 8\pi$, Ding et al. [13] found a critical point of $J_{8\pi}$ under certain conditions on Σ and h. For the cases where $\rho = 8\pi$, $h \geqslant 0$ or h changes the sign, we refer the readers to [31, 32, 39].

In a celebrated paper, Djadli [17] was able to find a solution of (1.2) for all $\rho \in (8k\pi, 8(k+1)\pi)$ $(k \in \mathbb{N}^*)$ and arbitrary genera of Σ, by adapting a min-max scheme introduced by Djadli and Malchiodi [18]. In particular, Chen-Lin's existence result for (1.2) was improved by Djadli [17] to arbitrary possible $\chi(\Sigma)$. Let us summarize the procedure in [17]. Denote the family of formal sums by

$$\Sigma_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geqslant 0, \sum_{i=1}^k t_i = 1, x_i \in \Sigma \right\},$$

endowed with the weak topology of distributions, say the topology of $(C^1(\Sigma))^*$. This is known in the literature as the formal set of barycenters of Σ. The first and most important step is to construct two continuous maps Ψ and Φ_λ between Σ_k and sub-levels of J_ρ, say,

$$\Sigma_k \xrightarrow{\Phi_\lambda} J_\rho^{-(\rho - 8k\pi)\ln\lambda} \xrightarrow{\Psi} \Sigma_k$$

for $\lambda \geqslant \lambda_L = e^{L/(\rho - 8k\pi)}$ and large $L > 0$; moreover $\lim_{\lambda \to +\infty} \Psi \circ \Phi_\lambda = \mathrm{Id}$, and in particular, $\Psi \circ \Phi_\lambda$ is homotopic to the identity on Σ_k provided $\lambda \geqslant \lambda_L$. Here, J_ρ^a stands for a set of all the functions $u \in W^{1,2}(\Sigma)$ with $J_\rho(u) \leqslant a$ for any real number a. The second step is to set the suitable min-max value for J_ρ, namely,

$$\alpha_{\lambda,\rho} = \inf_{\gamma \in \Gamma_\lambda} \sup_{(\sigma,t) \in \widehat{\Sigma}_k} J_\rho(\gamma(\sigma, t)),$$

where $\widehat{\Sigma}_k = \Sigma_k \times [0,1]/(\Sigma_k \times \{0\})$ is a topological cone, and Γ_λ is a set of paths

$$\Gamma_\lambda = \{\gamma \in C^0(\widehat{\Sigma}_k, W^{1,2}(\Sigma)) : \gamma(\sigma, 1) = \Phi_\lambda(\sigma), \forall\, \sigma \in \Sigma_k\}.$$

The hypothesis $\rho \in (8k\pi, 8(k+1)\pi)$ and the fact that Σ_k is non-contractible lead to $\alpha_{\lambda,\rho} > -\infty$ for sufficiently large λ. The third step is to obtain critical points of J_ρ for $\rho \in \Lambda$, where Λ is a dense subset of $(8k\pi, 8(k+1)\pi)$, by using the monotonicity of $\alpha_{\lambda,\rho}/\rho$. The final step is to find critical points of J_ρ for any $\rho \in (8k\pi, 8(k+1)\pi)$, by using a compactness result of Li and Shafrir [24] and an improved Trudinger-Moser inequality due to Chen and Li [11]. Note that the last two steps are essentially done by Ding et al. [15].

This method was extensively used to deal with the problems of elliptic equations or systems involving exponential growth nonlinearities. For Toda systems, we refer the readers to [1,3,25,26] and the references therein. Recently, Sun et al. [30] extended Djadli's result to the case of a generalized mean field equation. Marchis et al. [12] employed it to find critical points of a Trudinger-Moser functional.

In this paper, we are concerned with the boundary value problems on the mean field equation. From now on, we let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$. Our first aim is to generalize the results of Ding et al. [15] and Chen and Lin [9]. Precisely, we have the following theorem.

Theorem 1.1. *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$, Δ_g be the Laplace-Beltrami operator, and $h : \overline{\Sigma} \to \mathbb{R}$ be a positive smooth function. If Σ is not simply connected, then for any $\rho \in (8k\pi, 8(k+1)\pi)$ with k a positive integer, the Dirichlet problem*

$$\begin{cases} \Delta_g u = \rho \dfrac{he^u}{\int_\Sigma he^u dv_g} & \text{in } \Sigma, \\ u = 0 & \text{on } \partial\Sigma \end{cases} \tag{1.3}$$

has a solution.

The proof of Theorem 1.1 is based on the min-max theorem [34, Theorem 2.8], which was also used by Ding et al. [15] and Djadli [17], compactness analysis, and an improved Trudinger-Moser inequality. All of the three parts are quite different from those of [15, 17]. On the choice of the metric space, we use $\widehat{\Sigma}_{\epsilon,k}$ (see (2.44) below) instead of $\widehat{\widehat{\Sigma}}_k$ or $\widehat{\Sigma}_k$; on compactness analysis, we use a reflection method different from that of Chen and Lin [9] to show that the blow-up phenomenon cannot occur on the boundary $\partial\Sigma$; moreover, we need to prove an improved Trudinger-Moser inequality for functions with boundary value zero, nor is it the original one in [11].

We also consider the Neumann boundary value problem on the mean field equation. In this regard, our second result is the following theorem.

Theorem 1.2. *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$, $\boldsymbol{v}$ be the unit normal outward vector on $\partial\Sigma$, Δ_g be the Laplace-Beltrami operator, and $h : \overline{\Sigma} \to \mathbb{R}$ be a positive smooth function. If $\rho \in (4k\pi, 4(k+1)\pi)$ with k a positive integer, then the Neumann boundary value problem*

$$\begin{cases} \Delta_g u = \rho \left(\dfrac{he^u}{\int_\Sigma he^u dv_g} - \dfrac{1}{|\Sigma|} \right) & \text{in } \Sigma, \\ \partial u / \partial \boldsymbol{v} = 0 & \text{on } \partial\Sigma \end{cases} \tag{1.4}$$

has a solution.

We remark that in Theorem 1.2, Σ does not need to be non-contractible. For the proof of Theorem 1.2, we choose a metric space $\widehat{\mathscr{S}_k}$ (see (3.27) below), which is non-contractible, whether Σ is non-contractible or not. Concerning the compactness of solutions to (1.4), if it has a sequence of blow-up solutions, then we show that $\rho = 4k\pi$ for $k \in \mathbb{N}^*$. Also we derive an improved Trudinger-Moser inequality for functions with integral mean zero, which is important in our analysis.

Before ending Section 1, we mention a recent result of Zhang et al. [40]. Using the min-max scheme of Djadli [17] and Djadli and Malchiodi [18], they obtained the existence of solutions to the equation

$$\begin{cases} \Delta_g u = 0 & \text{in } \Sigma, \\ \partial u / \partial \boldsymbol{v} = \rho \dfrac{he^u}{\int_{\partial\Sigma} he^u ds_g} & \text{on } \partial\Sigma \end{cases}$$

for any $\rho \in (2k\pi, 2(k+1)\pi)$, $k \in \mathbb{N}^*$, and any positive smooth function h. This improved an early result of Guo and Liu [19].

The rest of this paper is organized as follows. Theorems 1.1 and 1.2 are proved by the min-max method in Sections 2 and 3, respectively. Throughout this paper, the sequence and the subsequence are not distinguished. We often denote various constants by the same C from line to line, even in the same line. Sometimes we write $C_k, C_{k,\epsilon}, C(\epsilon), \ldots$ to emphasize the dependence of these constants.

2 The Dirichlet boundary value problem

In this section, Theorem 1.1 is proved. This will be divided into several subsections. In Subsection 2.1, we analyze the compactness of solutions to the Dirichlet problem (1.3). In Subsection 2.2, we derive an improved Trudinger-Moser inequality for functions $u \in W_0^{1,2}(\Sigma)$. In Subsection 2.3, we construct two continuous maps between sub-levels J_ρ^{-L} with sufficiently large L and the topological space Σ_k. In Subsection 2.4, we construct min-max levels of J_ρ, and ensure these min-max levels are finite. In Subsection 2.5, several uniform estimates on min-max levels of J_ρ are obtained. In Subsection 2.6, adapting the argument of [15, Lemma 3.2], we prove that J_ρ has a critical point for ρ in a dense subset of $(8k\pi, 8(k+1)\pi)$. In Subsection 2.7, using compactness of solutions to the Dirichlet problem (1.3), we conclude that J_ρ has a critical point for any $\rho \in (8k\pi, 8(k+1)\pi)$.

2.1 Compactness analysis

Let (ρ_n) be a sequence of numbers tending to ρ, (h_n) be a function sequence converging to h in $C^1(\overline{\Sigma})$, and (u_n) be a sequence of solutions to

$$\begin{cases} \Delta_g u_n = \rho_n \dfrac{h_n e^{u_n}}{\int_\Sigma h_n e^{u_n} dv_g} & \text{in } \Sigma, \\ u_n = 0 & \text{on } \partial\Sigma. \end{cases} \tag{2.1}$$

Define $v_n = u_n - \log \int_\Sigma h_n e^{u_n} dv_g$. Then $\Delta_g v_n = \rho_n h_n e^{v_n}$ and $\int_\Sigma h_n e^{v_n} dv_g = 1$.

Lemma 2.1. *Assume that ρ is a positive number and h is a positive function. Up to a subsequence, one of the following alternatives holds*:

(i) *(u_n) is bounded in $L^\infty(\overline{\Sigma})$;*

(ii) *(v_n) converges to $-\infty$ uniformly in $\overline{\Sigma}$;*

(iii) *there exists a finite singular set $\mathcal{S} = \{p_1, \ldots, p_m\} \subset \Sigma$ such that for any $1 \leqslant j \leqslant m$, there is a sequence of points $\{p_{j,n}\} \subset \Sigma$ satisfying $p_{j,n} \to p_j$, $u_n(p_{j,n}) \to +\infty$, and v_n converges to $-\infty$ uniformly on any compact subset of $\overline{\Sigma} \setminus \mathcal{S}$ as $n \to \infty$. Moreover,*

$$\rho_n \int_\Sigma h_n e^{v_n} dv_g \to 8m\pi.$$

Proof. Note that $\overline{\Sigma} = \Sigma \cup \partial\Sigma$, where Σ is an open set including all the inner points of $\overline{\Sigma}$, and $\partial\Sigma$ is its boundary. The compactness analysis on (u_n) will be divided into two parts.

Part I. Analysis in the interior domain Σ.

According to an observation in [37, Subsection 4.1] (compared with [2, Theorem 4.17]), by the Green representation formula for functions with boundary value zero, we have

$$\|u_n\|_{W_0^{1,q}(\Sigma)} \leqslant C_q, \quad \forall\, 1 < q < 2. \tag{2.2}$$

We claim that there exists some constant $c_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\int_\Sigma h_n e^{u_n} dv_g \geqslant c_0. \tag{2.3}$$

Suppose the claim is not true. By Jensen's inequality,

$$\mathrm{e}^{\frac{1}{|\Sigma|}\int_\Sigma u_n dv_g} \leqslant \frac{1}{|\Sigma|}\int_\Sigma \mathrm{e}^{u_n} dv_g \to 0.$$

Thus $\int_\Sigma u_n dv_g \to -\infty$, which contradicts (2.2), and concludes our claim (2.3).

To proceed, we assume that $\rho_n h_n \mathrm{e}^{v_n} dv_g$ converges to some nonnegative measure μ. If $\mu(x^*) < 4\pi$ for some $x^* \in \Sigma$, then there exist two positive constants ϵ_0 and r_0 such that

$$\int_{B_{x^*}(r_0)} \rho_n h_n \mathrm{e}^{v_n} dv_g \leqslant 4\pi - \epsilon_0.$$

In view of (2.1), by Brezis-Merle's theorem [5, Theorem 1] and elliptic estimates, we have that (u_n) is bounded in $L^\infty(B_{x^*}(r_0/2))$. This leads to $\mu(x^*) = 0$. Define a set $\mathcal{S} = \{x \in \Sigma : \mu(x) \geqslant 4\pi\}$.

If $\mathcal{S} \neq \emptyset$, then we shall show that for any compact set $A \subset \Sigma \setminus \mathcal{S}$, it holds that

$$v_n \to -\infty \quad \text{uniformly in } x \in A. \tag{2.4}$$

It suffices to prove that

$$\int_\Sigma h_n \mathrm{e}^{u_n} dv_g \to +\infty. \tag{2.5}$$

Suppose that (2.5) does not hold. In view of (2.3), there is a constant c_1 such that up to a subsequence,

$$0 < c_0 \leqslant \int_\Sigma h_n \mathrm{e}^{u_n} dv_g \leqslant c_1.$$

Choose $x_0 \in \mathcal{S}$ and $0 < r_0 < \mathrm{dist}(x_0, \partial\Sigma)$ satisfying $B_{x_0}(r_0) \cap \mathcal{S} = \{x_0\}$. Note that (u_n) is locally uniformly bounded in $\Sigma \setminus \mathcal{S}$. There exists a positive constant c_2 depending on x_0 and r_0 such that $|v_n(x)| \leqslant c_2$ for all $x \in \partial B_{x_0}(r_0)$. Let w_n be a solution to

$$\begin{cases} \Delta_g w_n = \rho_n h_n \mathrm{e}^{v_n} & \text{in } B_{x_0}(r_0), \\ w_n = -c_2 & \text{on } \partial B_{x_0}(r_0). \end{cases}$$

Then the maximum principle implies that $w_n \leqslant v_n$ in $B_{x_0}(r_0)$. By the Green formula, w_n converges to w weakly in $W^{1,q}(B_{x_0}(r_0))$ and a.e. in $B_{x_0}(r_0)$. Moreover, w is a solution of

$$\begin{cases} \Delta_g w = \mu & \text{in } B_{x_0}(r_0), \\ w = -c_2 & \text{on } \partial B_{x_0}(r_0). \end{cases}$$

Let G_{x_0} be a distributional solution of

$$\begin{cases} \Delta_g G_{x_0} = 4\pi\delta_{x_0} & \text{in } B_{x_0}(r_0), \\ G_{x_0} = -c_2 & \text{on } \partial B_{x_0}(r_0). \end{cases}$$

Clearly, G_{x_0} is represented by

$$G_{x_0}(x) = -2\log \mathrm{dist}(x, x_0) + A_{x_0} + o(1), \tag{2.6}$$

where A_{x_0} is a constant, and $o(1) \to 0$ as $x \to x_0$. Since

$$\begin{cases} \Delta_g(w - G_{x_0}) \geqslant 0 & \text{in } B_{x_0}(r_0), \\ w - G_{x_0} = 0 & \text{on } \partial B_{x_0}(r_0), \end{cases}$$

it follows from the maximum principle that

$$w(x) \geqslant G_{x_0}(x) \quad \text{for all } x \in B_{x_0}(r_0) \setminus \{x_0\}. \tag{2.7}$$

Combining (2.6), (2.7) and the fact that $w_n \to w$ a.e. in $B_{x_0}(r_0)$, by Fatou's lemma, we calculate

$$+\infty = \int_{B_{x_0}(r_0)} \mathrm{e}^{G_{x_0}} dv_g \leqslant \int_{B_{x_0}(r_0)} \mathrm{e}^{w} dv_g \leqslant \liminf_{n\to\infty} \int_{B_{x_0}(r_0)} \mathrm{e}^{w_n} dv_g \leqslant \liminf_{n\to\infty} \int_{B_{x_0}(r_0)} \mathrm{e}^{v_n} dv_g \leqslant C.$$

This is impossible and excludes the possibility of (2.5). Hence we conclude (2.4).

We may assume $\mathcal{S} = \{x_1, \dots, x_m\}$. Then it holds that $\mu(x_i) = 8\pi$ for all $1 \leqslant i \leqslant m$. Without loss of generality, it suffices to prove $\mu(x_1) = 8\pi$. Choose an isothermal coordinate system $\phi : U \to \mathbb{B}_1(0)$ near x_1. In such coordinates, the metric g and the Laplace-Beltrami operator Δ_g are represented by $g = \mathrm{e}^{\psi(y)}(dy_1^2 + dy_2^2)$ and $\Delta_g = -\mathrm{e}^{-\psi(y)}\Delta_{\mathbb{R}^2}$, respectively, where ψ is a smooth function with $\psi(0,0) = 0$, and $\Delta_{\mathbb{R}^2} = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ denotes the standard Laplacian on $\mathbb{R}^2$. Set $\widetilde{u} = u \circ \phi^{-1}$ for any function $u : U \to \mathbb{R}$. Since (u_n) is a sequence of solutions to (2.1), $\widetilde{u}_n = u_n \circ \phi^{-1}$ satisfies

$$-\Delta_{\mathbb{R}^2}\widetilde{u}_n(y) = \mathrm{e}^{\psi(y)}\rho_n \widetilde{h}_n(y)\mathrm{e}^{\widetilde{v}_n(y)}, \quad y \in \mathbb{B}_1. \tag{2.8}$$

Multiplying both sides of (2.8) by $y \cdot \nabla_{\mathbb{R}^2}\widetilde{u}_n(y)$, by integration by parts, we have

$$\begin{aligned}\frac{r}{2}\int_{\partial\mathbb{B}_r} |\nabla_{\mathbb{R}^2}\widetilde{u}_n|^2 d\sigma - r\int_{\partial\mathbb{B}_r} \langle\nabla_{\mathbb{R}^2}\widetilde{u}_n, \nu\rangle^2 d\sigma = r\int_{\partial\mathbb{B}_r} \mathrm{e}^{\psi}\rho_n\widetilde{h}_n\mathrm{e}^{\widetilde{v}_n} d\sigma - \int_{\mathbb{B}_r} \mathrm{e}^{\widetilde{v}_n}\rho_n\langle\nabla_{\mathbb{R}^2}(\mathrm{e}^{\psi}\widetilde{h}_n), y\rangle dy \\ - 2\int_{\mathbb{B}_r} \mathrm{e}^{\psi}\rho_n\widetilde{h}_n\mathrm{e}^{\widetilde{v}_n} dy,\end{aligned} \tag{2.9}$$

where $\mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 < r\}$, $\partial\mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = r\}$, and ν denotes the unit outward vector on $\partial\mathbb{B}_r$. In view of (2.4), (u_n) converges to a Green function $G(x, \cdot)$ weakly in $W_0^{1,q}(\Sigma)$ and in $C^2_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$. Locally, $G(x_1, \cdot)$ satisfies

$$\Delta_{g,z}G(x_1, z) = \mu(x_1)\delta_{x_1}(z), \quad \forall\, z \in \phi^{-1}(\mathbb{B}_1).$$

Clearly,

$$\widetilde{G}(y) = G(x_1, \phi^{-1}(y)) = -\frac{\mu(x_1)}{2\pi}\log|y| + \eta(y)$$

for some $\eta \in C^2(\mathbb{B}_1)$. Passing to the limit $n \to \infty$ first, and then $r \to 0$ in (2.9), we obtain

$$\mu(x_1) = \lim_{r\to 0}\left(\frac{r}{2}\int_{\partial\mathbb{B}_r} \langle\nabla_{\mathbb{R}^2}\widetilde{G}, \nu\rangle^2 d\sigma - \frac{r}{4}\int_{\partial\mathbb{B}_r} |\nabla_{\mathbb{R}^2}\widetilde{G}|^2 d\sigma\right) = \frac{(\mu(x_1))^2}{8\pi}.$$

This immediately leads to $\mu(x_1) = 8\pi$.

Part II. Analysis on the boundary $\partial\Sigma$.

Let $x^* \in \Sigma$ be fixed. Note that $\rho_n h_n \mathrm{e}^{v_n} dv_g$ converges to the nonnegative Radon measure μ on $\overline{\Sigma}$. If $\mu(x^*) < 2\pi$, there exist a neighborhood V of x^* and a number $\gamma_0 > 0$ such that

$$\int_V \rho_n h_n \mathrm{e}^{v_n} dv_g \leqslant 2\pi - \gamma_0. \tag{2.10}$$

Without loss of generality, we take an isothermal coordinate system $(V, \phi, \{y_1, y_2\})$ such that $\phi(x^*) = (0,0)$, and $\phi : V \to \mathbb{B}_1^+ \cup \Gamma = \{(y_1, y_2) : y_2 \geqslant 0\}$, where $\Gamma = \{(y_1, y_2) : |y_1| < 1, y_2 = 0\}$. Moreover, in this coordinate system, the metric $g = \mathrm{e}^{\psi(y)}(dy_1^2 + dy_2^2)$ and the Laplace-Beltrami operator $\Delta_g = -\mathrm{e}^{-\psi(y)}\Delta_{\mathbb{R}^2}$, where $\psi : \mathbb{B}_1^+ \cup \Gamma \to \mathbb{R}$ is a smooth function with $\psi(0,0) = 0$. For more details about isothermal coordinates on the boundary, we refer the readers to [38]. Now the local version of (2.1) is

$$\begin{cases} -\Delta_{\mathbb{R}^2}(u_n \circ \phi^{-1})(y) = \mathrm{e}^{\psi(y)}\rho_n(h_n \circ \phi^{-1})(y)\mathrm{e}^{(v_n\circ\phi^{-1})(y)} & \text{in } \mathbb{B}_1^+, \\ u_n \circ \phi^{-1}(y) = 0 & \text{on } \Gamma. \end{cases} \tag{2.11}$$

For any function $u : V \to \mathbb{R}$, we define a function $\widetilde{u} : \mathbb{B}_1^+ \cup \Gamma \to \mathbb{R}$ by

$$\widetilde{u}(y_1, y_2) = \begin{cases} u \circ \phi^{-1}(y_1, y_2), & \text{if } y_2 \geqslant 0, \\ -u \circ \phi^{-1}(y_1, -y_2), & \text{if } y_2 < 0. \end{cases} \tag{2.12}$$

One can easily check that $\widetilde{u}_n$ is a distributional solution of

$$-\Delta_{\mathbb{R}^2}\widetilde{u}_n(y) = \widetilde{f}_n(y), \quad y \in \mathbb{B}_1, \tag{2.13}$$

where $\widetilde{f}_n$ is defined as in (2.12) and for $y \in \mathbb{B}_1^+ \cup \Gamma$,

$$f_n \circ \phi^{-1}(y) = \mathrm{e}^{\psi(y)}\rho_n(h_n \circ \phi^{-1})(y)\mathrm{e}^{(v_n\circ\phi^{-1})(y)}.$$

In view of (2.10) and the fact that $\psi(0,0)=0$, there would exist a number $r_0 \in (0,1)$ such that

$$\int_{\mathbb{B}_{r_0}} |\widetilde{f}_n(y)|dy \leqslant 4\pi - \gamma_0.$$

Let w_n be a solution of

$$\begin{cases} -\Delta_{\mathbb{R}^2} w_n = \widetilde{f}_n & \text{in } \mathbb{B}_{r_0}, \\ w_n = 0 & \text{on } \partial\mathbb{B}_{r_0}. \end{cases}$$

By Brezis-Merle's theorem [5, Theorem 1], there exists some constant C depending only on γ_0 and r_0 such that

$$\int_{\mathbb{B}_{r_0}} \exp\left(\frac{(4\pi - \gamma_0/2)|w_n|}{\|\widetilde{f}_n\|_{L^1(\mathbb{B}_{r_0})}}\right)dy \leqslant C.$$

Hence, there exists some $q_0 > 1$ such that

$$\|\mathrm{e}^{|w_n|}\|_{L^{q_0}(\mathbb{B}_{r_0})} \leqslant C. \tag{2.14}$$

Let $\eta_n = \widetilde{u}_n - w_n$. Then η_n satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2} \eta_n = 0 & \text{in } \mathbb{B}_{r_0}, \\ \eta_n = \widetilde{u}_n & \text{on } \partial\mathbb{B}_{r_0}. \end{cases} \tag{2.15}$$

Noticing (2.2) and (2.14), we have by applying elliptic estimates to (2.15) that

$$\|\eta_n\|_{L^\infty(\mathbb{B}_{r_0/2})} \leqslant C. \tag{2.16}$$

Combining (2.3), (2.14) and (2.16), we conclude $\|\widetilde{f}_n\|_{L^{q_0}(\mathbb{B}_{r_0/2})} \leqslant C$. Applying elliptic estimates to (2.13), we obtain that $\|\widetilde{u}_n\|_{L^\infty(\mathbb{B}_{r_0/4})} \leqslant C$, which implies $\|u_n\|_{L^\infty(\phi^{-1}(\mathbb{B}^+_{r_0/4}))} \leqslant C$. In conclusion, we have that if $\mu(x^*) < 2\pi$, then (u_n) is uniformly bounded near x^*. This also leads to $\mu(x^*) = 0$.

If $\mu(x^*) \geqslant 2\pi$, in the same coordinate system $(V, \phi, \{y_1, y_2\})$ as above, $\widetilde{f}_n(y)dy$ converges to a Radon measure $\widetilde{\mu}$ with $\widetilde{\mu}(0,0) \geqslant 4\pi$. Obviously, there exists some $r_1 > 0$ such that for any $x \in \mathbb{B}_{r_1} \setminus \{(0,0)\}$, $\widetilde{\mu}(x) = 0$. Using the same argument as the proof of (2.4), we conclude that for any compact set $A \subset \mathbb{B}_{r_1} \setminus \{(0,0)\}$, $\widetilde{v}_n$ converges to $-\infty$ uniformly in A. This leads to $\widetilde{f}_n(y)dy$ converging to the Dirac measure $\widetilde{\mu}(0,0)\delta_{(0,0)}(y)$. Recalling (2.2), we have that $\widetilde{u}_n$ converges to $\widetilde{G}_0$ weakly in $W^{1,q}(\mathbb{B}_{r_1})$ and a.e. in $\mathbb{B}_{r_1}$, where $\widetilde{G}_0$ satisfies

$$-\Delta_{\mathbb{R}^2}\widetilde{G}_0(y) = \widetilde{\mu}(0,0)\delta_{(0,0)}(y), \quad y \in \mathbb{B}_{r_1}.$$

Clearly, $\widetilde{G}_0$ is represented by

$$\widetilde{G}_0(y) = -\frac{\widetilde{\mu}(0,0)}{2\pi}\log|y| + A_0 + O(|y|) \tag{2.17}$$

as $y \to 0$, where A_0 is a constant. Noting that $\widetilde{v}_n$ converges to $-\infty$ locally uniformly in $\mathbb{B}_{r_1} \setminus \{(0,0)\}$, we have by applying elliptic estimates to (2.13) that

$$\widetilde{u}_n \to \widetilde{G}_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{B}_{r_1} \setminus \{(0,0)\}). \tag{2.18}$$

By (2.11), $\widetilde{u}_n(y_1, 0) = 0$ for all $|y_1| < 1$, which together with (2.18) leads to $\widetilde{G}_0(y_1, 0) = 0$ for all $0 < |y_1| < r_1$. This contradicts (2.17). Therefore,

$$\{x \in \partial\Sigma : \mu(x) \geqslant 2\pi\} = \emptyset.$$

Combining Parts I and II, we conclude the lemma. □

2.2 An improved Trudinger-Moser inequality

In this subsection, we shall derive an improved Trudinger-Moser inequality, which is an analog of that of Chen and Li [11]. It is known (see, for example, [20]) that

$$\log \int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dv_g + C, \quad \forall\, u \in W_0^{1,2}(\Sigma). \tag{2.19}$$

Lemma 2.2. *Let $b_0 > 0$ and $\gamma_0 > 0$ be two constants, and $\Omega_1, \ldots, \Omega_k$ be k domains of $\overline{\Sigma}$ satisfying $\mathrm{dist}(\Omega_i, \Omega_j) \geqslant b_0$ for all $1 \leqslant i < j \leqslant k$. Then for any $\epsilon > 0$, there exists some constant C depending only on b_0, γ_0, k and ϵ such that*

$$\log \int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{16k\pi - \epsilon} \int_\Sigma |\nabla_g u|^2 dv_g + C \tag{2.20}$$

for all $u \in W_0^{1,2}(\Sigma)$ with

$$\int_{\Omega_i} \mathrm{e}^u dv_g \geqslant \gamma_0 \int_\Sigma \mathrm{e}^u dv_g, \quad i = 1, \ldots, k. \tag{2.21}$$

Proof. We modify an argument of Chen and Li [11]. Take smooth functions $\phi_1, \ldots, \phi_k$ defined on $\overline{\Sigma}$ satisfying

$$\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j = \emptyset, \quad \forall\, 1 \leqslant i < j \leqslant k, \tag{2.22}$$

$$\phi_i \equiv 1 \quad \text{on } \Omega_i, \quad 0 \leqslant \phi_i \leqslant 1 \quad \text{on } \overline{\Sigma}, \quad \forall\, 1 \leqslant i \leqslant k, \tag{2.23}$$

and for some positive constant b_1 depending only on b_0 and g,

$$|\nabla_g \phi_i| \leqslant b_1, \quad \forall\, 1 \leqslant i \leqslant k. \tag{2.24}$$

For any $u \in W_0^{1,2}(\Sigma)$ satisfying (2.21), we have $\phi_i u \in W_0^{1,2}(\Omega)$ for all $1 \leqslant i \leqslant k$, and thus (2.19) implies

$$\begin{aligned}
\int_\Sigma \mathrm{e}^u dv_g &\leqslant \frac{1}{\gamma_0} \int_{\Omega_i} \mathrm{e}^u dv_g \\
&\leqslant \frac{1}{\gamma_0} \int_\Sigma \mathrm{e}^{\phi_i u} dv_g \\
&\leqslant \frac{1}{\gamma_0} \exp\left(\frac{1}{16\pi} \|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)} + C \right).
\end{aligned}$$

Recall an elementary inequality: if $a \leqslant a_i$ for nonnegative numbers a and a_i, $i = 1, \ldots, k$, then $a \leqslant (a_1 \cdots a_k)^{1/k}$. In view of (2.22)–(2.24), we have

$$\begin{aligned}
\int_\Sigma \mathrm{e}^u dv_g &\leqslant \frac{1}{\gamma_0} \left(\prod_{i=1}^k \exp\left(\frac{1}{16\pi} \|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)} + C \right) \right)^{1/k} \\
&= \frac{\mathrm{e}^C}{\gamma_0} \exp\left(\frac{1}{16k\pi} \sum_{i=1}^k \|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)} \right) \\
&= \frac{\mathrm{e}^C}{\gamma_0} \exp\left(\frac{1}{16k\pi} \left\| \nabla_g \left(u \sum_{i=1}^k \phi_i \right) \right\|^2_{L^2(\Sigma)} \right) \\
&\leqslant C \exp\left(\frac{1}{16k\pi} (1 + \epsilon_1) \|\nabla_g u\|^2_{L^2(\Sigma)} + C(\epsilon_1) \|u\|^2_{L^2(\Sigma)} \right).
\end{aligned} \tag{2.25}$$

Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_\ell \leqslant \lambda_{\ell+1} \leqslant \cdots$ be all the eigenvalues of the Laplace-Beltrami operator with respect to the Dirichlet boundary condition with $\lambda_i \to +\infty$ as $i \to \infty$, and $\{e_i\}_{i=1}^\infty$ be the corresponding unit normal eigenfunctions, i.e., $\Delta_g e_i = \lambda_i e_i$ and $\int_\Sigma e_i e_j dv_g = \delta_{ij}$ for $i, j = 1, 2, \ldots$ It is known that

$W_0^{1,2}(\Sigma) = E_\ell \oplus E_\ell^\perp$, where $E_\ell = \mathrm{span}\{e_1, \ldots, e_\ell\}$ and $E_\ell^\perp = \{e_{\ell+1}, e_{\ell+2}, \ldots\}$. Let $u \in W_0^{1,2}(\Sigma)$ be as above. Write $u = v + w$ with $v \in E_\ell$ and $w \in E_\ell^\perp$. Thus the Poincaré inequality implies

$$\|v\|_{C^0(\overline{\Sigma})} \leqslant \sum_{i=1}^{\ell} \|e_i\|_{C^0(\overline{\Sigma})} \int_\Sigma |u||e_i| dv_g \leqslant C_\ell \|\nabla_g u\|_{L^2(\Sigma)},$$

while by the definition of the $(\ell+1)$-th eigenvalue,

$$\int_\Sigma w^2 dv_g \leqslant \frac{1}{\lambda_{\ell+1}} \int_\Sigma |\nabla_g w|^2 dv_g.$$

Having the above two estimates and applying (2.25) to w, we have

$$\begin{aligned}\int_\Sigma \mathrm{e}^u dv_g &\leqslant \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \int_\Sigma \mathrm{e}^w dv_g \\ &\leqslant C \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \exp\left(\frac{1}{16k\pi}(1+\epsilon_1)\|\nabla_g w\|_{L^2(\Sigma)}^2 + \frac{C(\epsilon_1)}{\lambda_{\ell+1}}\|\nabla_g w\|_{L^2(\Sigma)}^2\right) \\ &\leqslant C \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \exp\left(\frac{1}{16k\pi}\left(1+\epsilon_1\frac{C(\epsilon_1)}{\lambda_{\ell+1}}\right)\|\nabla_g u\|_{L^2(\Sigma)}^2\right).\end{aligned}$$

This together with Young's inequality gives

$$\log \int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{16k\pi}\left(1 + \epsilon_1 \frac{C(\epsilon_1)}{\lambda_{\ell+1}} + \epsilon_1\right) \int_\Sigma |\nabla_g u|^2 dv_g + C_{\ell,k,\epsilon_1}. \tag{2.26}$$

Let $\epsilon > 0$ be any given number. Choosing $\epsilon_1 = \epsilon/(32k\pi - 2\epsilon)$, and then taking a sufficiently large ℓ such that $C(\epsilon_1)/\lambda_{\ell+1} \leqslant 1$ in (2.26), we immediately have

$$\log \int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{16k\pi - \epsilon} \int_\Sigma |\nabla_g u|^2 dv_g + C,$$

where C is a constant depending only on b_0, γ_0, k and ϵ. This is exactly (2.20). □

For any $\rho > 0$, we define a functional $J_\rho : W_0^{1,2}(\Sigma) \to \mathbb{R}$ by

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 dv_g - \rho \log \int_\Sigma h\mathrm{e}^u dv_g. \tag{2.27}$$

Clearly $J_\rho \in C^2(W_0^{1,2}(\Sigma), \mathbb{R})$. To find solutions of the mean field equation (1.3), it suffices to find critical points of J_ρ. For any $a \in \mathbb{R}$, the sub-level of J_ρ is written as $J_\rho^a = \{u \in W_0^{1,2}(\Sigma) : J_\rho(u) \leqslant a\}$.

Let Σ_k be the formal set of barycenters of Σ (of order k), which is

$$\Sigma_k = \left\{\sum_{i=1}^k t_i \delta_{x_i} : t_i \geqslant 0, x_i \in \Sigma, \sum_{i=1}^k t_i = 1\right\}. \tag{2.28}$$

It is endowed with the weak topology of distributions. In computation, we use on Σ_k the metric given by $(C^1(\overline{\Sigma}))^*$ inducing the same topology. Similarly, we may define

$$\overline{\Sigma}_k = \left\{\sum_{i=1}^k t_i \delta_{x_i} : t_i \geqslant 0, x_i \in \overline{\Sigma}, \sum_{i=1}^k t_i = 1\right\}. \tag{2.29}$$

2.3 Continuous maps between sub-levels of J_ρ and Σ_k

Let J_ρ, Σ_k and $\overline{\Sigma}_k$ be defined as in (2.27)–(2.29), respectively. In this subsection, we shall construct continuous maps between sub-levels of J_ρ and Σ_k (or $\overline{\Sigma}_k$).

Lemma 2.3. *For any $k \geqslant 1$, both Σ_k and $\overline{\Sigma}_k$ are non-contractible.*

Proof. Since Σ_k is homotopic to $\overline{\Sigma}_k$, we only need to prove that Σ_k is non-contractible. Let $\chi(\Sigma)$ be the Euler characteristic of Σ. By [21, Corollary 1.4(a)],

$$\chi(\Sigma_k) = 1 - \frac{1}{k!}(1-\chi(\Sigma))\cdots(k-\chi(\Sigma)). \tag{2.30}$$

Denote the genus of Σ by $\mathfrak{g}$ and the number of the connected components of $\partial\Sigma$ by m. Notice that Σ is simply connected if and only if the Euler characteristic of Σ equals 1. By the assumption, it then follows that

$$\chi(\Sigma) = 2 - 2\mathfrak{g} - m \leqslant 0. \tag{2.31}$$

Inserting (2.31) into (2.30), we have

$$\chi(\Sigma_k) \leqslant 0. \tag{2.32}$$

On the other hand, it holds that

$$\chi(\Sigma_k) = \sum_{j=0}^{3k-1}(-1)^j \mathrm{dim} H_j(\Sigma_k, \mathbb{Z}). \tag{2.33}$$

Suppose that Σ_k is contractible. Then $\mathrm{dim}H_j(\Sigma_k,\mathbb{Z}) = 0$ for all $j \geqslant 1$, while $\mathrm{dim}H_0(\Sigma_k,\mathbb{Z}) = 1$, since Σ_k is connected. Hence (2.33) gives $\chi(\Sigma_k) = 1$, contradicting (2.32). Hence Σ_k is non-contractible. □

Lemma 2.4. *Let $\rho \in (8k\pi, 8(k+1)\pi)$. Then for any sufficiently large $L > 0$, there exists a continuous retraction*

$$\Psi : J_\rho^{-L} = \{u \in W_0^{1,2}(\Sigma) : J_\rho(u) \leqslant -L\} \to \overline{\Sigma}_k.$$

Moreover, if $(u_n) \subset W_0^{1,2}(\Sigma)$ satisfies $\frac{\mathrm{e}^{u_n}}{\int_\Sigma \mathrm{e}^{u_n} dv_g} dv_g \to \sigma \in \overline{\Sigma}_k$, then $\Psi(u_n) \to \sigma \in \overline{\Sigma}_k$.

Proof. By [17, Proposition 4.1], for any $\epsilon > 0$, there exists an $L_0 > 0$ such that for all $u \in J_\rho^{-L_0}$,

$$\frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u dv_g} dv_g \in \{\sigma \in \mathcal{D}(\overline{\Sigma}) : \mathbf{d}(\sigma, \overline{\Sigma}_k) < \epsilon\},$$

where $\mathcal{D}(\overline{\Sigma})$ denotes the set of all the distributions on $\overline{\Sigma}$. If $\epsilon_0 > 0$ is sufficiently small, then there exists a continuous retraction

$$\psi_k : \{\sigma \in \mathcal{D}(\overline{\Sigma}) : \mathbf{d}(\sigma, \overline{\Sigma}_k) < \epsilon_0\} \to \overline{\Sigma}_k. \tag{2.34}$$

For sufficiently large $L > 0$, we set

$$\Psi(u) = \psi_k\left(\frac{\mathrm{e}^u}{\int_\Sigma \mathrm{e}^u dv_g} dv_g\right), \quad \forall\, u \in J_\rho^{-L}.$$

As a consequence, we have a continuous map $\Psi : J_\rho^{-L} \to \overline{\Sigma}_k$.

Moreover, if $(u_n) \subset W_0^{1,2}(\Sigma)$ satisfies $\frac{\mathrm{e}^{u_n}}{\int_\Sigma \mathrm{e}^{u_n} dv_g} dv_g \to \sigma \in \overline{\Sigma}_k$, then as $n \to \infty$,

$$\Psi(u_n) = \psi_k\left(\frac{\mathrm{e}^{u_n}}{\int_\Sigma \mathrm{e}^{u_n} dv_g} dv_g\right) \to \psi_k(\sigma) = \sigma,$$

as desired. □

Let $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$ be fixed. Take a smooth increasing function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying $\eta(t) = t$ for $t \leqslant 1$, and $\eta(t) = 2$ for $t \geqslant 2$. Set $\eta_r(t) = r\eta(t/r)$ for $r > 0$. For $\lambda > 0$ and $x \in \overline{\Sigma}$, we define

$$\widetilde{\phi}_{\lambda,\sigma}(x) = \log\left(\sum_{i=1}^k t_i \frac{8\lambda^2}{(1+\lambda^2\eta_r^2(\mathrm{dist}(x,x_i)))^2}\right) \tag{2.35}$$

and

$$\phi_{\lambda,\sigma}(x) = \widetilde{\phi}_{\lambda,\sigma}(x) - \log\frac{8\lambda^2}{(1+4\lambda^2 r^2)^2}. \tag{2.36}$$

Lemma 2.5. *Let $\rho \in (8k\pi, 8(k+1)\pi)$ and $\epsilon > 0$. If $\lambda > 0$ is chosen sufficiently large, and $r > 0$ is chosen sufficiently small, then for any $\sigma \in \Sigma_k$ with $\mathrm{dist}(\mathrm{supp}\,\sigma, \partial\Sigma) \geqslant \epsilon$, it holds that*

$$J_\rho(\phi_{\lambda,\sigma}) \leqslant (8k\pi - \rho)\log\lambda \tag{2.37}$$

and

$$\frac{\mathrm{e}^{\phi_{\lambda,\sigma}}}{\int_\Sigma \mathrm{e}^{\phi_{\lambda,\sigma}} dv_g} dv_g \to \sigma \quad \text{as } \lambda \to +\infty. \tag{2.38}$$

Proof. Given $\sigma \in \Sigma_k$, without loss of generality, we assume $\mathrm{supp}\,\sigma = \{x_1, \ldots, x_k\} \subset \Sigma$. Let $\widetilde{\phi}_{\lambda,\sigma}$ and $\phi_{\lambda,\sigma}$ be defined as in (2.35) and (2.36), respectively, where $\lambda > 0$ and $0 < r < \epsilon/4$. Write $r_i = r_i(x) = \mathrm{dist}(x, x_i)$ for $x \in \overline{\Sigma}$. A simple observation gives

$$\widetilde{\phi}_{\lambda,\sigma}(x) = \begin{cases} \log \dfrac{8\lambda^2}{(1+4\lambda^2 r^2)^2} & \text{for } x \in \Sigma \setminus \bigcup_{i=1}^k B_{2r}(x_i), \\ \log\left(\dfrac{8\lambda^2 t_i}{(1+\lambda^2\eta_r^2(r_i))^2} + \dfrac{8\lambda^2(1-t_i)}{(1+4\lambda^2 r^2)^2}\right) & \text{for } x \in B_{2r}(x_i). \end{cases} \tag{2.39}$$

As a consequence, $\phi_{\lambda,\sigma} \in W_0^{1,2}(\Sigma)$. For $x \in B_{2r}(x_i)$, a straightforward calculation shows

$$\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x) = \frac{\frac{8\lambda^2 t_i}{(1+\lambda^2\eta_r^2(r_i))^2}}{\frac{8\lambda^2 t_i}{(1+\lambda^2\eta_r^2(r_i))^2} + \frac{8\lambda^2(1-t_i)}{(1+4\lambda^2r^2)^2}} \frac{4\lambda^2\eta_r(r_i)\eta_r'(r_i)\nabla_g r_i}{1+\lambda^2\eta_r^2(r_i)},$$

and thus

$$|\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x)| \leqslant \frac{4\lambda^2\eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)}.$$

In view of (2.39), it holds that $\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x) = 0$ for $x \in \Sigma \setminus \bigcup_{i=1}^k B_{2r}(x_i)$. Hence,

$$\begin{aligned} \int_\Sigma |\nabla_g \widetilde{\phi}_{\lambda,\sigma}|^2 dv_g &= \int_{\bigcup_{i=1}^k B_{2r}(x_i)} |\nabla_g \widetilde{\phi}_{\lambda,\sigma}|^2 dv_g \\ &\leqslant \sum_{i=1}^k \int_{B_{2r}(x_i)} \left(\frac{4\lambda^2\eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)}\right)^2 dv_g \\ &= \sum_{i=1}^k 16\pi(1+O(r^2))\left(\log(1+\lambda^2r^2) + \frac{1}{1+\lambda^2r^2} - 1\right) + O(1) \\ &\leqslant 16k\pi(1+O(r^2))\log\lambda^2 + C \end{aligned} \tag{2.40}$$

for some constant C independent of r and λ. Moreover, for any s with

$$0 < s < \min\Big\{r, \min_{1\leqslant i<j\leqslant k} \mathrm{dist}(x_i, x_j)\Big\},$$

it holds that

$$\begin{aligned} \int_{\bigcup_{i=1}^k B_{2r}(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g &= \int_{\bigcup_{i=1}^k B_s(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g + O\left(\frac{1}{\lambda^2 s^2}\right) \\ &= \sum_{i=1}^k \int_{B_s(x_i)} \frac{8\lambda^2 t_i}{(1+\lambda^2 r_i^2)^2} dv_g + O\left(\frac{1}{\lambda^2 s^2}\right) \\ &= 8\pi(1+O(s^2)) + O\left(\frac{1}{\lambda^2 s^2}\right) \end{aligned}$$

and

$$\int_{\Sigma\setminus\bigcup_{i=1}^k B_{2r}(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = O\left(\frac{1}{\lambda^2 r^4}\right).$$

It follows that

$$\int_{\Sigma} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = 8\pi(1+O(s^2)) + O\bigg(\frac{1}{\lambda^2 s^2}\bigg) + O\bigg(\frac{1}{\lambda^2 r^4}\bigg). \tag{2.41}$$

Passing to the limit $\lambda \to +\infty$ first, and then $s \to 0+$, we have

$$\lim_{\lambda\to+\infty} \int_{\Sigma} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = 8\pi. \tag{2.42}$$

Hence,

$$\int_{\Sigma} \mathrm{e}^{\phi_{\lambda,\sigma}} dv_g = \int_{\Sigma} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g \frac{(1+4\lambda^2 r^2)^2}{8\lambda^2} = (8\pi + o_\lambda(1))\lambda^2 r^4. \tag{2.43}$$

Combining (2.40) and (2.43), we obtain

$$\begin{aligned} J_\rho(\phi_{\lambda,\sigma}) &= \frac{1}{2}\int_{\Sigma} |\nabla_g \phi_{\lambda,\sigma}|^2 dv_g - \rho \log \int_{\Sigma} h\mathrm{e}^{\phi_{\lambda,\sigma}} dv_g \\ &\leqslant (16k\pi - 2\rho + O(r^2))\log\lambda + C_r. \end{aligned}$$

Since $\rho > 8k\pi$, choosing $r > 0$ sufficiently small and $\lambda > 0$ sufficiently large, we conclude (2.37).

Finally, we prove (2.38). Let $\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$ be as above. For any $\varphi \in C^1(\overline{\Sigma})$, similar to (2.41), we calculate

$$\int_{\Sigma} \varphi \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = 8\pi \sum_{i=1}^k t_i \varphi(x_i) + O(s^2) + O\bigg(\frac{1}{\lambda^2 s^2}\bigg) + O\bigg(\frac{1}{\lambda^2 r^4}\bigg).$$

Letting $\lambda \to +\infty$ first, and then $s \to 0+$, we obtain

$$\lim_{\lambda\to+\infty} \int_{\Sigma} \varphi \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = 8\pi \sum_{i=1}^k t_i \varphi(x_i).$$

This together with (2.42) implies (2.38). □

Lemma 2.6. *Let Ψ and $L > 0$ be as in Lemma 2.4. If $\lambda > 0$ is chosen sufficiently large, then there exists a continuous map $\Phi_\lambda : \Sigma_k \to J_\rho^{-L}$ such that $\Psi \circ \Phi_\lambda$ is homotopic to the identity map* $\mathrm{Id} : \Sigma_k \to \Sigma_k$.

Proof. Let $\phi_{\lambda,\sigma}$ be constructed as in Lemma 2.5. For any $\sigma \in \Sigma_k$, we define $\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}$. Clearly, the map $\Phi_\lambda : \Sigma_k \to W_0^{1,2}(\Sigma)$ is continuous. By (2.37), if $\lambda \geqslant \mathrm{e}^{L/(\rho-8k\pi)}$, then $J_\rho(\phi_{\lambda,\sigma}) \leqslant -L$. Thus $\Phi_\lambda(\sigma) \in J_\rho^{-L}$. By Lemma 2.4 and (2.38), it holds that

$$\Psi \circ \Phi_\lambda(\sigma) \;=\; \Psi(\phi_{\lambda,\sigma}) = \psi_k\bigg(\frac{\mathrm{e}^{\phi_{\lambda,\sigma}}}{\int_\Sigma \mathrm{e}^{\phi_{\lambda,\sigma}} dv_g} dv_g\bigg) \to \sigma$$

as $\lambda \to +\infty$. Hence, $\Psi \circ \Phi_\lambda$ is homotopic to $\mathrm{Id} : \Sigma_k \to \Sigma_k$. □

2.4 Min-max values

In this subsection, we shall construct the suitable min-max value of J_ρ for $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}^*$. Recalling that Σ is non-contractible, we can take a sufficiently small $\epsilon > 0$ such that

$$\Sigma_\epsilon = \{x \in \Sigma : \mathrm{dist}(x, \partial\Sigma) \geqslant \epsilon\}$$

is non-contractible. Let

$$\Sigma_{\epsilon,k} = \{\sigma \in \Sigma_k : \mathrm{dist}(\mathrm{supp}\,\sigma, \partial\Sigma) \geqslant \epsilon\}.$$

According to Lemma 2.3, we see that $\Sigma_{\epsilon,k}$ is also non-contractible. Let

$$\widehat{\Sigma}_{\epsilon,k} = \Sigma_{\epsilon,k} \times [0,1]/(\Sigma_{\epsilon,k} \times \{0\}) \tag{2.44}$$

be the topological cone over $\Sigma_{\epsilon,k}$. A path set associated with the metric space $\widehat{\Sigma}_{\epsilon,k}$ is defined by

$$\Gamma_\lambda = \{\gamma \in C^0(\widehat{\Sigma}_{\epsilon,k}, W_0^{1,2}(\Sigma)) : \gamma|_{\Sigma_{\epsilon,k}\times\{1\}} \in \Gamma_{\lambda,0}\}, \tag{2.45}$$

where $\Gamma_{\lambda,0}$ is given by

$$\Gamma_{\lambda,0} = \{\gamma \in C^0(\Sigma_{\epsilon,k}\times\{1\}, W_0^{1,2}(\Sigma)) : \gamma(\sigma,1) = \Phi_\lambda(\sigma), \forall\, \sigma \in \Sigma_{\epsilon,k}\}.$$

If we write a path $\overline{\gamma} : \widehat{\Sigma}_{\epsilon,k} \to W_0^{1,2}(\Sigma)$ by $\overline{\gamma}(\sigma,t) = t\phi_{\lambda,\sigma}$, then $\overline{\gamma} \in \Gamma_\lambda$, and thus $\Gamma_\lambda \neq \emptyset$.

For any real numbers λ and ρ, we set

$$\alpha_{\lambda,\rho} = \inf_{\gamma\in\Gamma_\lambda} \sup_{(\sigma,t)\in\widehat{\Sigma}_{\epsilon,k}} J_\rho(\gamma(\sigma,t)) \tag{2.46}$$

and

$$\beta_{\lambda,\rho} = \sup_{\gamma\in\Gamma_{\lambda,0}} \sup_{(\sigma,t)\in\Sigma_{\epsilon,k}\times\{1\}} J_\rho(\gamma(\sigma,t)). \tag{2.47}$$

Lemma 2.7. *Let $\rho \in (8k\pi, 8(k+1)\pi)$, and $\epsilon > 0$ be as above. If λ is chosen sufficiently large, and r is chosen sufficiently small, then $-\infty < \beta_{\lambda,\rho} < \alpha_{\lambda,\rho} < +\infty$.*

Proof. If $L > 0$ is large enough, $\Psi : J_\rho^{-L} \to \overline{\Sigma}_k$ is well defined (see Lemma 2.4 above). It follows from Lemmas 2.5 and 2.6 that for sufficiently large $\lambda > 0$ and sufficiently small $r > 0$, it holds that $\Phi_\lambda(\sigma) \in J_\rho^{-4L}$ for all $\sigma \in \Sigma_{\epsilon,k}$. This together with (2.47) implies

$$\beta_{\lambda,\rho} \leqslant -4L. \tag{2.48}$$

Now we claim $\alpha_{\lambda,\rho} > -2L$. If not, $\alpha_{\lambda,\rho} \leqslant -2L$. By the definition of $\alpha_{\lambda,\rho}$, namely (2.46), there exists some $\gamma_1 \in \Gamma_\lambda$ such that $\sup_{(\sigma,t)\in\widehat{\Sigma}_{\epsilon,k}} J_\rho(\gamma_1(\sigma,t)) \leqslant -3L/2$. As a consequence,

$$J_\rho(\gamma_1(\sigma,t)) \leqslant -\frac{3}{2}L \quad \text{for all } (\sigma,t) \in \widehat{\Sigma}_{\epsilon,k}.$$

Since $\Psi : J_\rho^{-L} \to \overline{\Sigma}_k$ is continuous, the map $\Psi\circ\gamma_1 : \widehat{\Sigma}_{\epsilon,k} \to \overline{\Sigma}_k$ is also continuous. Note that $\gamma_1(\sigma,1) = \Phi_\lambda(\sigma)$ and $\gamma_1(\sigma,0) \equiv u_0 \in J_\rho^{-L}$ for all $\sigma \in \Sigma_{\epsilon,k}$. If we let $\pi : \overline{\Sigma}_k \to \Sigma_{\epsilon,k}$ be a continuous projection, then $\pi\circ\Psi\circ\Phi_\lambda : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$ is homotopic to a constant map $\pi\circ\Psi\circ\gamma_1(\cdot,0) : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$. Moreover, by Lemma 2.6, $\pi\circ\Psi\circ\Phi_\lambda$ is homotopic to $\mathrm{Id} : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$. Hence the identity map $\mathrm{Id} : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$ is homotopic to the constant map $\pi\circ\Psi\circ\gamma_1(\cdot,0) : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$, which contradicts the fact that $\Sigma_{\epsilon,k}$ is non-contractible. Therefore,

$$\alpha_{\lambda,\rho} > -2L. \tag{2.49}$$

Since $J_\rho \in C^2(W_0^{1,2}(\Sigma), \mathbb{R})$ and $\widehat{\Sigma}_{\epsilon,k}$ is a compact metric space, we immediately have that $\beta_{\lambda,\rho} > -\infty$ and $\alpha_{\lambda,\rho} < +\infty$. This together with (2.48) and (2.49) concludes the lemma. □

To proceed, we need several uniform estimates for functionals J_ρ.

2.5 Uniform estimates with respect to ρ

Let $[a,b] \subset (8k\pi, 8(k+1)\pi)$ be any closed interval. Let $L > 0$ be sufficiently large such that

$$\Psi : J_b^{-L} \to \overline{\Sigma}_k \tag{2.50}$$

is a continuous map defined as in Lemma 2.4. Let $\Sigma_{\epsilon,k}$ be given as in the previous subsection. Choose a sufficiently large $\lambda > 0$ such that for all $\sigma \in \Sigma_{\epsilon,k}$, $\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}$ satisfies

$$J_a(\phi_{\lambda,\sigma}) \leqslant -(a - 8k\pi)\log\lambda \leqslant -4L, \tag{2.51}$$

where $\phi_{\lambda,\sigma}$ is defined as in (2.36). It should be remarked that the choice of λ depends not only on L, k and a, but also on ϵ. Let Γ_λ and $\alpha_{\lambda,\rho}$ be given as in (2.45) and (2.46), respectively.

Lemma 2.8. *Let $\rho \in [a,b]$. Then $\Psi : J_\rho^{-L} \to \overline{\Sigma}_k$ is well defined uniformly with respect to ρ. Moreover, for all $\rho \in [a,b]$, it holds that*

$$J_\rho(\phi_{\lambda,\sigma}) \leqslant -4L, \quad \forall\, \sigma \in \Sigma_{\epsilon,k}.$$

Proof. Let $\rho \in [a,b]$. If $u \in J_\rho^{-L}$, then $J_\rho(u) \leqslant -L$. This implies

$$\log \int_\Sigma h e^u dv_g > 0.$$

It follows that $J_b(u) \leqslant J_\rho(u) \leqslant -L$ and $u \in J_b^{-L}$. As a consequence $J_\rho^{-L} \subset J_b^{-L}$, and thus by (2.50), $\Psi : J_\rho^{-L} \to \overline{\Sigma}_k$ is well defined.

Let $\sigma \in \Sigma_{\epsilon,k}$ and $\phi_{\lambda,\sigma}$ satisfy (2.51). As above, we have that by (2.51),

$$\begin{aligned} J_\rho(\phi_{\lambda,\sigma}) &= \frac{1}{2}\int_\Sigma |\nabla_g \phi_{\lambda,\sigma}|^2 dv_g - \rho \int_\Sigma h e^{\phi_{\lambda,\sigma}} dv_g \\ &\leqslant \frac{1}{2}\int_\Sigma |\nabla_g \phi_{\lambda,\sigma}|^2 dv_g - a \int_\Sigma h e^{\phi_{\lambda,\sigma}} dv_g \\ &= J_a(\phi_{\lambda,\sigma}) \leqslant -4L. \end{aligned}$$

This ends the proof of the lemma. □

For simplicity, we denote $\alpha_{\lambda,\rho}$ by α_ρ. By Lemmas 2.7 and 2.8, α_ρ is a real number for any $\rho \in [a,b]$. Then we have an analog of [15, Lemma 2.4].

Lemma 2.9. *α_ρ/ρ is decreasing in $\rho \in [a,b]$.*

Proof. Let $a \leqslant \rho_1 < \rho_2 \leqslant b$. Then for any $(\sigma,t) \in \widehat{\Sigma}_{\epsilon,k}$ and any $\gamma \in \Gamma_\lambda$, it holds that

$$\frac{J_{\rho_1}(\gamma(\sigma,t))}{\rho_1} - \frac{J_{\rho_2}(\gamma(\sigma,t))}{\rho_2} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \int_\Sigma |\nabla_g \gamma(\sigma,t)| dv_g \geqslant 0.$$

It then follows that $\alpha_{\rho_1}/\rho_1 \geqslant \alpha_{\rho_2}/\rho_2$. □

By Lemma 2.9, α_ρ/ρ is differentiable almost everywhere in $[a,b] \subset (8k\pi, 8(k+1)\pi)$. Define

$$\Lambda_{a,b} = \left\{\rho \in (a,b) : \frac{\alpha_\rho}{\rho} \text{ is differentiable at } \rho\right\}. \tag{2.52}$$

Then $\Lambda_{a,b}$ is a dense subset of $[a,b]$.

2.6 Existence for a dense set

In this subsection, we shall prove that J_ρ has a critical point for any $\rho \in \Lambda_{a,b}$. The argument we shall use is adapted from Ding et al. [15]. For readers' convenience, we provide the details here.

Lemma 2.10. *If $\rho \in \Lambda_{a,b}$, then α_ρ is differentiable at ρ. In particular, if $\rho \in \Lambda_{a,b}$, then we have*

$$\alpha_{\tilde\rho} = \alpha_\rho + O(\tilde\rho - \rho) \quad \textit{as } \tilde\rho \to \rho.$$

Proof. In view of (2.52), it suffices to notice that $\alpha_\rho = \rho(\alpha_\rho/\rho)$. □

As an analog of [15, Lemma 3.2], we have the following lemma.

Lemma 2.11. *If $\rho \in \Lambda_{a,b}$, then α_ρ is a critical value of J_ρ.*

Proof. Let $(\rho_n) \subset [a,b]$ be an increasing sequence converging to $\rho \in \Lambda_{a,b}$. By the definition of α_{ρ_n}, there must be a path $\gamma_n \in \Gamma_\lambda$ such that

$$\sup_{u \in \gamma_n(\widehat{\Sigma}_{\epsilon,k})} J_{\rho_n}(u) \leqslant \alpha_{\rho_n} + \rho - \rho_n. \tag{2.53}$$

Also we use the definition of α_ρ to find some $u_n \in \gamma_n(\widehat{\Sigma}_{\epsilon,k}) \subset W_0^{1,2}(\Sigma)$ with

$$J_\rho(u_n) \geqslant \alpha_\rho - (\rho - \rho_n). \tag{2.54}$$

For the above u_n, by Lemma 2.10, we have

$$\frac{1}{2}\int_\Sigma |\nabla_g u_n|^2 dv_g = \frac{\frac{J_{\rho_n}(u_n)}{\rho_n} - \frac{J_\rho(u_n)}{\rho}}{\frac{1}{\rho_n} - \frac{1}{\rho}}$$
$$\leqslant \rho\frac{\alpha_{\rho_n} - \alpha_\rho}{\rho - \rho_n} + \left(\frac{1}{\rho_n} - \frac{1}{\rho}\right)\alpha_\rho + \rho_n + \rho$$
$$\leqslant c_0 \tag{2.55}$$

for some constant c_0 depending only on ρ, α_ρ and $(\alpha_\rho/\rho)'$. Moreover, by Lemmas 2.9 and 2.10, and the estimate (2.53), one finds

$$J_\rho(u_n) \leqslant \frac{\rho}{\rho_n}J_{\rho_n}(u_n) \leqslant \frac{\rho}{\rho_n}(\alpha_{\rho_n} + \rho - \rho_n) \leqslant \alpha_\rho + C(\rho - \rho_n) \tag{2.56}$$

for some constant C independent of n.

Suppose that α_ρ is not a critical value of J_ρ. Since any bounded Palais-Smale sequence must converge to a critical point of J_ρ (see [15, Lemma 3.1]), there would exist a $\delta > 0$ such that

$$\|dJ_\rho(u)\|_{(W_0^{1,2}(\Sigma))^*} \geqslant 2\delta \tag{2.57}$$

for all $u \in \mathscr{N}_\delta$, where

$$\mathscr{N}_\delta = \left\{u \in W_0^{1,2}(\Sigma) : \int_\Sigma |\nabla_g u|^2 dv_g \leqslant 2c_0,\, |J_\rho(u) - \alpha_\rho| < \delta\right\}. \tag{2.58}$$

It follows from (2.54)–(2.56) that $\mathscr{N}_\delta \neq \emptyset$. Let $\boldsymbol{X}_\rho : \mathscr{N}_\delta \to W_0^{1,2}(\Sigma)$ be a pseudo-gradient vector field for J_ρ in $\mathscr{N}_\delta$, namely, a locally Lipschitz vector field satisfying $\|\boldsymbol{X}_\rho\|_{W_0^{1,2}(\Sigma)} \leqslant 1$ and

$$dJ_\rho(u)(\boldsymbol{X}_\rho(u)) \leqslant -\delta. \tag{2.59}$$

Here, we have used (2.57). One can check that as $n \to \infty$, $dJ_{\rho_n}(u)$ converges to $dJ_\rho(u)$ in $(W_0^{1,2}(\Sigma))^*$ uniformly in u with $\int_\Sigma |\nabla_g u|^2 dv_g \leqslant c^*$. Thus, $\boldsymbol{X}_\rho$ is also a pseudo-gradient vector field for J_{ρ_n} in $\mathscr{N}_\delta$. Moreover, it holds that for all $u \in \mathscr{N}_\delta$ and sufficiently large n,

$$dJ_{\rho_n}(u)(\boldsymbol{X}_\rho(u)) \leqslant -\delta/2. \tag{2.60}$$

Take a Lipschitz continuous cut-off function η such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in $\mathscr{N}_{\delta/2}$, and $\eta \equiv 0$ outside $\mathscr{N}_\delta$. Let $\psi : W_0^{1,2}(\Sigma) \times [0, +\infty)$ be the flow generated by $\eta\boldsymbol{X}_\rho$, which satisfies

$$\begin{cases} \dfrac{\partial}{\partial s}\psi(u,s) = \eta(\psi(u,s))\boldsymbol{X}_\rho(\psi(u,s)), \\ \psi(u,0) = u. \end{cases}$$

This flow has long time existence because it always remains stationary outside $\mathscr{N}_\delta$. It follows from (2.59) that for all $u \in \mathscr{N}_{\delta/2}$,

$$\frac{d}{ds}\bigg|_{s=0} J_\rho(\psi(u,s)) = dJ_\rho(u)(\boldsymbol{X}_\rho(u)) \leqslant -\delta. \tag{2.61}$$

In view of (2.49) and (2.58), one easily sees $\phi_{\lambda,\sigma} \notin \mathscr{N}_\delta$ for all $\sigma \in \Sigma_{\epsilon,k}$. Since $J_\rho(\psi(\phi_{\lambda,\sigma}, s))$ is decreasing in s, it holds that $J_\rho(\psi(\phi_{\lambda,\sigma}, s)) \leqslant -4L$ for all $s \in [0, +\infty)$. Hence $\psi(\gamma_n(\sigma,1), s) \notin \mathscr{N}_\delta$, and thus $\psi(\gamma_n(\sigma,1), s) \equiv \psi(\gamma_n(\sigma,1), 0) = \phi_{\lambda,\sigma}$ for all $\sigma \in \Sigma_{\epsilon,k}$ and all $s \in [0, +\infty)$. As a consequence, if we write $\psi_s(\cdot) = \psi(\cdot, s)$, then we have $\psi_s \circ \gamma_n \in \Gamma_\lambda$. By (2.56) and the monotonicity of $J_\rho(\psi_s(u))$ in s, we obtain

$$\alpha_\rho \leqslant \sup_{u \in \psi_s \circ \gamma_n(\widehat{\Sigma}_{\epsilon,k})} J_\rho(u) \leqslant \sup_{u \in \gamma_n(\widehat{\Sigma}_{\epsilon,k})} J_\rho(u) \leqslant \alpha_\rho + C(\rho - \rho_n). \tag{2.62}$$

We now claim that

$$\sup_{u\in\psi_s\circ\gamma_n(\widehat{\Sigma}_{\epsilon,k})} J_\rho(u) \text{ is achieved in } \mathscr{N}_{\delta/2}. \tag{2.63}$$

In fact, since $\widehat{\Sigma}_{\epsilon,k}$ is a compact metric space, the continuous function $J_\rho(\psi_s\circ\gamma_n(\cdot,\cdot))$ attains its supremum at some $(\sigma_0,t_0)\in\widehat{\Sigma}_{\epsilon,k}$. As a result, the function $u_{s,n}=\psi_s\circ\gamma_n(\sigma_0,t_0)$ achieves $\sup_{u\in\psi_s\circ\gamma_n(\widehat{\Sigma}_{\epsilon,k})}J_\rho(u)$. If n is chosen sufficiently large, (2.62) implies that $\alpha_\rho\leqslant J_\rho(u_{s,n})\leqslant\alpha_\rho+\delta/2$. By (2.60), $J_{\rho_n}(\psi_s(u))$ is decreasing in s, which together with (2.53) gives $J_{\rho_n}(u_{s,n})\leqslant\alpha_{\rho_n}+\rho-\rho_n$. It then follows that

$$\begin{aligned}\frac{1}{2}\int_\Sigma|\nabla_g u_{s,n}|^2dv_g&=\left(\frac{1}{\rho_n}-\frac{1}{\rho}\right)^{-1}\left(\frac{J_{\rho_n}(u_{s,n})}{\rho_n}-\frac{J_\rho(u_{s,n})}{\rho}\right)\\&\leqslant\left(\frac{1}{\rho_n}-\frac{1}{\rho}\right)^{-1}\left(\frac{\alpha_{\rho_n}+\rho-\rho_n}{\rho_n}-\frac{\alpha_\rho}{\rho}\right)\\&\leqslant c_0,\end{aligned}$$

where c_0 is the same constant as in (2.55). Therefore, $u_{s,n}\in\mathscr{N}_{\delta/2}$, and our claim is confirmed.

Let $\bar{s}>0$ and $\overline{\gamma}_n=\psi_{\bar{s}}\circ\gamma_n$. Then by (2.61) and (2.63), we have

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=\bar{s}}\sup_{(\sigma,t)\in\widehat{\Sigma}_{\epsilon,k}}J_\rho(\psi_s\circ\gamma_n(\sigma,t))&=\frac{d}{ds}\bigg|_{s=\bar{s}}\sup_{(\sigma,t)\in\widehat{\Sigma}_{\epsilon,k}}J_\rho(\psi_{s-\bar{s}}\circ\overline{\gamma}_n(\sigma,t))\\&=\frac{d}{ds}\bigg|_{s=0}\sup_{(\sigma,t)\in\widehat{\Sigma}_{\epsilon,k}}J_\rho(\psi_s\circ\overline{\gamma}_n(\sigma,t))\\&\leqslant\sup_{u\in\mathscr{N}_{\delta/2}}\frac{d}{ds}\bigg|_{s=0}J_\rho(\psi_s(u))\\&\leqslant-\delta.\end{aligned}\tag{2.64}$$

Using the Newton-Leibniz formula, we conclude from (2.62) and (2.64) that

$$\sup_{u\in\psi_s\circ\gamma_n(\widehat{\Sigma}_{\epsilon,k})}J_\rho(u)<\alpha_\rho,$$

if $s>0$ is sufficiently large. This contradicts the definition of α_ρ, and ends the proof of the lemma. □

2.7 Existence for all $\rho\in(8k\pi,8(k+1)\pi)$

In this subsection, we use the previous analysis to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For any $\rho\in(8k\pi,8(k+1)\pi)$ and $k\in\mathbb{N}^*$, there are two constants a and b with $8k\pi<a<\rho<b<8(k+1)\pi$. By Lemma 2.11, we may take an increasing sequence of numbers $(\rho_n)\subset\Lambda_{a,b}$ such that $\rho_n\to\rho$ and α_{ρ_n} is achieved by $u_n\in W_0^{1,2}(\Sigma)$. Moreover, u_n satisfies

$$\Delta_g u_n=\rho_n\frac{he^{u_n}}{\int_\Sigma he^{u_n}dv_g}\quad\text{in }\Sigma. \tag{2.65}$$

By Lemma 2.9,

$$\alpha_{\rho_n}\leqslant\frac{b}{a}\alpha_a. \tag{2.66}$$

Denoting $v_n=u_n-\log\int_\Sigma he^{u_n}dv_g$, by (2.65), we have

$$\begin{cases}\Delta_g v_n=\rho_n he^{v_n},\\ \displaystyle\int_\Sigma he^{v_n}dv_g=1.\end{cases}$$

By Lemma 2.1, (u_n) is bounded in $L^\infty(\overline{\Sigma})$. Let $\Omega_1, \ldots, \Omega_{k+1}$ be disjoint sub-domains of $\overline{\Sigma}$. By Lemma 2.2,

$$\log \int_\Sigma \mathrm{e}^{u_n} dv_g \leqslant \frac{1}{16(k+1)\pi - \epsilon} \int_\Sigma |\nabla_g u_n|^2 dv_g + C_\epsilon$$

for any $\epsilon > 0$ and some constant $C_\epsilon > 0$. This together with (2.66) implies that for $0 < \epsilon < 16(k+1)\pi - 2b$,

$$\begin{aligned}\frac{1}{2} \int_\Sigma |\nabla_g u_n|^2 dv_g &= J_{\rho_n}(u_n) + \rho_n \log \int_\Sigma h \mathrm{e}^{u_n} dv_g \\ &\leqslant \frac{b}{16(k+1)\pi - \epsilon} \int_\Sigma |\nabla_g u_n|^2 dv_g + C.\end{aligned}$$

It then follows that (u_n) is bounded in $W_0^{1,2}(\Sigma)$. Without loss of generality, we assume that u_n converges to u_ρ weakly in $W_0^{1,2}(\Sigma)$, strongly in $L^p(\Sigma)$ for any $p > 1$, and almost everywhere in Σ. Moreover, e^{u_n} converges to e^{u_ρ} strongly in $L^p(\Sigma)$ for any $p > 1$. By (2.65), u_ρ is a distributional solution of (1.3). Hence, u_ρ is a critical point of J_ρ. □

3 The Neumann boundary value problem

In this section, we shall prove Theorem 1.2 by the min-max method. Since part of the proof is analogous to that of Theorem 1.1, we only give its outline but stress the difference. In Subsection 3.1, we prove a compactness result for solutions of (1.4). In Subsection 3.2, we derive an improved Trudinger-Moser inequality for functions $u \in W^{1,2}(\Sigma)$ with $\int_\Sigma u dv_g = 0$. In Subsection 3.3, we construct two continuous maps between sub-levels of J_ρ and the topological space $\mathscr{S}_k$, where J_ρ and $\mathscr{S}_k$ are defined as in (3.26) and (3.27), respectively. In Subsection 3.4, we construct min-max levels of J_ρ. The remaining part of the proof of Theorem 1.2 is outlined in Subsection 3.5.

3.1 Compactness analysis

Let (ρ_n) be a number sequence tending to $\rho \in \mathbb{R}$, (h_n) be a function sequence converging to h in $C^1(\overline{\Sigma})$, and (u_n) be a sequence of solutions to

$$\begin{cases} \Delta_g u_n = \rho_n \left(\dfrac{h_n \mathrm{e}^{u_n}}{\int_\Sigma h_n \mathrm{e}^{u_n} dv_g} - \dfrac{1}{|\Sigma|} \right) & \text{in } \Sigma, \\ \partial u_n / \partial \boldsymbol{v} = 0 & \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma u_n dv_g = 0. \end{cases} \tag{3.1}$$

Define

$$v_n = u_n - \log \int_\Sigma h_n \mathrm{e}^{u_n} dv_g.$$

Then $\Delta_g v_n = \rho_n (h_n \mathrm{e}^{v_n} - 1/|\Sigma|)$ and $\int_\Sigma h_n \mathrm{e}^{v_n} dv_g = 1$. Concerning the compactness of (u_n), we have an analog of Lemma 2.1.

Lemma 3.1. *Assume that ρ is a positive number and h is a positive function. Up to a subsequence, one of the following alternatives holds*:

(i) (u_n) *is bounded in* $L^\infty(\overline{\Sigma})$;

(ii) (v_n) *converges to* $-\infty$ *uniformly in* $\overline{\Sigma}$;

(iii) *there exists a finite singular set* $\mathcal{S} = \{p_1, \ldots, p_m\} \subset \overline{\Sigma}$ *such that for any* $1 \leqslant j \leqslant m$, *there is a sequence of points* $\{p_{j,n}\} \subset \overline{\Sigma}$ *satisfying* $p_{j,n} \to p_j$, $u_n(p_{j,n}) \to +\infty$, *and* v_n *converges to* $-\infty$ *uniformly on any compact subset of* $\overline{\Sigma} \setminus \mathcal{S}$ *as* $n \to \infty$. *Moreover, if* $\mathcal{S}$ *has* ℓ *points in* Σ *and* $(m - \ell)$ *points on* $\partial\Sigma$, *then*

$$\rho_n \int_\Sigma h_n \mathrm{e}^{v_n} dv_g \to 4(m + \ell)\pi.$$

Proof. We modify arguments in the proof of Lemma 2.1, and divide the proof into two parts.

Part I. Analysis in the interior domain Σ.

Let (u_n) be a sequence of solutions to (3.1). By the Green representation formula (see [38]), we have

$$\|u_n\|_{W^{1,q}(\Sigma)} \leqslant C_q, \quad \forall\, 1 < q < 2. \tag{3.2}$$

Since (h_n) converges to $h > 0$ in $C^1(\overline{\Sigma})$, there exists some constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\int_\Sigma \mathrm{e}^{v_n} dv_g \leqslant C. \tag{3.3}$$

Moreover, Jensen's inequality implies

$$\liminf_{n\to\infty} \int_\Sigma \mathrm{e}^{u_n} dv_g \geqslant |\Sigma|.$$

Without loss of generality, we assume that $\rho_n h_n \mathrm{e}^{v_n} dv_g$ converges to some nonnegative measure μ on $\overline{\Sigma}$. If $\mu(x^*) < 4\pi$ for some $x^* \in \Sigma$, then there exist two positive constants ϵ_0 and r_0 verifying

$$\int_{B_{x^*}(r_0)} \rho_n h_n \mathrm{e}^{v_n} dv_g \leqslant 4\pi - \epsilon_0.$$

In view of (3.1), by a result of Brezis and Merle [5, Theorem 1] and elliptic estimates, we have that (u_n) is bounded in $L^\infty(B_{x^*}(r_0/2))$. This leads to $\mu(x^*) = 0$. Define a set $\mathcal{S} = \{x \in \Sigma : \mu(x) \geqslant 4\pi\}$. If $\mathcal{S} \neq \emptyset$, then by almost the same argument as the proof of (2.4), we conclude that for any compact set $A \subset \Sigma \setminus \mathcal{S}$, it holds that

$$v_n \to -\infty \quad \text{uniformly in } x \in A. \tag{3.4}$$

Assume $\mathcal{S} = \{x_1, \ldots, x_j\}$ for some positive integer j. We shall show that $\mu(x_i) = 8\pi$ for all $1 \leqslant i \leqslant j$. Without loss of generality, it suffices to prove $\mu(x_1) = 8\pi$. For this purpose, we choose an isothermal coordinate system $\phi : U \to \mathbb{B}_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$ near x_1. In such coordinates, the metric g and the Laplace-Beltrami operator Δ_g are represented by $g = \mathrm{e}^{\psi(y)}(dy_1^2 + dy_2^2)$ and $\Delta_g = -\mathrm{e}^{-\psi(y)}\Delta_{\mathbb{R}^2}$, respectively, where ψ is a smooth function with $\psi(0,0) = 0$, and $\Delta_{\mathbb{R}^2} = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ denotes the standard Laplacian on $\mathbb{R}^2$. Set $\widetilde{u} = u \circ \phi^{-1}$ for any function $u : U \to \mathbb{R}$. Since (u_n) satisfies (3.1), $\widetilde{u}_n = u_n \circ \phi^{-1}$ satisfies

$$-\Delta_{\mathbb{R}^2} \widetilde{u}_n(y) = \mathrm{e}^{\psi(y)} \rho_n (\widetilde{h}_n(y) \mathrm{e}^{\widetilde{v}_n(y)} - |\Sigma|^{-1}), \quad y \in \mathbb{B}_1. \tag{3.5}$$

Multiplying both sides of (3.5) by $y \cdot \nabla_{\mathbb{R}^2} \widetilde{u}_n(y)$, by integration by parts, we have

$$\begin{aligned}\frac{r}{2}\int_{\partial\mathbb{B}_r} |\nabla_{\mathbb{R}^2}\widetilde{u}_n|^2 d\sigma - r\int_{\partial\mathbb{B}_r} \langle \nabla_{\mathbb{R}^2}\widetilde{u}_n, \nu\rangle^2 d\sigma &= r\int_{\partial\mathbb{B}_r} \mathrm{e}^{\psi}\rho_n \widetilde{h}_n \mathrm{e}^{\widetilde{v}_n} d\sigma - \int_{\mathbb{B}_r} \mathrm{e}^{\widetilde{v}_n}\rho_n \langle \nabla_{\mathbb{R}^2}(\mathrm{e}^{\psi}\widetilde{h}_n), y\rangle dy \\ &\quad - 2\int_{\mathbb{B}_r} \mathrm{e}^{\psi}\rho_n \widetilde{h}_n \mathrm{e}^{\widetilde{v}_n} dy + \frac{\rho_n}{|\Sigma|}\int_{\mathbb{B}_r} \mathrm{e}^{\psi} y \cdot \nabla_{\mathbb{R}^2}\widetilde{u}_n dy, \end{aligned} \tag{3.6}$$

where $\mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 < r\}$, $\partial\mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = r\}$, and ν denotes the unit outward vector on $\partial\mathbb{B}_r$. In view of (3.4), (u_n) converges to a Green function $G(x_1, \cdot)$ weakly in $W^{1,q}(\Sigma)$ and in $C^2_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$. Locally, $G(x_1, \cdot)$ satisfies

$$\Delta_{g,z} G(x_1, z) = \mu(x_1)\delta_{x_1}(z) - \rho|\Sigma|^{-1}, \quad \forall\, z \in \phi^{-1}(\mathbb{B}_1).$$

Clearly,

$$\widetilde{G}(y) = G(x_1, \phi^{-1}(y)) = -\frac{\mu(x_1)}{2\pi}\log|y| + \eta(y)$$

for some $\eta \in C^2(\mathbb{B}_1)$. Passing to the limit $n \to \infty$ first, and then $r \to 0$ in (3.6), we obtain

$$\mu(x_1) = \lim_{r\to 0}\left(\frac{r}{2}\int_{\partial\mathbb{B}_r} \langle \nabla_{\mathbb{R}^2}\widetilde{G}, \nu\rangle^2 d\sigma - \frac{r}{4}\int_{\partial\mathbb{B}_r} |\nabla_{\mathbb{R}^2}\widetilde{G}|^2 d\sigma\right) = \frac{(\mu(x_1))^2}{8\pi}. \tag{3.7}$$

This immediately leads to $\mu(x_1) = 8\pi$. In conclusion, we have

$$\mu(x_i) = 8\pi \quad \text{for all } 1 \leqslant i \leqslant j. \tag{3.8}$$

Part II. Analysis on the boundary $\partial\Sigma$.

Let $x^* \in \overline{\Sigma}$ be fixed. Note that $\rho_n h_n \mathrm{e}^{v_n} dv_g$ converges to the nonnegative Radon measure μ on $\overline{\Sigma}$ as $n \to \infty$. If $\mu(x^*) < 2\pi$, there exist a neighborhood V of x^* and a number $\gamma_0 > 0$ such that

$$\int_V \rho_n h_n \mathrm{e}^{v_n} dv_g \leqslant 2\pi - \gamma_0. \tag{3.9}$$

Without loss of generality, we take an isothermal coordinate system $(V, \phi, \{y_1, y_2\})$ such that $\phi(x^*) = (0,0)$, and $\phi : V \to \mathbb{B}_1^+ \cup \Gamma = \{(y_1, y_2) : y_1^2 + y_2^2 < 1, y_2 \geqslant 0\}$, where $\Gamma = \{(y_1, y_2) : |y_1| < 1, y_2 = 0\}$. Moreover, in this coordinate system, the metric $g = \mathrm{e}^{\psi(y)}(dy_1^2 + dy_2^2)$ and the Laplace-Beltrami operator $\Delta_g = -\mathrm{e}^{-\psi(y)}\Delta_{\mathbb{R}^2}$, where $\psi : \mathbb{B}_1^+ \cup \Gamma \to \mathbb{R}$ is a smooth function with $\psi(0,0) = 0$; moreover, $\partial/\partial \boldsymbol{v} = \mathrm{e}^{-\psi(y)/2}\partial/\partial y_2$. For more details about isothermal coordinates on the boundary, we refer the readers to [38, Section 2]. Now the local version of (3.1) is

$$\begin{cases} -\Delta_{\mathbb{R}^2}(u_n \circ \phi^{-1})(y) = \mathrm{e}^{\psi(y)}\rho_n((h_n \circ \phi^{-1})(y)\mathrm{e}^{(v_n \circ \phi^{-1})(y)} - |\Sigma|^{-1}) & \text{in } \mathbb{B}_1^+, \\ \dfrac{\partial}{\partial y_2}(u_n \circ \phi^{-1})(y) = 0 & \text{on } \Gamma. \end{cases} \tag{3.10}$$

For any function $u : V \to \mathbb{R}$, we define a function $\widetilde{u} : \mathbb{B}_1 \to \mathbb{R}$ by

$$\widetilde{u}(y_1, y_2) = \begin{cases} u \circ \phi^{-1}(y_1, y_2), & \text{if } y_2 \geqslant 0, \\ u \circ \phi^{-1}(y_1, -y_2), & \text{if } y_2 < 0. \end{cases} \tag{3.11}$$

One can easily derive from (3.10) that $\widetilde{u}_n$ is a distributional solution of

$$-\Delta_{\mathbb{R}^2}\widetilde{u}_n(y) = \widetilde{f}_n(y), \quad y \in \mathbb{B}_1, \tag{3.12}$$

where $\widetilde{f}_n$ is defined as in (3.11) and for $y \in \mathbb{B}_1^+ \cup \Gamma$,

$$f_n \circ \phi^{-1}(y) = \mathrm{e}^{\psi(y)}\rho_n((h_n \circ \phi^{-1})(y)\mathrm{e}^{(v_n \circ \phi^{-1})(y)} - |\Sigma|^{-1}).$$

In view of (3.9) and the fact $\psi(0,0) = 0$, there exists a number $r_0 \in (0,1)$ such that

$$\int_{\mathbb{B}_{r_0}} |\widetilde{f}_n(y)| dy \leqslant 4\pi - \gamma_0.$$

Let w_n be a solution of

$$\begin{cases} -\Delta_{\mathbb{R}^2} w_n = \widetilde{f}_n & \text{in } \mathbb{B}_{r_0}, \\ w_n = 0 & \text{on } \partial\mathbb{B}_{r_0}. \end{cases}$$

By [5, Theorem 1], there exists some constant C depending only on ϵ_0 and r_0 such that

$$\int_{\mathbb{B}_{r_0}} \exp\left(\frac{(4\pi - \gamma_0/2)|w_n|}{\|\widetilde{f}_n\|_{L^1(\mathbb{B}_{r_0})}}\right) dy \leqslant C.$$

Hence, there exists some $q_0 > 1$ such that

$$\|\mathrm{e}^{|w_n|}\|_{L^{q_0}(\mathbb{B}_{r_0})} \leqslant C. \tag{3.13}$$

Let $\eta_n = \widetilde{u}_n - w_n$. Then η_n satisfies

$$\begin{cases} -\Delta_{\mathbb{R}^2}\eta_n = 0 & \text{in } \mathbb{B}_{r_0}, \\ \eta_n = \widetilde{u}_n & \text{on } \partial\mathbb{B}_{r_0}. \end{cases} \tag{3.14}$$

Noticing (3.2) and (3.13), we have by applying elliptic estimates to (3.14) that

$$\|\eta_n\|_{L^\infty(\mathbb{B}_{r_0/2})} \leqslant C. \tag{3.15}$$

Combining (3.3), (3.13) and (3.15), we conclude $\|\widetilde{f}_n\|_{L^{q_0}(\mathbb{B}_{r_0/2})} \leqslant C$. Applying elliptic estimates to (3.12), we obtain $\|\widetilde{u}_n\|_{L^\infty(\mathbb{B}_{r_0/4})} \leqslant C$, which implies $\|u_n\|_{L^\infty(\phi^{-1}(\mathbb{B}^+_{r_0/4}))} \leqslant C$. In conclusion, we have that if $\mu(x^*) < 2\pi$, then (u_n) is uniformly bounded near x^*. This also leads to $\mu(x^*) = 0$.

If $\mu(x^*) \geqslant 2\pi$, in the same coordinate system $(V, \phi, \{y_1, y_2\})$ as above, $\widetilde{f}_n(y)dy$ converges to a Radon measure $\widetilde{\mu}$ with $\widetilde{\mu}(0,0) = 2\mu(x^*) \geqslant 4\pi$. Obviously, there exists some $r_1 > 0$ such that for any $x \in \mathbb{B}_{r_1} \setminus \{(0,0)\}$, $\widetilde{\mu}(x) = 0$. Using the same argument as the proof of (3.4), we conclude that for any compact set $A \subset \mathbb{B}_{r_1} \setminus \{(0,0)\}$, $\widetilde{v}_n$ converges to $-\infty$ uniformly in A. This leads to that $\widetilde{f}_n(y)dy$ converges to the Dirac measure $\widetilde{\mu}(0,0)\delta_{(0,0)}(y)$. Recalling (3.2), we have that $\widetilde{u}_n$ converges to $\widetilde{G}_0$ weakly in $W^{1,q}(\mathbb{B}_{r_1})$ and a.e. in $\mathbb{B}_{r_1}$, where $\widetilde{G}_0$ satisfies

$$-\Delta_{\mathbb{R}^2}\widetilde{G}_0(y) = \widetilde{\mu}(0,0)\delta_{(0,0)}(y) - \rho|\Sigma|^{-1}, \quad y \in \mathbb{B}_{r_1}.$$

Clearly, $\widetilde{G}_0$ is represented by

$$\widetilde{G}_0(y) = -\frac{\widetilde{\mu}(0,0)}{2\pi}\log|y| + A_0 + O(|y|)$$

as $y \to 0$, where A_0 is a constant. Noting that $\widetilde{v}_n$ converges to $-\infty$ locally uniformly in $\mathbb{B}_{r_1} \setminus \{(0,0)\}$, we have by applying elliptic estimates to (3.12) that

$$\widetilde{u}_n \to \widetilde{G}_0 \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{B}_{r_1} \setminus \{(0,0)\}).$$

Multiplying both sides of (3.12) by $y \cdot \nabla_{\mathbb{R}^2}\widetilde{u}_n(y)$, completely analogous to (3.6) and (3.7), we obtain $\widetilde{\mu}(0,0) = 8\pi$, and thus

$$\mu(x^*) = 4\pi. \tag{3.16}$$

Note that if $\mu(x_i) > 0$ for some $x_i \in \overline{\Sigma}$, then there must exist $x_{i,n} \subset \overline{\Sigma}$ satisfying $u_n(x_{i,n}) \to +\infty$. If not, (u_n) would be uniformly bounded near x, which leads to $\mu(x) = 0$. The lemma then follows from (3.8) and (3.16) immediately. □

3.2 An improved Trudinger-Moser inequality

For a compact surface with smooth boundary, it was proved by Yang [36] that

$$\sup_{u \in W^{1,2}(\Sigma),\, \int_\Sigma |\nabla_g u|^2 dv_g \leqslant 1,\, \int_\Sigma u dv_g = 0} \int_\Sigma \mathrm{e}^{2\pi u^2} dv_g < \infty. \tag{3.17}$$

Define $\overline{u} = \frac{1}{|\Sigma|}\int_\Sigma u dv_g$. By (3.17) and Young's inequality, we obtain

$$\begin{aligned}\log \int_\Sigma \mathrm{e}^{u-\overline{u}} dv_g &\leqslant \log \int_\Sigma \mathrm{e}^{2\pi \frac{(u-\overline{u})^2}{\|\nabla_g u\|_2^2} + \frac{1}{8\pi}\|\nabla_g u\|_2^2} dv_g \\ &= \frac{1}{8\pi}\int_\Sigma |\nabla_g u|^2 dv_g + \log \int_\Sigma \mathrm{e}^{2\pi \frac{(u-\overline{u})^2}{\|\nabla_g u\|_2^2}} dv_g \\ &\leqslant \frac{1}{8\pi}\int_\Sigma |\nabla_g u|^2 dv_g + C.\end{aligned}$$

Hence,

$$\log \int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{8\pi}\int_\Sigma |\nabla_g u|^2 dv_g + \frac{1}{|\Sigma|}\int_\Sigma u dv_g, \quad \forall\, u \in W^{1,2}(\Sigma). \tag{3.18}$$

Lemma 3.2. *Let b_0 and γ_0 be two positive constants, and $\Omega_1,\ldots,\Omega_k$ be k domains of $\overline{\Sigma}$ with $\mathrm{dist}(\Omega_i,\Omega_j)\geqslant b_0$ for all $1\leqslant i<j\leqslant k$. Then for any $\epsilon>0$, there exists some constant C depending only on b_0, γ_0, k and ϵ such that*

$$\log\int_\Sigma \mathrm{e}^u dv_g \leqslant \frac{1}{8k\pi-\epsilon}\int_\Sigma |\nabla_g u|^2 dv_g + \frac{1}{|\Sigma|}\int_\Sigma u dv_g + C \tag{3.19}$$

for all $u\in W^{1,2}(\Sigma)$ with

$$\int_{\Omega_i} \mathrm{e}^u dv_g \geqslant \gamma_0 \int_\Sigma \mathrm{e}^u dv_g, \quad i=1,\ldots,k. \tag{3.20}$$

Proof. We follow the lines of Chen and Li [11]. Take smooth functions $\phi_1,\ldots,\phi_k$ defined on $\overline{\Sigma}$ satisfying

$$\mathrm{supp}\,\phi_i\cap \mathrm{supp}\,\phi_j=\emptyset, \quad \forall\, 1\leqslant i<j\leqslant k, \tag{3.21}$$

$$\phi_i\equiv 1 \quad \text{on } \Omega_i, \quad 0\leqslant \phi_i\leqslant 1 \quad \text{on } \overline{\Sigma}, \quad \forall\, 1\leqslant i\leqslant k, \tag{3.22}$$

and for some positive constant b_1 depending only on b_0 and the metric g,

$$|\nabla_g\phi_i|\leqslant b_1, \quad \forall\, 1\leqslant i\leqslant k. \tag{3.23}$$

For any $u\in W^{1,2}(\Sigma)$ satisfying (3.20), we have $\phi_i u\in W^{1,2}(\Omega)$ for all $1\leqslant i\leqslant k$, and thus (3.18) implies

$$\begin{aligned}\int_\Sigma \mathrm{e}^u dv_g &\leqslant \frac{1}{\gamma_0}\int_{\Omega_i}\mathrm{e}^u dv_g\\ &\leqslant \frac{1}{\gamma_0}\int_\Sigma \mathrm{e}^{\phi_i u} dv_g\\ &\leqslant \frac{1}{\gamma_0}\exp\left(\frac{1}{8\pi}\|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)}+\frac{1}{|\Sigma|}\int_\Sigma \phi_i u dv_g + C\right).\end{aligned}$$

Note that (3.21) gives

$$\sum_{i=1}^k \|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)} = \left\|\nabla_g\left(u\sum_{i=1}^k\phi_i\right)\right\|^2_{L^2(\Sigma)},$$

and (3.22) implies

$$\sum_{i=1}^k\int_\Sigma \phi_i u dv_g \leqslant \int_\Sigma |u| dv_g \leqslant \frac{1}{|\Sigma|^{1/2}}\|u\|_{L^2(\Sigma)}.$$

Combining the above three estimates, (3.23), Young's inequality and an elementary inequality

$$a\leqslant (a_1\cdots a_k)^{1/k}, \quad \text{if } 0\leqslant a\leqslant a_i, \quad i=1,\ldots,k,$$

we obtain

$$\begin{aligned}\int_\Sigma \mathrm{e}^u dv_g &\leqslant \frac{1}{\gamma_0}\left(\prod_{i=1}^k \exp\left(\frac{1}{8\pi}\|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)}+\frac{1}{|\Sigma|}\int_\Sigma \phi_i u dv_g + C\right)\right)^{1/k}\\ &= \frac{\mathrm{e}^C}{\gamma_0}\exp\left(\frac{1}{8k\pi}\sum_{i=1}^k\|\nabla_g(\phi_i u)\|^2_{L^2(\Sigma)}+\frac{1}{k}\frac{1}{|\Sigma|}\sum_{i=1}^k\int_\Sigma\phi_i u dv_g\right)\\ &= \frac{\mathrm{e}^C}{\gamma_0}\exp\left(\frac{1}{8k\pi}\left\|\nabla_g\left(u\sum_{i=1}^k\phi_i\right)\right\|^2_{L^2(\Sigma)}+\frac{1}{k}\frac{1}{|\Sigma|}\sum_{i=1}^k\int_\Sigma\phi_i u dv_g\right)\\ &\leqslant C\exp\left(\frac{1}{8k\pi}(1+\epsilon_1)\|\nabla_g u\|^2_{L^2(\Sigma)}+C(\epsilon_1)\|u\|^2_{L^2(\Sigma)}\right).\end{aligned} \tag{3.24}$$

Let $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_\ell\leqslant\lambda_{\ell+1}\leqslant\cdots$ be all the eigenvalues of the Laplace-Beltrami operator with respect to the Neumann boundary condition. Clearly $\lambda_i\to+\infty$ as $i\to\infty$. Let $\{e_i\}_{i=1}^\infty$ be the corresponding unit normal eigenfunctions, i.e., $\Delta_g e_i = \lambda_i e_i$, $\int_\Sigma e_i dv_g = 0$ and $\int_\Sigma e_i e_j dv_g = \delta_{ij}$ for $i, j = 1, 2, \ldots$ It is known that $\mathscr{H} := \{u \in W^{1,2}(\Sigma) : \overline{u} = 0\} = E_\ell \oplus E_\ell^\perp$, where $E_\ell = \mathrm{span}\{e_1, \ldots, e_\ell\}$ and $E_\ell^\perp = \mathrm{span}\{e_{\ell+1}, e_{\ell+2}, \ldots\}$. Let $u \in W^{1,2}(\Sigma)$ be given as above. We decompose $u - \overline{u} = v + w$ with $v \in E_\ell$ and $w \in E_\ell^\perp$. Thus, the Poincaré inequality implies

$$\|v\|_{C^0(\overline{\Sigma})} \leqslant \sum_{i=1}^{\ell} \|e_i\|_{C^0(\overline{\Sigma})} \int_\Sigma |u - \overline{u}||e_i| dv_g \leqslant C_\ell \|\nabla_g u\|_{L^2(\Sigma)},$$

while by the definition of the $(\ell+1)$-th eigenvalue $\lambda_{\ell+1}$,

$$\int_\Sigma w^2 dv_g \leqslant \frac{1}{\lambda_{\ell+1}} \int_\Sigma |\nabla_g w|^2 dv_g.$$

Having the above two estimates and applying (3.24) to w, we have

$$\begin{aligned}
\int_\Sigma \mathrm{e}^{u-\overline{u}} dv_g &\leqslant \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \int_\Sigma \mathrm{e}^w dv_g \\
&\leqslant C \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \exp\left(\frac{1}{8k\pi}(1+\epsilon_1)\|\nabla_g w\|^2_{L^2(\Sigma)} + \frac{C(\epsilon_1)}{\lambda_{\ell+1}}\|\nabla_g w\|^2_{L^2(\Sigma)}\right) \\
&\leqslant C \mathrm{e}^{C_\ell \|\nabla_g u\|_{L^2(\Sigma)}} \exp\left(\frac{1}{8k\pi}\left(1+\epsilon_1+\frac{C(\epsilon_1)}{\lambda_{\ell+1}}\right)\|\nabla_g u\|^2_{L^2(\Sigma)}\right).
\end{aligned}$$

This together with Young's inequality gives

$$\log \int_\Sigma \mathrm{e}^{u-\overline{u}} dv_g \leqslant \frac{1}{8k\pi}\left(1 + \epsilon_1 \frac{C(\epsilon_1)}{\lambda_{\ell+1}} + \epsilon_1\right) \int_\Sigma |\nabla_g u|^2 dv_g + C_{\ell,k,\epsilon_1}. \tag{3.25}$$

Let $0 < \epsilon < 8k\pi$ be any given number. Choosing $\epsilon_1 = \epsilon/(16k\pi - 2\epsilon)$, and then taking a sufficiently large ℓ such that $C(\epsilon_1)/\lambda_{\ell+1} \leqslant 1$, by (3.25), we have

$$\log \int_\Sigma \mathrm{e}^{u-\overline{u}} dv_g \leqslant \frac{1}{8k\pi - \epsilon} \int_\Sigma |\nabla_g u|^2 dv_g + C,$$

where C is a constant depending only on b_0, γ_0, k and ϵ. This is exactly (3.19). □

Define a functional $J_\rho : W^{1,2}(\Sigma) \to \mathbb{R}$ by

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 dv_g - \rho \log \int_\Sigma h \mathrm{e}^u dv_g + \frac{\rho}{|\Sigma|} \int_\Sigma u dv_g. \tag{3.26}$$

From now on to the end of this section, J_ρ is always given as in (3.26).

3.3 Continuous maps between sub-levels of J_ρ and Σ_k

Let $\overline{\Sigma}_k$ be defined as in (2.29), and $(\partial\Sigma)_k$ be defined by

$$(\partial\Sigma)_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geqslant 0, \sum_{i=1}^k = 1, x_i \in \partial\Sigma \right\}.$$

Lemma 3.3. *Let $\rho \in (4k\pi, 4(k+1)\pi)$. Then for any sufficiently large $L > 0$, there exists a continuous retraction*

$$\Psi : J_\rho^{-L} = \{u \in W^{1,2}(\Sigma) : J_\rho(u) \leqslant -L\} \to \overline{\Sigma}_k.$$

Moreover, if $(u_n) \subset W^{1,2}(\Sigma)$ satisfies $\frac{\mathrm{e}^{u_n}}{\int_\Sigma \mathrm{e}^{u_n} dv_g} dv_g \to \sigma \in \overline{\Sigma}_k$, then $\Psi(u_n) \to \sigma \in \overline{\Sigma}_k$.

Proof. Since the proof is almost the same as that of Lemma 2.4, we omit the details here. □

For any finite set E, we denote the number of all the distinct points of E by $\sharp E$. We define

$$\mathscr{S}_k = \{\sigma \in \overline{\Sigma}_k : \sharp(\mathrm{supp}\,\sigma \cap \Sigma) + \sharp\mathrm{supp}\,\sigma \leqslant k\}. \tag{3.27}$$

Let us explain what $\mathscr{S}_k$ means. Clearly, $\mathscr{S}_1 = (\partial\Sigma)_1 = \partial\Sigma$. For $k = 2$,

$$\mathscr{S}_2 = \{\delta_x : x \in \Sigma\} \cup \{t\delta_{x_1} + (1-t)\delta_{x_2} : 0 \leqslant t \leqslant 1, x_1, x_2 \in \partial\Sigma\} = (\partial\Sigma)_2 \cup \Sigma.$$

For $k = 3$, we write

$$\begin{aligned}\mathscr{S}_3 &= \{t\delta_{x_1} + (1-t)\delta_{x_2} : 0 \leqslant t \leqslant 1, x_1 \in \Sigma, x_2 \in \partial\Sigma\}\\ &\quad \cup \{t_1\delta_{x_1} + t_2\delta_{x_2} + t_3\delta_{x_3} : 0 \leqslant t_i \leqslant 1, x_i \in \partial\Sigma, 1 \leqslant i \leqslant 3, t_1 + t_2 + t_3 = 1\}\\ &= (\partial\Sigma)_3 \cup \mathscr{A}_3,\end{aligned}$$

where $\mathscr{A}_3 = \{t\delta_{x_1} + (1-t)\delta_{x_2} : 0 \leqslant t \leqslant 1, x_1 \in \Sigma, x_2 \in \partial\Sigma\}$. We observe that $\dim\mathscr{A}_3 < \dim(\partial\Sigma)_3 = 5$, since $(\partial\Sigma)_3 \setminus (\partial\Sigma)_2$ is a smooth 5-dimensional manifold, and $\dim\mathscr{A}_3 \leqslant 4$.

Lemma 3.4. *For any $k \geqslant 1$, $\mathscr{S}_k$ is non-contractible.*

Proof. Obviously, $\mathscr{S}_1 = \partial\Sigma$ is non-contractible. For $k \geqslant 2$, based on the above observation, an induction argument shows $\mathscr{S}_k = (\partial\Sigma)_k \cup \mathscr{A}_k$, where $\dim\mathscr{A}_k < \dim(\partial\Sigma)_k = 2k-1$. Though $(\partial\Sigma)_k$ is a combination of several branches of different dimensions, we still denote the maximum dimension of those branches by $\dim(\partial\Sigma)_k$. Arguing as in [17, Lemma 4.7], we have that $(\partial\Sigma)_k$ is non-contractible. (This was also noticed by Zhang et al. [40].) In fact, we have

$$H_{2k-1}((\partial\Sigma)_k, \mathbb{Z}_2) \neq \{0\}. \tag{3.28}$$

Give any $(2k-1)$-dimensional closed chain $\mathcal{C}_{2k-1}$ and any $(2k-1)$-dimensional boundary chain $\mathcal{E}_{2k-1}$ of $(\partial\Sigma)_k$. Since $(\partial\Sigma)_k$ is a closed sub-topological space in $\mathscr{S}_k$, and $\dim((\partial\Sigma)_k) = \dim(\mathscr{S}_k) = 2k-1$, one easily sees that $\mathcal{C}_{2k-1}$ is also a closed chain of $\mathscr{S}_k$ and $\mathcal{E}_{2k-1} = 0$ is also the boundary $(2k-1)$-chain of $\mathscr{S}_k$. Hence,

$$H_{2k-1}((\partial\Sigma)_k, \mathbb{Z}_2) \subset H_{2k-1}(\mathscr{S}_k, \mathbb{Z}_2),$$

which together with (3.28) implies that $\mathscr{S}_k$ is non-contractible. □

Take a smooth increasing function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying $\eta(t) = t$ for $t \leqslant 1$ and $\eta(t) = 2$ for $t \geqslant 2$. Set $\eta_r(t) = r\eta(t/r)$ for $r > 0$. For $\lambda > 0$, $x \in \overline{\Sigma}$, and $1 \leqslant \ell < m$, we define

$$\widetilde{\phi}_{\lambda,\sigma}(x) = \log\left(\sum_{i=1}^{\ell} \frac{t_i}{2} \frac{8\lambda^2}{(1+\lambda^2\eta_r^2(\mathrm{dist}(x,x_i)))^2} + \sum_{i=\ell+1}^{m} t_i \frac{8\lambda^2}{(1+\lambda^2\eta_r^2(\mathrm{dist}(x,x_i)))^2}\right) \tag{3.29}$$

and

$$\phi_{\lambda,\sigma}(x) = \widetilde{\phi}_{\lambda,\sigma}(x) - \frac{1}{|\Sigma|}\int_\Sigma \widetilde{\phi}_{\lambda,\sigma} dv_g. \tag{3.30}$$

Lemma 3.5. *Let $\rho \in (4k\pi, 4(k+1)\pi)$. If $\lambda > 0$ is chosen sufficiently large, and $r > 0$ is chosen sufficiently small, then for any $\sigma \in \mathscr{S}_k$, it holds that*

$$J_\rho(\phi_{\lambda,\sigma}) \leqslant (4k\pi - \rho)\log\lambda \tag{3.31}$$

and

$$\frac{\mathrm{e}^{\phi_{\lambda,\sigma}}}{\int_\Sigma \mathrm{e}^{\phi_{\lambda,\sigma}} dv_g} dv_g \to \sigma \quad \text{as } \lambda \to +\infty. \tag{3.32}$$

Proof. Both the cases where $\operatorname{supp}\sigma \cap \Sigma = \emptyset$ and $\operatorname{supp}\sigma \cap \Sigma \neq \emptyset$ can be dealt with in the same way. Give $\sigma \in \mathscr{S}_k$. Without loss of generality, we assume $\operatorname{supp}\sigma = \{x_1, \ldots, x_m\} \subset \overline{\Sigma}$, $\operatorname{supp}\sigma \cap \Sigma = \{x_1, \ldots, x_\ell\}$ and $m + \ell \leqslant k$. Let $\widetilde{\phi}_{\lambda,\sigma}$ and $\phi_{\lambda,\sigma}$ be defined as in (3.29) and (3.30), respectively, where $\lambda > 0$ and $r > 0$. Write $r_i = r_i(x) = \mathrm{dist}(x, x_i)$ for $x \in \overline{\Sigma}$. A simple observation gives

$$\widetilde{\phi}_{\lambda,\sigma}(x) = \begin{cases} \log\dfrac{8\lambda^2}{(1+4\lambda^2r^2)^2} & \text{for } x \in \overline{\Sigma} \setminus \bigcup_{i=1}^{k} B_{2r}(x_i),\\ \log\left(\dfrac{8\lambda^2\bar{t}_i}{(1+\lambda^2\eta_r^2(r_i))^2} + \dfrac{8\lambda^2(1-\bar{t}_i)}{(1+4\lambda^2r^2)^2}\right) & \text{for } x \in B_{2r}(x_i),\end{cases} \tag{3.33}$$

where $\bar{t}_i = t_i/2$ for $1 \leqslant i \leqslant \ell$, $\bar{t}_i = t_i$ for $\ell+1 \leqslant i \leqslant m$, and $B_{2r}(x_i) = \{x \in \overline{\Sigma} : \mathrm{dist}(x, x_i) < 2r\}$ denotes a geodesic ball centered at x_i with radius $2r$. One easily sees $\phi_{\lambda,\sigma} \in W^{1,2}(\Sigma)$ and $\int_\Sigma \phi_{\lambda,\sigma} dv_g = 0$. For $x \in B_{2r}(x_i)$ and $i = 1, \ldots, m$, a straightforward calculation shows

$$\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x) = \frac{\frac{8\lambda^2 \bar{t}_i}{(1+\lambda^2\eta_r^2(r_i))^2}}{\frac{8\lambda^2 \bar{t}_i}{(1+\lambda^2\eta_r^2(r_i))^2} + \frac{8\lambda^2(1-\bar{t}_i)}{(1+4\lambda^2 r^2)^2}} \frac{4\lambda^2 \eta_r(r_i)\eta_r'(r_i)\nabla_g r_i}{1+\lambda^2\eta_r^2(r_i)},$$

and thus,

$$|\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x)| \leqslant \frac{4\lambda^2 \eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)}.$$

In view of (3.33), it holds that $\nabla_g \widetilde{\phi}_{\lambda,\sigma}(x) = 0$ for $x \in \overline{\Sigma} \setminus \bigcup_{i=1}^m B_{2r}(x_i)$. We calculate that for $1 \leqslant i \leqslant \ell$,

$$\int_{B_{2r(x_i)}} \left(\frac{4\lambda^2 \eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)} \right)^2 dv_g = 16\pi(1+O(r^2)) \left(\log(1+\lambda^2 r^2) + \frac{1}{1+\lambda^2 r^2} - 1 \right),$$

and for $\ell+1 \leqslant i \leqslant m$,

$$\int_{B_{2r(x_i)}} \left(\frac{4\lambda^2 \eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)} \right)^2 dv_g = 8\pi(1+O(r^2)) \left(\log(1+\lambda^2 r^2) + \frac{1}{1+\lambda^2 r^2} - 1 \right).$$

For a fixed $r > 0$, since $x_1, \ldots, x_m$ are arbitrary, one sees that $\{B_{2r}(x_1), \ldots, B_{2r}(x_m)\}$ may have nonempty intersections, and thus,

$$\begin{aligned}
\int_\Sigma |\nabla_g \widetilde{\phi}_{\lambda,\sigma}|^2 dv_g &= \int_{\bigcup_{i=1}^m B_{2r}(x_i)} |\nabla_g \widetilde{\phi}_{\lambda,\sigma}|^2 dv_g \\
&\leqslant \sum_{i=1}^m \int_{B_{2r(x_i)}} \left(\frac{4\lambda^2 \eta_r(r_i)\eta_r'(r_i)}{1+\lambda^2\eta_r^2(r_i)} \right)^2 dv_g \\
&\leqslant ((16\pi\ell + 8\pi(m-\ell))(1+O(r^2))) \left(\log(1+\lambda^2 r^2) + \frac{1}{1+\lambda^2 r^2} - 1 \right) + O(1) \\
&\leqslant 8k\pi(1+O(r^2)) \log \lambda^2 + C
\end{aligned} \tag{3.34}$$

for some constant C depending only on r. Moreover, for any s and

$$0 < s < \min \left\{ r, \frac{1}{2} \min_{1 \leqslant i < j \leqslant m} \mathrm{dist}(x_i, x_j) \right\},$$

it holds that

$$\begin{aligned}
\int_{\bigcup_{i=1}^m B_{2r}(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g &= \int_{\bigcup_{i=1}^m B_s(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g + O\left(\frac{1}{\lambda^2 s^2} \right) \\
&= \sum_{i=1}^m \int_{B_s(x_i)} \frac{8\lambda^2 \bar{t}_i}{(1+\lambda^2 r_i^2)^2} dv_g + O\left(\frac{1}{\lambda^2 s^2} \right) \\
&= 4\pi(1+O(s^2)) + O\left(\frac{1}{\lambda^2 s^2} \right)
\end{aligned}$$

and

$$\int_{\Sigma \setminus \bigcup_{i=1}^m B_{2r}(x_i)} \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = O\left(\frac{1}{\lambda^2 r^4} \right).$$

It follows that

$$\int_\Sigma \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}} dv_g = 4\pi(1+O(s^2)) + O\left(\frac{1}{\lambda^2 s^2} \right) + O\left(\frac{1}{\lambda^2 r^4} \right). \tag{3.35}$$

Passing to the limit $\lambda \to +\infty$ first, and then $s \to 0+$, we have

$$\lim_{\lambda\to+\infty}\int_\Sigma \mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}}dv_g = 4\pi. \tag{3.36}$$

Note that there exists some constant C depending only on r such that

$$\frac{1}{|\Sigma|}\int_\Sigma \widetilde{\phi}_{\lambda,\sigma}dv_g \leqslant -\log\lambda^2 + C. \tag{3.37}$$

Hence by (3.36) and (3.37),

$$\int_\Sigma \mathrm{e}^{\phi_{\lambda,\sigma}}dv_g \geqslant C(1+o_\lambda(1))\lambda^2.$$

This together with (3.34) gives

$$\begin{aligned} J_\rho(\phi_{\lambda,\sigma}) &= \frac{1}{2}\int_\Sigma |\nabla_g\phi_{\lambda,\sigma}|^2 dv_g - \rho\log\int_\Sigma h\mathrm{e}^{\phi_{\lambda,\sigma}}dv_g \\ &\leqslant (4k\pi - \rho + O(r^2))\log\lambda^2 + C_r. \end{aligned}$$

Since $\rho > 4k\pi$, choosing $r > 0$ sufficiently small and $\lambda > 0$ sufficiently large, we conclude (3.31).

Finally we prove (3.32). Let $\sigma = \sum_{i=1}^m t_i\delta_{x_i} \in \mathscr{S}_k$ be as above. For any $\varphi \in C^1(\overline{\Sigma})$, similar to (3.35), we calculate

$$\int_\Sigma \varphi\mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}}dv_g = 4\pi\sum_{i=1}^k t_i\varphi(x_i) + O(s^2) + O\left(\frac{1}{\lambda^2 s^2}\right) + O\left(\frac{1}{\lambda^2 r^4}\right).$$

Letting $\lambda \to +\infty$ first, and then $s \to 0+$, we obtain

$$\lim_{\lambda\to+\infty}\int_\Sigma \varphi\mathrm{e}^{\widetilde{\phi}_{\lambda,\sigma}}dv_g = 4\pi\sum_{i=1}^k t_i\varphi(x_i).$$

This together with (3.36) implies (3.32). □

Similar to (2.34), for a sufficiently small $\epsilon_0 > 0$, we have a continuous retraction

$$\mathfrak{p} : \{\sigma \in \mathcal{D}(\Sigma) : \mathbf{d}(\sigma, \mathscr{S}_k) < \epsilon_0\} \to \mathscr{S}_k.$$

Lemma 3.6. *Let Ψ and L be as in Lemma 3.3. If $\lambda > 0$ is chosen sufficiently large, then there exists a continuous map $\Phi_\lambda : \mathscr{S}_k \to J_\rho^{-L}$ such that $\mathfrak{p}\circ\Psi\circ\Phi_\lambda : \mathscr{S}_k \to \mathscr{S}_k$ is homotopic to the identity map $\mathrm{Id} : \mathscr{S}_k \to \mathscr{S}_k$.*

Proof. Let $\phi_{\lambda,\sigma}$ be constructed as in Lemma 3.5. For any $\sigma \in \mathscr{S}_k$, we define $\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}$ for large $\lambda > 0$. Clearly, the map $\Phi_\lambda : \mathscr{S}_k \to W^{1,2}(\Sigma)$ is continuous. By (3.31), if $\lambda \geqslant \mathrm{e}^{L/(\rho-4k\pi)}$, then $J_\rho(\phi_{\lambda,\sigma}) \leqslant -L$. Thus $\Phi_\lambda(\sigma) \in J_\rho^{-L}$. By Lemma 3.3 and (3.32), it holds that

$$\begin{aligned} \mathfrak{p}\circ\Psi\circ\Phi_\lambda(\sigma) &= \mathfrak{p}\circ\Psi(\phi_{\lambda,\sigma}) \\ &= \mathfrak{p}\circ\psi_k\left(\frac{\mathrm{e}^{\phi_{\lambda,\sigma}}}{\int_\Sigma \mathrm{e}^{\phi_{\lambda,\sigma}}dv_g}dv_g\right) \\ &\to \sigma \end{aligned}$$

as $\lambda \to +\infty$. Hence, $\mathfrak{p}\circ\Psi\circ\Phi_\lambda$ is homotopic to $\mathrm{Id} : \mathscr{S}_k \to \mathscr{S}_k$. □

3.4 Min-max values

Let

$$\mathcal{H} = \left\{u \in W^{1,2}(\Sigma) : \int_\Sigma u dv_g = 0\right\}$$

and

$$\widehat{\mathscr{S}_k} = \mathscr{S}_k \times [0,1]/(\mathscr{S}_k \times \{0\})$$

be the topological cone over $\mathscr{S}_k$. A path set associated with the metric space $\widehat{\mathscr{S}_k}$ is defined by

$$\Gamma_\lambda = \{\gamma \in C^0(\widehat{\mathscr{S}_k}, \mathcal{H}) : \gamma|_{\widehat{\mathscr{S}_k}\times\{1\}} \in \Gamma_{\lambda,0}\},$$

where $\Gamma_{\lambda,0}$ is given by

$$\Gamma_{\lambda,0} = \{\gamma \in C^0(\mathscr{S}_k \times \{1\}, \mathcal{H}) : \gamma(\sigma,1) = \Phi_\lambda(\sigma), \forall\, \sigma \in \mathscr{S}_k\}.$$

If we write a path $\overline{\gamma} : \widehat{\mathscr{S}_k} \to \mathcal{H}$ by $\overline{\gamma}(\sigma,t) = t\phi_{\lambda,\sigma}$, then $\overline{\gamma} \in \Gamma_\lambda$, and thus $\Gamma_\lambda \neq \emptyset$.

For real numbers λ and ρ, we set

$$\alpha_{\lambda,\rho} = \inf_{\gamma\in\Gamma_\lambda} \sup_{(\sigma,t)\in\widehat{\mathscr{S}_k}} J_\rho(\gamma(\sigma,t))$$

and

$$\beta_{\lambda,\rho} = \sup_{\gamma\in\Gamma_{\lambda,0}} \sup_{(\sigma,t)\in\mathscr{S}_k\times\{1\}} J_\rho(\gamma(\sigma,t)).$$

Lemma 3.7. *Let $\rho \in (4k\pi, 4(k+1)\pi)$. If λ is chosen sufficiently large, and r is chosen sufficiently small, then $-\infty < \beta_{\lambda,\rho} < \alpha_{\lambda,\rho} < +\infty$.*

Proof. The proof is very similar to that of Lemma 2.7. It suffices to use Lemma 3.6 instead of the fact that $\pi\circ\Psi\circ\Phi_\lambda : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$ is homotopic to $\mathrm{Id} : \Sigma_{\epsilon,k} \to \Sigma_{\epsilon,k}$, and use Lemma 3.4 instead of the non-contractibility of $\Sigma_{\epsilon,k}$. □

3.5 Completion of the proof of Theorem 1.2

Define $\alpha_\rho = \alpha_{\lambda,\rho}$ for sufficiently large $\lambda > 0$. Similar to Lemma 2.9, α_ρ/ρ is decreasing in $\rho \in (4k\pi, 4(k+1)\pi)$. Let

$$\Lambda_k = \left\{\rho \in (4k\pi, 4(k+1)\pi) : \frac{\alpha_\rho}{\rho} \text{ is differentiable at } \rho\right\}.$$

In view of an analog of Lemma 2.11, α_ρ is a critical value of J_ρ for any $\rho \in \Lambda_k$.

Now we let $\rho \in (4k\pi, 4(k+1)\pi)$. Take an increasing sequence of numbers $(\rho_n) \subset \Lambda_k$ such that $\rho_n \to \rho$, $(\rho_n) \subset [a,b] \subset (4k\pi, 4(k+1)\pi)$, and α_{ρ_n} is achieved by $u_n \in \mathcal{H}$. Moreover, u_n satisfies the Euler-Lagrange equation

$$\Delta_g u_n = \rho_n\left(\frac{he^{u_n}}{\int_\Sigma he^{u_n}dv_g} - \frac{1}{|\Sigma|}\right). \tag{3.38}$$

Since α_ρ/ρ is decreasing in $\rho \in [a,b]$,

$$\alpha_{\rho_n} \leqslant \frac{b}{a}\alpha_a. \tag{3.39}$$

Define $v_n = u_n - \log\int_\Sigma he^{u_n}dv_g$, and we have

$$\begin{cases} \Delta_g v_n = \rho_n(he^{v_n} - |\Sigma|^{-1}), \\ \displaystyle\int_\Sigma he^{v_n}dv_g = 1. \end{cases}$$

By Lemma 3.1, (u_n) is bounded in $L^\infty(\overline{\Sigma})$. Let $\Omega_1,\ldots,\Omega_{k+1}$ be disjoint closed sub-domains of $\overline{\Sigma}$. It follows from Lemma 3.2 that

$$\log\int_\Sigma e^{u_n}dv_g \leqslant \frac{1}{8(k+1)\pi-\epsilon}\int_\Sigma |\nabla_g u_n|^2 dv_g + C_\epsilon$$

for any $\epsilon > 0$ and some constant $C_\epsilon > 0$. This together with (3.39) implies that for $0 < \epsilon < 8(k+1)\pi - 2b$,

$$\frac{1}{2}\int_\Sigma |\nabla_g u_n|^2 dv_g = J_{\rho_n}(u_n) + \rho_n \log \int_\Sigma h e^{u_n} dv_g$$
$$\leqslant \frac{b}{8(k+1)\pi - \epsilon}\int_\Sigma |\nabla_g u_n|^2 dv_g + C.$$

Then it follows that (u_n) is bounded in $\mathcal{H}$. Without loss of generality, we assume that u_n converges to u_0 weakly in $\mathcal{H}$, strongly in $L^p(\Sigma)$ for any $p > 1$, and almost everywhere in $\overline{\Sigma}$. Moreover, e^{u_n} converges to e^{u_0} strongly in $L^p(\Sigma)$ for any $p > 1$. By (3.38), u_0 satisfies

$$\Delta_g u_0 = \rho\bigg(\frac{h e^{u_0}}{\int_\Sigma h e^{u_0} dv_g} - \frac{1}{|\Sigma|}\bigg)$$

in the distributional sense. In particular, u_0 is a critical point of J_ρ.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11721101) and the National Key Research and Development Project (Grant No. SQ2020YFA070080). The second author was supported by Hubei Provincial Natural Science Foundation of China (Grant No. 2021CFB400) and National Natural Science Foundation of China (Grant No. 11971358).

References

1 Ao W W, Jevnikar A, Yang W. On the boundary behavior for the blow-up solutions of the sinh-Gordon equation and rank N Toda systems in bounded domains. Int Math Res Not IMRN, 2020, 2020: 9386–9419

2 Aubin T. Nonlinear Analysis on Manifolds. Monge-Ampère Equations. Grundlehren der mathematischen Wissenschaften, vol. 252. New York: Springer-Verlag, 1982

3 Battaglia L, Jevnikar A, Malchiodi A, et al. A general existence result for the Toda system on compact surfaces. Adv Math, 2015, 285: 937–979

4 Berger M S. On Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds. J Differential Geom, 1971, 5: 325–332

5 Brezis H, Merle F. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. Comm Partial Differential Equations, 1991, 16: 1223–1253

6 Caffarelli L A, Yang Y S. Vortex condensation in the Chern-Simons Higgs model: An existence theorem. Comm Math Phys, 1995, 168: 321–336

7 Chang S Y A, Yang P C. Prescribing Gaussian curvature on S^2. Acta Math, 1987, 159: 215–259

8 Chang S Y A, Yang P C. Conformal deformation of metrics on S^2. J Differential Geom, 1988, 27: 259–296

9 Chen C C, Lin C S. Topological degree for a mean field equation on Riemann surfaces. Comm Pure Appl Math, 2003, 56: 1667–1727

10 Chen W X, Ding W Y. Scalar curvatures on S^2. Trans Amer Math Soc, 1987, 303: 365–382

11 Chen W X, Li C M. Prescribing Gaussian curvatures on surfaces with conical singularities. J Geom Anal, 1991, 1: 359–372

12 De Marchis F, Malchiodi A, Martinazzi L, et al. Critical points of the Moser-Trudinger functional on closed surfaces. Invent Math, 2022, 230: 1165–1248

13 Ding W, Jost J, Li J, et al. The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. Asian J Math, 1997, 1: 230–248

14 Ding W, Jost J, Li J, et al. An analysis of the two-vortex case in the Chern-Simons Higgs model. Calc Var Partial Differential Equations, 1998, 7: 87–97

15 Ding W, Jost J, Li J, et al. Existence results for mean field equations. Ann Inst H Poincaré Anal Non Linéaire, 1999, 16: 653–666

16 Ding W, Jost J, Li J, et al. Multiplicity results for the two-vortex Chern-Simons Higgs model on the two-sphere. Comment Math Helv, 1999, 74: 118–142

17 Djadli Z. Existence result for the mean field problem on Riemann surfaces of all genuses. Commun Contemp Math, 2008, 10: 205–220

18 Djadli Z, Malchiodi A. Existence of conformal metrics with constant Q-curvature. Ann of Math (2), 2008, 168: 813–858

19 Guo Y X, Liu J Q. Blow-up analysis for solutions of the Laplacian equation with exponential Neumann boundary condition in dimension two. Commun Contemp Math, 2006, 8: 737–761

20 Jiang N. The equation $\Delta u = 8\pi(1 - he^u)$ on compact Riemann surface with boundary. Master's Thesis. Beijing: Institue of Mathematics, Chinese Academy of Sciences, 1998

21 Kallel S, Karoui R. Symmetric joins and weighted barycenters. Adv Nonlinear Stud, 2011, 11: 117–143

22 Kazdan J L, Warner F W. Curvature functions for compact 2-manifolds. Ann of Math (2), 1974, 99: 14–47

23 Li Y Y. Harnack type inequality: The method of moving planes. Comm Math Phys, 1999, 200: 421–444

24 Li Y Y, Shafrir I. Blow-up analysis for solutions of $-\Delta u = V\mathrm{e}^u$ in dimension two. Indiana Univ Math J, 1994, 43: 1255–1270

25 Malchiodi A, Ndiaye C B. Some existence results for the Toda system on closed surfaces. Atti Accad Naz Lincei Rend Lincei Mat Appl, 2007, 18: 391–412

26 Malchiodi A, Ruiz D. A variational analysis of the Toda system on compact surfaces. Comm Pure Appl Math, 2013, 66: 332–371

27 Marchioro C, Pulvirenti M. Mathematical Theory of Incompressible Nonviscous Fluids. Applied Mathematical Sciences, vol. 96. New York: Springer-Verlag, 1994

28 Struwe M. The evolution of harmonic mappings with free boundaries. Manuscripta Math, 1991, 70: 373–384

29 Struwe M, Tarantello G. On multivortex solutions in Chern-Simons gauge theory. Boll Unione Mat Ital, 1998, 1: 109–121

30 Sun L, Wang Y, Yang Y. Existence results for a generalized mean field equation on a closed Riemann surface. arXiv:2101.03859, 2021

31 Sun L, Zhu J. Existence of Kazdan-Warner equation with sign-changing prescribed function. arXiv:2012.12840, 2020

32 Sun L, Zhu J. Global existence and convergence of a flow to Kazdan-Warner equation with non-negative prescribed function. Calc Var Partial Differential Equations, 2021, 60: 42

33 Tarantello G. Multiple condensate solutions for the Chern-Simons-Higgs theory. J Math Phys, 1996, 37: 3769–3796

34 Willem M. Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Boston: Birkhäuser, 1996

35 Yang Y S. On a system of nonlinear elliptic equations arising in theoretical physics. J Funct Anal, 2000, 170: 1–36

36 Yang Y Y. Extremal functions for Moser-Trudinger inequalities on 2-dimensional compact Riemannian manifolds with boundary. Internat J Math, 2006, 17: 313–330

37 Yang Y Y. On a sharp inequality of L. Fontana for compact Riemannian manifolds. Manuscripta Math, 2018, 157: 51–79

38 Yang Y Y, Zhou J. Blow-up analysis involving isothermal coordinates on the boundary of compact Riemann surface. J Math Anal Appl, 2021, 504: 125440

39 Yang Y Y, Zhu X B. A remark on a result of Ding-Jost-Li-Wang. Proc Amer Math Soc, 2017, 145: 3953–3959

40 Zhang T, Zhou C, Zhou C. Existence of solutions for the Laplacian equation with exponential Neumann boundary condition. Front Math China, 2022, in press