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Calculus of Variations

An optimal pinching theorem of minimal Legendrian submanifolds in the unit sphere

Yong Luo[1] · Linlin Sun[2] · Jiabin Yin[3,4]

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Abstract
In this paper, we study the rigidity theorem of closed minimally immersed Legendrian submanifolds in the unit sphere. Utilizing the maximum principle, we obtain a new characterization of the Calabi torus in the unit sphere which is the minimal Calabi product Legendrian immersion of a point and the totally geodesic Legendrian sphere. We also establish an optimal Simons' type integral inequality in terms of the second fundamental form of three-dimensional closed minimal Legendrian submanifolds in the unit sphere. Our optimal rigidity results for minimal Legendrian submanifolds in the unit sphere are new and also can be applied to minimal Lagrangian submanifolds in the complex projective space.

Mathematics Subject Classification 53C24 · 53C40

1 Introduction

Let M be an n-dimensional closed minimally immersed submanifold in the unit sphere $\mathbb{S}^{n+m}$ of dimension $n+m$. Let $\mathbf{B}$ be the second fundamental form of this immersion. Simons [27], Chern, do Carmo and Kobayashi [7], Lawson [16] proved that under the pinching condition $|\mathbf{B}|^2 \leq \frac{n}{2-\frac{1}{m}}$, M must be either one of the Clifford minimal tori $\mathbf{S}^p\left(\sqrt{\frac{p}{n}}\right) \times \mathbf{S}^{n-p}\left(\sqrt{\frac{n-p}{n}}\right)$ in $\mathbb{S}^{n+1}$ or the Veronese surface in $\mathbb{S}^4$ unless M is the totally geodesic sphere $\mathbb{S}^n$ in $\mathbb{S}^{n+1}$. Li and Li [17] improved Simons' pinching constant to $\frac{2n}{3}$ for higher codimension $m \geq 3$.

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✉ Linlin Sun
sunll@whu.edu.cn

Extended author information available on the last page of the article

Chen and Xu [6] obtained the same pinching constant independently by using a different method. These rigidity results mentioned above can be viewed as an intrinsic rigidity theorem for pinching of the scalar curvature according to the Gauss equation. The intrinsic rigidity theorem for pinching of the sectional curvature was obtained by Yau [32], for pinching of the Ricci curvature by Ejiri [11]. The extrinsic rigidity theorem for pinching of the second fundamental form was obtained by Gauchman [13].

There are many papers on the particularly interesting case of closed minimal Legendrian submanifolds in the unit sphere $\mathbb{S}^{2n+1}$ or closed minimal Lagrangian submanifolds in $\mathbb{CP}^n$ (for an incomplete list, see e.g. [2, 5, 22, 30] for pinching of the scalar curvature, [3, 10, 23, 28, 31, 32] for pinching of the sectional curvature, [19] for pinching of the Ricci curvature). Inspired by papers of Ros [24, 25] on pinching and rigidity of Kähler submanifolds, Gauchman [14] and Xia [29] studied pinching of the geometric quantity

$$\Theta(p) := \max_{X \in T_pM,\ |X|=1} |\mathbf{B}(X, X)|$$

for closed Lagrangian submanifolds in $\mathbb{CP}^n$. In particular, Xia [29] proved that if $\Theta^2 \leq 1/2$, then either M is totally geodesic or $\Theta^2 \equiv 1/2$ and the last case was classified completely.

These curvature pinching and characterization results were proved based on analysis of a Simons' type formula. This formula is related to a special sort of submanifolds, those that have parallel second fundamental form. Lagrangian submanifolds in $\mathbb{CP}^n$ with parallel second fundamental form were completely classified by Naitoh [20, 21] for the irreducible case and by Dillen, Li, Vrancken and Wang [8] in the general case. The classification theorem of Dillen, Li, Vrancken and Wang states that Lagrangian submanifolds with parallel second fundamental form in $\mathbb{CP}^n$ are one of the following:

(a) totally geodesic submanifolds;
(b) embedded submanifolds which are locally congruent to one of the following standard embeddings in $\mathbb{CP}^n$:

$$\begin{aligned} \mathrm{SU}(k)/\mathrm{SO}(k), &\quad n = (k-1)(k+2)/2, \quad k \geq 3, \\ \mathrm{SU}(k), &\quad n = k^2 - 1, \quad k \geq 3, \\ \mathrm{SU}(2k)/\mathrm{Sp}(k), &\quad n = 2k^2 - k - 1, \quad k \geq 3, \\ \mathrm{E}_6/\mathrm{F}_4, &\quad n = 26; \end{aligned}$$

(c) locally a finite Riemannian covering of the unique flat torus, minimally embedded in $\mathbb{CP}^2$ with parallel second fundamental form;
(d) locally the Calabi product of a point with a lower dimensional Lagrangian submanifold with parallel second fundamental form;
(e) locally the Calabi product of two lower dimensional Lagrangian submanifolds with parallel second fundamental form.

The examples of a)-c) are minimal Lagrangian submanifolds, but examples of d)-e) contain both minimal and non-minimal ones. Furthermore the unique minimal submanifold in d) is the so called Calabi torus, which is the image of Example 1.1 by the Hopf fibration of $\mathbb{S}^{2n+1}$ to $\mathbb{CP}^n$.

Example 1.1 (Calabi torus) Let $\gamma = (\gamma_1, \gamma_2) : \mathbb{S}^1 \longrightarrow \mathbb{S}^3$ be a Legendrian curve in $\mathbb{S}^3 \subset \mathbb{C}^2$ defined by

$$\gamma(t) = \left(\sqrt{\frac{n}{n+1}} \exp\left(\sqrt{-1} \sqrt{\frac{1}{n}} t \right), \sqrt{\frac{1}{n+1}} \exp\left(-\sqrt{-1} \sqrt{n} t \right) \right)$$

and $\phi : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ the totally geodesic Legendrian sphere. Then $F := (\gamma_1\phi, \gamma_2) : \mathbb{S}^1 \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ is a minimal Legendrian immersion. Denote by $M := F\left(\mathbb{S}^1 \times \mathbb{S}^{n-1}\right)$. We call this minimal Legendrian submanifold M the *Calabi torus*. One can choose a local orthonormal frame of TM such that the second fundamental form $\mathbf{B}$ satisfies

$$\mathbf{B}\left(e_1, e_j\right) = -\sqrt{\frac{1}{n}} J e_j + \delta_{1j}\sqrt{n} J e_1, \quad j \in \{1, \ldots, n\},$$

$$\mathbf{B}\left(e_i, e_j\right) = -\delta_{ij}\sqrt{\frac{1}{n}} J e_1, \quad i, j \in \{2, \ldots, n\}.$$

One can check that

$$|\mathbf{B}|^2 = \frac{(n-1)(n+2)}{n}, \quad \Theta = \max_{X \in TM,\ |X|=1} |\mathbf{B}(X, X)| = \frac{n-1}{\sqrt{n}}.$$

Thus

$$|\mathbf{B}|^2 = \frac{n+2}{\sqrt{n}}\Theta,$$

and for $n = 3$

$$|\mathbf{B}|^2 = 2 + \Theta^2 = \frac{10}{7}\left(1 + \Theta^2\right).$$

The above mentioned papers in paragraph 2 gave various curvature pinching and characterization results for compact minimal Lagrangian submanifolds of a) (cf. [2, 3, 5, 23, 28, 30]); a) and c) (cf. [10, 22, 31, 32]); and a), b) when $k = 3$ and c) (cf. [14, 29]). Nevertheless according to our knowledge such kind of result was missing for the examples in d) and e). Bewaring of this, Luo and Sun [18] conjectured that *if M is a closed minimal Legendrian submanifold in $\mathbb{S}^{2n+1}$ and $|\mathbf{B}|^2 \leq (n+2)(n-1)/n$, then M is either the totally geodesic sphere or the Calabi torus* (cf. Example 1.1). In this paper we aim to get a curvature pinching and characterization result for the Calabi torus and we obtain the following theorem.

Theorem 1.1 *Let M be a closed minimal Legendrian submanifold in the unit sphere $\mathbb{S}^{2n+1}(n \geq 2)$ and $\mathbf{B}$ its second fundamental form. Assume the following pinching condition holds pointwisely,*

$$|\mathbf{B}|^2 \leq \frac{n+2}{\sqrt{n}}\Theta. \tag{1.1}$$

Then M is either the totally geodesic sphere or the Calabi torus.

If $n = 3$, the pinching condition can be changed weakly to

$$|\mathbf{B}|^2 \leq 2 + \Theta^2.$$

Remark 1.1 Checking the proof step by step, we can prove that a closed minimal Legendrian submanifold M in the unit sphere $\mathbb{S}^7$ with $\Theta^2 \leq 2/3$ must be the totally geodesic sphere. Therefore we improve Xia's result [29] for minimal Lagrangian submanifolds in $\mathbb{CP}^3$. From this, in case of dimension 3, we can obtain Li and Li's type pinching result [17] , i.e., M is totally geodesic if $|\mathbf{B}|^2 \leq 8/3$.

It is worth noting that this theorem could be stated similarly for closed minimal Lagrangian submanifolds in $\mathbb{CP}^n$, due to the well known correspondence of minimal Legendrian submanifolds in $\mathbb{S}^{2n+1}$ and minimal Lagrangian submanifolds in $\mathbb{CP}^n$ (cf. [4]), or by proofs with similar arguments.

We prove Theorem 1.1 by applying a maximum principle for tensors and a Simons' type formula of closed minimal Legendrian submanifolds in the unit sphere. We will also use an integral method to get an integral inequality of three-dimensional closed minimal Legendrian submanifolds in $\mathbb{S}^7$, which implies another pinching and rigidity result for the three-dimensional Calabi torus in Example 1.1 (cf. Theorem 3.1).

In sect. 2 we give some preliminaries on Legendrian submanifolds of the unit sphere, including a Simons' type formula. In sect. 3 we prove an integral inequality of closed Legendrian submanifolds in $\mathbb{S}^7$. Theorem 1.1 is proved in sect. 4. In the Appendix we prove an integral inequality of closed Lagrangian submanifolds in the nearly Kähler $\mathbb{S}^6$ by similar arguments used in the proof of Theorem 3.1, which improves the main theorem of Hu, Yin and Yin [15].

2 Preliminaries

Here we briefly record several facts about Legendrian submanifolds in the unit sphere. We refer readers to [1] for more material on contact geometry.

Let M be a closed n-dimensional submanifold of the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. We say that M is Legendrian if

$$JTM \subset T^{\perp}M, \quad JF \in \Gamma\left(T^{\perp}M\right)$$

where $F : M \longrightarrow \mathbb{S}^{2n+1}$ is the position vector and J is the complex structure of $\mathbb{C}^{n+1}$. We say that M is a minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$ if M is a minimal and Legendrian submanifold of $\mathbb{S}^{2n+1}$. Define

$$\sigma(X, Y, Z) := \langle \mathbf{B}(X, Y), JZ \rangle, \quad \forall X, Y, Z \in TM.$$

The Weingarten equation implies that

$$\sigma(X, Y, Z) = \sigma(Y, X, Z).$$

Moreover, by definition, one can check that σ is a tri-linear symmetric tensor, i.e.,

$$\sigma(X, Y, Z) = \sigma(Y, X, Z) = \sigma(X, Z, Y).$$

The Gauss equation, Codazzi equation and Ricci equation become

$$\begin{aligned}
R(X, Y, Z, W) &= \langle X, Z\rangle \langle Y, W\rangle - \langle X, W\rangle \langle Y, Z\rangle + \sum_i \sigma(X, Z, e_i)\sigma(Y, W, e_i) \\
&\quad - \sum_i \sigma(X, W, e_i)\sigma(Y, Z, e_i), \\
(\nabla_X \sigma)(Y, Z, W) &= (\nabla_Y \sigma)(X, Z, W), \\
R^{\perp}(X, Y, JZ, JW) &= \sum_i \sigma(X, Z, e_i)\sigma(Y, W, e_i) - \sum_i \sigma(X, W, e_i)\sigma(Y, Z, e_i),
\end{aligned}$$

where $\{e_i\}$ is an orthonormal basis of TM. The Codazzi equation implies

$$(\nabla_X \sigma)(Y, Z, W) = (\nabla_Y \sigma)(X, Z, W) = (\nabla_X \sigma)(Z, Y, W) = (\nabla_X \sigma)(Y, W, Z),$$

i.e., $\nabla\sigma$ is a four-linear symmetric tensor.

We will need the following Simons' identity (cf. [27], see also [5, 30]).

Lemma 2.1 (Simons' identity) *Assume that M is a minimal Legendrian submanifold in* $\mathbb{S}^{2n+1}$. *Then*

$$\begin{aligned}\Delta\sigma_{ijk} := & \sum_l \sigma_{ijk,ll} \\ = & (n+1)\sigma_{ijk} + 2\sum_{l,s,t}\sigma_{isl}\sigma_{jlt}\sigma_{kts} - \sum_{l,s,t}\sigma_{tli}\sigma_{tls}\sigma_{jks} \\ & - \sum_{l,s,t}\sigma_{tlj}\sigma_{tls}\sigma_{iks} - \sum_{l,s,t}\sigma_{tlk}\sigma_{tls}\sigma_{ijs}.\end{aligned} \tag{2.1}$$

Consequently,

$$\frac{1}{2}\Delta|\sigma|^2 = |\nabla\sigma|^2 + (n+1)|\sigma|^2 - \sum_{i,j}\langle\sigma_i,\sigma_j\rangle^2 - \sum_{i,j}\left|[\sigma_i,\sigma_j]\right|^2, \tag{2.2}$$

where $\sigma_i = \left(\sigma_{ijk}\right)_{1\le j,k\le n}$.

Proof The proof is classical. For readers' convenience, we provide details of the proof here. The Ricci identity yields

$$\sigma_{ijk,lm} = \sigma_{ijk,ml} + \sum_t \sigma_{tjk}R_{tilm} + \sum_t \sigma_{itk}R_{tjlm} + \sum_t \sigma_{ijt}R_{tklm}.$$

Therefore,

$$\begin{aligned}\Delta\sigma_{ijk} = & \sum_l \sigma_{ijk,ll} \\ = & \sum_l \sigma_{ijl,kl} \\ = & \sum_l \sigma_{ijl,lk} + \sum_{l,t}\sigma_{tjl}R_{tikl} + \sum_{l,t}\sigma_{itl}R_{tjkl} + \sum_{l,t}\sigma_{ijt}R_{tlkl} \\ = & \mu_{i,jk} + \sum_{l,t}\sigma_{tjl}R_{tikl} + \sum_{l,t}\sigma_{itl}R_{tjkl} + \sum_{l,t}\sigma_{ijt}R_{tlkl}.\end{aligned}$$

Here $\mu_i = \operatorname{tr}\sigma_i$. Thus by the Gauss equation,

$$\begin{aligned}\Delta\sigma_{ijk} = & \mu_{i,jk} + \sum_{l,t}\sigma_{tjl}\left(\delta_{tk}\delta_{il} - \delta_{tl}\delta_{ik} + \sigma_{tks}\sigma_{ils} - \sigma_{tls}\sigma_{iks}\right) \\ & + \sum_{l,t}\sigma_{til}\left(\delta_{tk}\delta_{jl} - \delta_{tl}\delta_{jk} + \sigma_{tks}\sigma_{jls} - \sigma_{tls}\sigma_{jks}\right) \\ & + \sum_{l,t}\sigma_{ijt}\left((n-1)\delta_{tk} + \sigma_{tks}\sigma_{lls} - \sigma_{tls}\sigma_{lks}\right) \\ = & \mu_{i,jk} + \sigma_{ijk} - \mu_j\delta_{ik} + \sum_{l,s,t}\sigma_{tjl}\left(\sigma_{tks}\sigma_{ils} - \sigma_{tls}\sigma_{iks}\right) \\ & + \sigma_{ijk} - \mu_i\delta_{jk} + \sum_{l,s,t}\sigma_{til}\left(\sigma_{tks}\sigma_{jls} - \sigma_{tls}\sigma_{jks}\right) \\ & + (n-1)\sigma_{ijk} + \sum_{l,s,t}\sigma_{ijt}\left(\sigma_{tks}\mu_s - \sigma_{tls}\sigma_{lks}\right) \\ = & \mu_{i,jk} - \mu_i\delta_{jk} - \mu_j\delta_{ik} + \sum_{s,t}\sigma_{ijt}\sigma_{tks}\mu_s\end{aligned}$$

$$
\begin{aligned}
&+(n+1)\sigma_{ijk}+2\sum_{l,s,t}\sigma_{tjl}\sigma_{tks}\sigma_{ils}-\sum_{l,s,t}\sigma_{tjl}\sigma_{tls}\sigma_{iks}\\
&-\sum_{l,s,t}\sigma_{til}\sigma_{tls}\sigma_{jks}-\sum_{l,s,t}\sigma_{tls}\sigma_{lks}\sigma_{ijt}\\
=&(n+1)\sigma_{ijk}+2\sum_{l,s,t}\sigma_{tjl}\sigma_{tks}\sigma_{ils}-\sum_{l,s,t}\sigma_{tjl}\sigma_{tls}\sigma_{iks}\\
&-\sum_{l,s,t}\sigma_{til}\sigma_{tls}\sigma_{jks}-\sum_{l,s,t}\sigma_{tls}\sigma_{lks}\sigma_{ijt},
\end{aligned}
$$

where we used the fact that $\mu_i = 0$ since M is minimal. □

3 An integral inequality for the three-dimensional case

In this section we prove an integral inequality for closed three-dimensional minimal Legendrian submanifolds in $\mathbb{S}^7$, which is inspired by a recent paper of Hu,Yin and Yin [15].

Theorem 3.1 *Let M be a closed minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7$. Then*

$$\int_M |\mathbf{B}|^2\left(|\mathbf{B}|^2-\frac{10}{7}\left(1+\Theta^2\right)\right)\geq 0. \tag{3.1}$$

Consequently, if

$$|\mathbf{B}|^2\leq\frac{10}{7}\left(1+\Theta^2\right),$$

then M is either the totally geodesic sphere or the Calabi torus.

Remark 3.1 We would like to point out that though the pinching result we obtain by integral estimates here is actually weaker than that we obtain in Theorem 1.1, we can not get any integral inequality like (3.1) by the maximum principle used in the proof of Theorem 1.1. Furthermore here we slightly refine the argument in [15] and use it to give an improvement of the main theorem in [15], please see Theorem Appendix A.1 in the Appendix for details. It seems that the maximum principle is not applicable in proving pinching results for Lagrangian submanifolds in the nearly Kähler $\mathbb{S}^6$.

Proof of Theorem 3.1 Consider an algebraic curvature $\hat{R}$ defined by

$$\hat{R}_{ijkl}=\left\langle\mathbf{B}\left(e_i,e_k\right),\mathbf{B}\left(e_j,e_l\right)\right\rangle-\left\langle\mathbf{B}\left(e_i,e_l\right),\mathbf{B}\left(e_j,e_k\right)\right\rangle,$$

i.e.,

$$\hat{R}_{ijkl}=\sum_a\left(\sigma_{ika}\sigma_{jla}-\sigma_{ila}\sigma_{jka}\right)=[\sigma_i,\sigma_j]_{kl}.$$

The algebraic Ricci curvature $\hat{R}ic$ and the algebraic scalar curvature $\hat{S}$ are given by

$$\hat{R}ic_{ij}=\sum_a\hat{R}_{iaja}=-\left\langle\sigma_i,\sigma_j\right\rangle,\quad \hat{S}=\sum_i\hat{R}ic_{ii}=-|\sigma|^2.$$

We then can rewrite Simons' identity (2.2) as follows

$$\frac{1}{2}\Delta|\sigma|^2=|\nabla\sigma|^2+(n+1)|\sigma|^2-\left|\hat{R}ic\right|^2-\left|\hat{R}\right|^2. \tag{3.2}$$

Recall the orthogonal decomposition for the algebraic curvature

$$\hat{R} = \hat{W} + \frac{1}{n-2}\mathring{\hat{R}ic} \owedge g + \frac{\hat{S}}{2n(n-1)} g \owedge g,$$

where $\hat{W}$ is the algebraic Weyl curvature and $\mathring{\hat{R}ic} = \hat{R}ic - \frac{\hat{S}}{n}g$ is the traceless algebraic Ricci curvature. We have the following identity

$$\begin{aligned}\left|\hat{R}\right|^2 &= \left|\hat{W}\right|^2 + \frac{4\left|\mathring{\hat{R}ic}\right|^2}{n-2} + \frac{2\hat{S}^2}{n(n-1)} \\ &= \left|\hat{W}\right|^2 + \frac{4\left|\hat{R}ic\right|^2}{n-2} - \frac{2\hat{S}^2}{(n-1)(n-2)}.\end{aligned}$$

For $n = 3$, the algebraic Weyl curvature $\hat{W}$ vanishes. It follows from (3.2) that

$$\begin{aligned}\frac{1}{2}\Delta |\sigma|^2 &= |\nabla\sigma|^2 + 4|\sigma|^2 - 5\left|\hat{R}ic\right|^2 + \left|\hat{S}\right|^2 \\ &= |\nabla\sigma|^2 + 4|\sigma|^2 - 5\sum_{i,j=1}^{3}\left\langle\sigma_i, \sigma_j\right\rangle^2 + |\sigma|^4.\end{aligned}$$

At a point p, choose e_1 such that

$$\sigma_{111} = \max_{X\in S_pM^3} \sigma(X, X, X),$$

then $\sigma_{112} = \sigma_{113} = 0$. Then we choose $\{e_2, e_3\}$ such that $\sigma_{123} = 0$. In other words, we may assume

$$\sigma_1 = \begin{pmatrix}\lambda_1+\lambda_2 & 0 & 0\\ 0 & -\lambda_1 & 0\\ 0 & 0 & -\lambda_2\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -\lambda_1 & 0\\ -\lambda_1 & \mu_1 & \mu_2\\ 0 & \mu_2 & -\mu_1\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}0 & 0 & -\lambda_2\\ 0 & \mu_2 & -\mu_1\\ -\lambda_2 & -\mu_1 & -\mu_2\end{pmatrix}.$$

A direct calculation yields

$$\begin{aligned}|\sigma|^2 =& 4\lambda_1^2 + 4\lambda_2^2 + 2\lambda_1\lambda_2 + 4\left(\mu_1^2+\mu_2^2\right) \\ =& \frac{5}{2}(\lambda_1+\lambda_2)^2 + \frac{3}{2}(\lambda_1-\lambda_2)^2 + 4\left(\mu_1^2+\mu_2^2\right), \\ \sum_{i,j}\left\langle\sigma_i,\sigma_j\right\rangle^2 =& 4\left(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2\right)^2 + 4\left(\lambda_1^2+\mu_1^2+\mu_2^2\right)^2 + 4\left(\lambda_2^2+\mu_1^2+\mu_2^2\right)^2 \\ &+ 2(\lambda_1-\lambda_2)^2\left(\mu_1^2+\mu_2^2\right) \\ =& \frac{11}{4}(\lambda_1+\lambda_2)^4 + \frac{3}{4}(\lambda_1-\lambda_2)^4 + 8\left(\mu_1^2+\mu_2^2\right)^2 \\ &+ \frac{9}{2}(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2 + 4(\lambda_1+\lambda_2)^2\left(\mu_1^2+\mu_2^2\right) \\ &+ 6(\lambda_1-\lambda_2)^2\left(\mu_1^2+\mu_2^2\right).\end{aligned}$$

Set

$$x = (\lambda_1+\lambda_2)^2, \quad y = (\lambda_1-\lambda_2)^2, \quad z = 4\left(\mu_1^2+\mu_2^2\right),$$

then

$$\begin{aligned}
|\sigma|^2 =& \frac{5}{2}x + \frac{3}{2}y + z, \\
\sum_{i,j} \langle \sigma_i, \sigma_j \rangle^2 =& \frac{11}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{2}z^2 + \frac{9}{2}xy + xz + \frac{3}{2}yz, \\
\frac{1}{5}|\sigma|^4 =& \frac{5}{4}x^2 + \frac{9}{20}y^2 + \frac{1}{5}z^2 + \frac{3}{2}xy + xz + \frac{3}{5}yz.
\end{aligned}$$

For every κ, we have

$$\begin{aligned}
\sum_{i,j} \langle \sigma_i, \sigma_j \rangle^2 - \frac{1}{5}|\sigma|^4 =& \frac{3}{2}x^2 + \frac{3}{10}y^2 + \frac{3}{10}z^2 + 3xy + \frac{9}{10}yz \\
=& \frac{3}{2}\left(x + \kappa y + \frac{2\kappa}{3}z\right)\left(x + \frac{3}{5}y + \frac{2}{3}z\right) + \frac{3}{10}y^2 \\
& + \frac{3}{10}z^2 + 3xy + \frac{9}{10}yz \\
& - \frac{9\kappa}{10}y^2 - \frac{2\kappa}{3}z^2 - \frac{15\kappa + 9}{10}xy - (1+\kappa)xz - \frac{8\kappa}{5}yz \\
=& \frac{3}{2}\left(x + \kappa y + \frac{2\kappa}{3}z\right)\left(x + \frac{3}{5}y + \frac{2}{3}z\right) \\
& - \frac{9\kappa - 3}{10}y^2 - \frac{20\kappa - 9}{30}z^2 - \frac{15\kappa - 21}{10}xy \\
& - (1+\kappa)xz - \frac{16\kappa - 9}{10}yz.
\end{aligned}$$

For $\kappa \geq \frac{7}{5}$,

$$\begin{aligned}
\sum_{i,j} \langle \sigma_i, \sigma_j \rangle^2 - \frac{1}{5}|\sigma|^4 \leq& \frac{3}{2}\left(x + \kappa y + \frac{2\kappa}{3}z\right)\left(x + \frac{3}{5}y + \frac{2}{3}z\right) \\
=& \frac{2\kappa}{5}\left(|\sigma|^2 - \frac{5\kappa - 3}{2\kappa}\Theta^2\right)|\sigma|^2.
\end{aligned}$$

Therefore we have the estimate

$$\frac{1}{2}\Delta|\sigma|^2 \geq |\nabla\sigma|^2 + 2\kappa\left(\frac{2}{\kappa} + \frac{5\kappa - 3}{2\kappa}\Theta^2 - |\sigma|^2\right)|\sigma|^2, \quad \forall \kappa \geq \frac{7}{5}.$$

In particular, taking $\kappa = \frac{7}{5}$ we obtain

$$\frac{1}{2}\Delta|\sigma|^2 \geq |\nabla\sigma|^2 + \frac{14}{5}\left(\frac{10}{7}\left(1 + \Theta^2\right) - |\sigma|^2\right)|\sigma|^2.$$

By integral by parts, we prove the first claim of the theorem.

If $|\mathbf{B}|^2 \leq \frac{10}{7}\left(1 + \Theta^2\right)$, we must have either $\mathbf{B} \equiv 0$ and M is totally geodesic or $|\mathbf{B}|^2 = \frac{10}{7}\left(1 + \Theta^2\right)$ and $\lambda_1 = \lambda_2 = \pm\frac{\sqrt{3}}{3}$, $\mu_1 = \mu_2 = 0$ and M must be the minimal Calabi torus by [18] since M is closed. ☐

4 Proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1. Firstly we need to prove several lemmas about the function Θ.

Let SM be the unit tangent bundle of M and S_pM the fibre of the unit tangent bundle of M at $p \in M$. We have the following characterization of the function Θ.

Lemma 4.1

$$\Theta(p) = \max_{X \in S_pM} \sigma\left(X, X, X\right). \tag{4.1}$$

Proof It is a straightforward verification. For readers' convenience, we list a proof here.

It suffices to prove that

$$\max_{X \in S_pM} \sigma\left(X, X, X\right) \geq \Theta(p).$$

Assume for some $u \in S_pM$,

$$\Theta(p) = \max_{X \in S_pM} \left|\mathbf{B}\left(X, X\right)\right| = \left|\mathbf{B}\left(u, u\right)\right|.$$

Choose a local orthonormal basis $\{e_i\}$ of T_pM such that $e_1 = u$. Applying the maximum principle,

$$\left\langle \mathbf{A}^{\mathbf{B}(u,u)}\left(e_1\right), e_j\right\rangle = \left\langle \mathbf{B}\left(e_1, e_1\right), \mathbf{B}\left(e_1, e_j\right)\right\rangle = 0, \quad \forall j > 1.$$

Here $\mathbf{A}^{\nu}$ is the shape operator associated with the normal vector ν. Thus,

$$\mathbf{A}^{Ju}\mathbf{A}^{Ju}\left(e_1\right) = \mathbf{A}^{Je_1}\left(-J\mathbf{B}\left(e_1, e_1\right)\right) = \mathbf{A}^{\mathbf{B}(u,u)}\left(e_1\right) = \left|\mathbf{B}\left(e_1, e_1\right)\right|^2 e_1.$$

We conclude that

$$\mathbf{A}^{Ju}\left(e_1\right) = \pm\left|\mathbf{B}\left(e_1, e_1\right)\right| e_1.$$

Consequently,

$$\begin{aligned} \Theta(p) &= \max_{X \in S_pM} \left|\mathbf{B}\left(X, X\right)\right| = \left|\mathbf{B}\left(e_1, e_1\right)\right| = \pm\left\langle \mathbf{A}^{Ju}\left(e_1\right), e_1\right\rangle = \pm\sigma\left(e_1, e_1, e_1\right) \\ &\leq \max_{X \in S_pM} \sigma\left(X, X, X\right). \end{aligned}$$

□

In the rest of this paper we will use the equivalent description (4.1) of Θ.

Lemma 4.2 *Θ is a nonnegative Lipschitz function on M.*

Proof It suffices to prove that for every $p_1, p_2 \in M$,

$$\Theta(p_1) \leq \Theta(p_2) + \max_{M} |\nabla\sigma| \operatorname{dist}(p_1, p_2).$$

Choose a geodesic $\gamma : [0, \rho] \longrightarrow M$ connecting p_1 and p_2, i.e., $\gamma(0) = p_1, \gamma(\rho) = p_2$, where $\rho = \operatorname{dist}(p_1, p_2)$. Assume

$$\Theta(p_1) = \sigma(e, e, e),$$

where $e \in S_{p_1}M$. We can extend e to a tangent unit vector field $e(t) \in T_{\gamma(t)}M$ along $\gamma(t)$ by parallel transport. Consider the function $f(t) = \sigma\,(e(t), e(t), e(t))$, then

$$\Theta(p_1) - \Theta(p_2) \le f(0) - f(\rho) = f'(t_0)\rho = \left(\nabla_{\dot\gamma(t_0)}\sigma\right)(e\,(t_0)\,, e\,(t_0)\,, e\,(t_0))\,\rho,$$

where $t_0 \in (0, \rho)$. Therefore,

$$\Theta(p_1) - \Theta(p_2) \le \max_M |\nabla\sigma|\,\mathrm{dist}(p_1, p_2).$$

□

Choose $e_1(p) \in S_pM$ such that

$$\Theta(p) = \sigma\,(e_1(p), e_1(p), e_1(p))\,.$$

Applying the method of Lagrange multipliers we obtain

$$\sigma\,(e_1(p), e_1(p), X) = \Theta(p)\,\langle e_1(p), X\rangle\,, \quad \forall X \in T_pM.$$

Hence we can choose an orthonormal basis $\{e_1(p), e_2(p), \dots, e_n(p)\}$ of T_pM such that

$$\sigma\left(e_1(p), e_i(p), e_j(p)\right) = \mu_i(p)\delta_{ij},$$

where $\Theta(p) = \mu_1(p) \ge \mu_2(p) \ge \cdots \ge \mu_n(p)$. Applying the maximum principle, one can check by definition directly that $\mu_1(p) \ge 2\mu_2(p)$. We say that $\Theta(p)$ is of *multiplicity one* if

$$e \in S_pM,\ \sigma\,(e, e, e) = \sigma\,(e_1(p), e_1(p), e_1(p)) = \Theta(p) \quad \Longrightarrow \quad e = e_1(p).$$

Lemma 4.3 *If $\mu_1(p_0) > 2\mu_2(p_0)$ and $\Theta(p_0)$ is of multiplicity one, then there is a unique smooth unit tangent vector field e around a neighborhood $U \subset M$ of p_0 with $e(p_0) = e_1(p_0)$ such that*

$$\Theta(p) = \sigma\,(e(p), e(p), e(p))\,, \quad \forall p \in U.$$

Proof Consider a smooth map

$$f : SM \times \mathbb{R} \longrightarrow TM, \quad ((p, u), \lambda) \mapsto \left(p, \lambda u - \sum_{j=1}^{n} \sigma\left(u, u, e_j(p)\right) e_j(p)\right)$$
$$=: (p, h(p, u, \lambda))\,,$$

where $\left\{e_j(p)\right\}_{j=1}^n$ is an orthonormal basis of T_pM. We compute

$$\begin{aligned} \mathrm{d}h\,(\theta_k) &= \lambda\theta_k - 2\sigma\left(u, \theta_k, e_j(p)\right) e_j(p), \quad k = 1, \dots, n-1,\\ \mathrm{d}h\left(\frac{\partial}{\partial\lambda}\right) &= u, \end{aligned}$$

where $\{\theta_k\}_{k=1}^{n-1}$ is an orthonormal basis of $T_u\left(S_pM\right)$. Notice that $\{e_2(p_0), \dots, e_n(p_0)\}$ is an orthonormal basis of $T_{e_1(p_0)}\left(S_{p_0}M\right)$. The assumption gives us

$$\det\left(\mathrm{d}f|_{((p_0, e_1(p_0)), \mu_1(p_0))}\right) = \Pi_{j=2}^n\left(\mu_1(p_0) - 2\mu_j(p_0)\right) \ne 0.$$

Apply the inverse function theory to conclude that $f : \Omega \longrightarrow f(\Omega)$ is a diffeomorphism for some neighborhood $\Omega \subset SM \times \mathbb{R}$ of $(p_0, e_1(p_0), \mu_1(p_0))$. In particular, for some neighborhood $\hat{U} \subset M$ of p_0, there is $\lambda \in C^\infty\left(\hat{U}\right)$ and $e \in \Gamma\left(S\hat{U}\right)$ such that $\lambda(p_0) = \Theta(p_0)$, $u(p_0) = e_1(p_0)$ and

$$\sigma\left(u(p), u(p), e_j(p)\right) e_j(p) = \lambda(p)u(p), \quad p \in \hat{U}.$$

Consider a second order tensor $\phi(\cdot,\cdot) = \sigma(u,\cdot,\cdot)$ on $\hat{U}$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of ϕ. One can check that $\lambda_1 = \lambda$ with eigenvector u and $\lambda_k(p_0) = \mu_k(p_0)$. Moreover, each λ_k is local Lipschitz in $\hat{U}$. Assume $\Theta(p) = \sigma(e, e, e)$ where $e = e(p) = \cos t u(p) + \sin t v \in T_pM$ and $v \perp u(p)$. By assumption, choose $\varepsilon > 0$ such that

$$\Theta(p_0) = \lambda_1(p_0) > (2+\varepsilon)\lambda_2(p_0).$$

If $\cos t < 1$, we claim that

$$\cos t \leq \frac{1}{1+\varepsilon},$$

in a neighborhood $\tilde{U} \subset \hat{U}$ of p_0. In fact, by definition of Θ, we have

$$\sigma(e, e, u(p)) = \Theta(p)\langle e, u(p)\rangle,$$

which implies that

$$\cos t\Theta(p) = \cos^2 t\lambda(p) + \sin^2 t\phi(v, v).$$

Without loss of generality, assume $0 < \cos t < 1$, then

$$\Theta(p) = \cos t\lambda(p) + \frac{\sin^2 t}{\cos t}\phi(v, v).$$

We get

$$\Theta(p) \leq \frac{1+\cos t}{\cos t}\phi(v, v) \leq \frac{1+\cos t}{\cos t}\lambda_2(p).$$

Thus $\lambda_2(p) > 0$ and

$$\cos t \leq \frac{\lambda_2(p)}{\Theta(p) - \lambda_2(p)}.$$

By the continuity of Θ and λ_2, we prove the claim.

Now we claim that there is a neighborhood $U \subset \tilde{U}$ of p_0 such that $e(p) = u(p)$ for all $p \in U$. Otherwise, there are sequences $\{p_n\} \subset U$, $\left\{v_n \in S_{p_n}M\right\}$, $\{t_n\} \subset [0, 2\pi]$ such that $e_n = \cos t_n u(p_n) + \sin t_n v_n$ satisfies

$$\Theta(p_n) = \sigma(e_n, e_n, e_n), \quad \cos t_n \leq \frac{1}{1+\varepsilon}, \quad \forall n.$$

Without loss of generality, assume $\lim_{n\to\infty} q_n = p$, $\lim_{n\to\infty} t_n = t$, $\lim_{n\to\infty} v_n = v$. Since Θ is continuous according to Lemma 4.2, $u := \lim_{n\to\infty} u_n = \cos t u(p_0) + \sin t v \neq e_1(p_0)$ satisfies

$$\Theta(p_0) = \sigma(u(p_0), u(p_0), u(p_0)),$$

which is a contradiction. □

Now we are prepared to give a proof of our main Theorem 1.1.

Proof of Theorem 1.1 When $n = 2$ we have $|\mathbf{B}|^2 = 4\Theta^2$, therefore (1.1) is equivalent to $|\mathbf{B}|^2 \leq 2$ and we get the conclusion from [31].

We need to check the case $n \geq 3$. Assume that M is not totally geodesic and Θ achieves its maximum value at p_0. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of $T_{p_0}M$ such that

$$\mu_1 = \sigma_{111}(p_0) = \Theta(p_0).$$

One can check that

$$\sigma_{11j}(p_0) = 0, \quad j = 2, \ldots, n.$$

Thus, we may assume that

$$\sigma_{1jk}(p_0) = \mu_j \delta_{jk}, \quad 1 \leq j, k \leq n.$$

For each e_i, choose a geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = p_0, \dot{\gamma}(0) = e_i$. We move $e_1 \in T_{p_0}M$ along the geodesic $\gamma(t)$ to $e_1(t) \in T_{\gamma(t)}M$ by parallel transport. Consider the function $f : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ defined by

$$f(t) = \sigma\left(e_1(t), e_1(t), e_1(t)\right).$$

Then $f(t)$ achieves its local maximum value at $t = 0$. The maximum principle gives

$$0 \geq f''(0) = \sigma_{111,ii}(p_0).$$

Thus

$$\Delta\sigma_{111}(p_0) \leq 0.$$

Now applying Simons' identity (2.1), we have at p_0

$$\begin{aligned} 0 &\geq (n+1)\,\sigma_{111} + 2\sigma_{1ab}\sigma_{1bc}\sigma_{1ca} - 3\sigma_{1ab}\sigma_{abc}\sigma_{11c} \\ &= (n+1)\,\mu_1 + 2\sum_j \mu_j^3 - 3\mu_1 \sum_j \mu_j^2 \\ &= (n+1)\,\mu_1 - \mu_1^3 + 2\sum_{j>1}\mu_j^3 - 3\mu_1\sum_{j>1}\mu_j^2. \end{aligned}$$

Without loss of generality, assume $\mu_1 > 0$. Set

$$a_j = -\mu_j, \quad j > 1.$$

Since $\sum_j \mu_j = 0$, we get

$$\begin{aligned} 0 &\geq (n+1)\sum_{j>1} a_j - \left(\sum_{j>1} a_j\right)^3 - 2\sum_{j>1} a_j^3 - 3\sum_j a_j \sum_{k>1} a_k^2 \\ &= (n+1)\sum_{j>1} a_j - 6\left(\sum_{j>1} a_j\right)^3 + 12\sum_{i>1} a_i \sum_{j>k>1} a_j a_k - 6\sum_{i>j>k>1} a_i a_j a_k. \end{aligned}$$

By Newton's inequality,

$$\sum_{i>j>k>1} a_i a_j a_k \leq \frac{2(n-3)}{3(n-2)} \frac{\left(\sum_{j>k>1} a_i a_j\right)^2}{\sum_{i>1} a_i},$$

the equality holds if and only if

$$\mu_2 = \mu_3 = \cdots = \mu_n.$$

We obtain

$$0 \geq n + 1 - 6 \left(\sum_{j>1} a_j \right)^2 + 12 \sum_{j>k>1} a_j a_k - \frac{4(n-3)}{n-2} \frac{\left(\sum_{j>k>1} a_j a_k \right)^2}{\left(\sum_{i>1} a_i \right)^2}. \tag{4.2}$$

Define β by

$$\beta = \mu_1^2 + 3 \sum_{j>1} \mu_j^2 = 4 \left(\sum_{j>1} a_j \right)^2 - 6 \sum_{j>k>1} a_j a_k,$$

then (4.2) gives

$$\begin{aligned} 0 \geq & n + 1 + 2\mu_1^2 - 2\beta - \frac{n-3}{9(n-2)} \left(4\mu_1 - \frac{\beta}{\mu_1} \right)^2 \\ = & n + 1 + \frac{2(n+6)}{9(n-2)} \left(\frac{5n-6}{2(n+6)} \frac{\beta}{\mu_1} - \mu_1 \right)^2 - \frac{3n}{2(n+6)} \frac{\beta^2}{\mu_1^2}. \end{aligned}$$

Since

$$\beta \geq \mu_1^2 + \frac{3}{n-1} \left(\sum_{j>1} \mu_j \right)^2 = \frac{n+2}{n-1} \mu_1^2,$$

we have

$$\frac{5n-6}{2(n+6)} \frac{\beta}{\mu_1} - \mu_1 \geq \frac{n-1}{n+2} \frac{\beta}{\mu_1} - \mu_1 \geq 0.$$

Therefore,

$$\frac{5n-6}{2(n+6)} \frac{\beta}{\mu_1} - \sqrt{\frac{27n(n-2)}{4(n+6)^2} \frac{\beta^2}{\mu_1^2} - \frac{9(n+1)(n-2)}{2(n+6)}} \leq \mu_1 \leq \frac{n-1}{n+2} \frac{\beta}{\mu_1},$$

which implies

$$\beta \geq \frac{n+2}{\sqrt{n}} \mu_1.$$

We conclude that

$$|\mathbf{B}|^2 \geq \beta \geq \frac{n+2}{\sqrt{n}} \mu_1.$$

If $n = 3$, we have the following estimate

$$|\mathbf{B}|^2 \geq \beta \geq 2 + \mu_1^2.$$

Therefore under the assumption (1.1), we must have $|\mathbf{B}|^2 = \frac{n+2}{\sqrt{n}} \mu_1$ at p_0 and

$$\sigma_{1jk}(p_0) = \mu_j \delta_{jk}, \quad j, k = 1, \ldots, n \tag{4.3}$$

$$\sigma_{ijk}(p_0) = 0, \quad i, j, k > 1. \tag{4.4}$$

Moreover, $\mu_2 = \cdots = \mu_n = -\frac{1}{n-1} \mu_1 < 0$.

Claim Θ is a constant on M.

If this claim is true, then the previous argument claims that conditions (4.3) and (4.4) hold everywhere. As an immediate consequence, M is the Calabi torus (cf. [18]) since M is closed.

Now we prove the above claim as follows.

Proof of the Claim Consider the nonempty subset Ω of M defined by

$$\Omega := \{p \in M : \Theta(p) = \mu_1\}.$$

Since Θ is continuous, we know that Ω is a closed subset of M. Then it suffices to prove that Ω is also an open subset of M since M is connected.

Firstly we claim that $\Theta(p_0)$ is of multiplicity one, i.e., the unit tangent vector $e \in T_{p_0}M$ with $\sigma(e, e, e) = \mu_1$ is unique. In fact, put $e = \sum_i x^i e_i$, then $\sum_j x^j x^j = 1$ and

$$\begin{aligned}
\mu_1 &= \sigma(e, e, e) \\
&= x^i x^j x^k \sigma_{ijk} \\
&= x^1 x^1 x^1 \mu_1 + 3x^1 \sum_{j>1} x^j x^j \mu_j \\
&= x^1 x^1 x^1 \mu_1 - \frac{\mu_1}{n-1} x^1 \left(1 - x^1 x^1\right).
\end{aligned}$$

We must have $x^1 = 1$ and $e = e_1$. Therefore, by Lemma 4.3 we can extend e_1 to a smooth tangent vector field still denoted by e_1 in a neighborhood U of p_0, such that for all $p \in U$ we have

$$\Theta(p) = \sigma(e_1, e_1, e_1)(p).$$

Secondly, we claim that Θ is subharmonic in U. In fact, we have

$$\nabla_{e_j}\Theta = \sigma_{111,j} + 3\sigma_{11i}\left\langle \nabla_{e_j} e_1, e_i \right\rangle = \sigma_{111,j} = \sigma_{11j,1},$$

and for $j > 1$,

$$\begin{aligned}
0 &= \nabla_{e_1}\sigma_{11j} \\
&= \sigma_{11j,1} + 2\sigma_{1kj}\left\langle \nabla_{e_1} e_1, e_k \right\rangle + \sigma_{11k}\left\langle \nabla_{e_1} e_j, e_k \right\rangle \\
&= \sigma_{11j,1} + 2\sum_{k>1}\sigma_{1kj}\left\langle \nabla_{e_1} e_1, e_k \right\rangle + \sigma_{111}\left\langle \nabla_{e_1} e_j, e_1 \right\rangle \\
&= \sigma_{11j,1} + 2\sum_{k>1}\sigma_{1kj}\left\langle \nabla_{e_1} e_1, e_k \right\rangle - \sigma_{111}\left\langle \nabla_{e_1} e_1, e_j \right\rangle.
\end{aligned}$$

At a considered point $p \in U$, we may assume $\sigma_{1jk} = \sigma_{1jj}\delta_{jk}$ and $\nabla_{e_i} e_j = 0$ for $i, j > 1$. We get

$$\begin{aligned}
\nabla_{e_1}\Theta &= \sigma_{111,1}, \\
\nabla_{e_j}\Theta &= \left(\Theta - 2\sigma_{1jj}\right)\left\langle \nabla_{e_1} e_1, e_j \right\rangle, \quad j > 1,
\end{aligned}$$

and

$$
\begin{aligned}
\Delta\Theta &= \nabla_{e_j}\nabla_{e_j}\Theta - \nabla_{\nabla_{e_j}e_j}\Theta \\
&= \left(\sigma_{111,11} + 4\sum_{j>1}\sigma_{111,j}\left\langle\nabla_{e_1}e_1, e_j\right\rangle\right) \\
&\quad + \sum_{j>1}\left(\sigma_{111,jj} + 3\sum_k \sigma_{11k,j}\left\langle\nabla_{e_j}e_1, e_k\right\rangle + \sum_k \sigma_{111,k}\left\langle\nabla_{e_j}e_j, e_k\right\rangle\right) \\
&\quad - \sum_{j>1}\nabla_{e_j}\Theta\left\langle\nabla_{e_1}e_1, e_j\right\rangle \\
&= \Delta\sigma_{111} + 3\sum_{j>1}\nabla_{e_j}\Theta\left\langle\nabla_{e_1}e_1, e_j\right\rangle \\
&= \Delta\sigma_{111} + 3\sum_{j>1}\left(\Theta - 2\sigma_{1jj}\right)\left\langle\nabla_{e_1}e_1, e_j\right\rangle^2,
\end{aligned}
$$

where in the second equality we used the symmetry of the four-linear tensor $(\sigma_{ijk,l})$. By assumption (1.1), the previous argument implies that

$$\Delta\sigma_{111} \geq 0.$$

Therefore, we have for all $p \in U$,

$$\Delta\Theta \geq 3\sum_{i>1}(\Theta - 2\sigma_{1ii})\left\langle\nabla_{e_1}e_1, e_i\right\rangle^2.$$

One can check that $\sigma_{11i}(p) = 0$ and $\Theta(p) \geq 2\sigma_{1ii}(p)$ for all $i > 1$ and all $p \in U$. Thus Θ is subharmonic in U. Since Θ achieves its local maximum value at $p_0 \in U$, the strong maximum principle implies that Θ locally must be a constant in U. We conclude that p_0 is an interior point of Ω. Thus Ω is an open subset of M. Therefore, Θ is constant on M. $\square$

At the end let us show that when $n = 3$, condition (1.1) implies $|\mathbf{B}|^2 \leq 2 + \Theta^2$. Since $\frac{5}{2}\Theta^2 \leq |\mathbf{B}|^2 \leq \frac{5}{\sqrt{3}}\Theta$, we have $\Theta \leq \frac{2}{\sqrt{3}}$, and $\Theta^2 + 2 - \frac{5}{\sqrt{3}}\Theta = \left(\Theta - \frac{2}{\sqrt{3}}\right)^2 + \frac{2}{3} - \frac{1}{\sqrt{3}}\Theta \geq 0$. Therefore $|\mathbf{B}|^2 \leq \frac{5}{\sqrt{3}}\Theta$ implies $|\mathbf{B}|^2 \leq 2 + \Theta^2$. This completes the proof of Theorem 1.1. $\square$

Appendix A. An application to lagrangian submanifolds in the nearly Kähler $\mathbb{S}^6$

Here we give a slight improvement of the main theorem in [15] as follows, by similar arguments used in the proof of Theorem 3.1.

Theorem Appendix A.1 *Let M be a closed Lagrangian submanifold in the homogeneous nearly Kähler $\mathbb{S}^6$. Then we have*

$$\int_M |\mathbf{B}|^2\left(|\mathbf{B}|^2 - \frac{75}{56} - \frac{10}{7}\Theta^2\right) \geq 0. \tag{4.5}$$

Moreover, the equality in (4.5) holds if and only if M is either the totally geodesic sphere, or the Dillen-Verstraelen-Vrancken's Berger sphere (see [9, Theorem 5.1]) which satisfies $|\mathbf{B}|^2 = \frac{75}{56} + \frac{10}{7}\Theta^2$ with $|\mathbf{B}|^2 \equiv \frac{25}{8}$ and $\Theta \equiv \frac{\sqrt{5}}{2}$.

Proof Here we only give a brief sketch. For more details please see [15]. We identify $\mathbb{R}^7$ as the imaginary Cayley numbers. The Cayley multiplication induces a cross product " $\times$ " on $\mathbb{R}^7$. The almost complex structure J on $\mathbb{S}^6 \subset \mathbb{R}^7$ is then given by

$$JX := x \times X, \quad \forall X \in T_x\mathbb{S}^6.$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\mathbb{S}^6$, then $\left(\bar{\nabla}_X J\right) X = 0$ for all $X \in T\mathbb{S}^6$. Then $\omega_{ijk} = \left\langle \left(\bar{\nabla}_{e_i} J\right) e_j, Je_k \right\rangle$ is the volume form of M. Since M is Lagrangian, i.e., $JTM \subset T^{\perp}M$, we have ([26, Lemma 3.2])

$$\mathbf{B}\left(e_i, \left(\bar{\nabla}_{e_j} J\right) e_k\right) = J\left(\bar{\nabla}_{\mathbf{B}(e_i,e_j)} J\right) e_k + J\left(\bar{\nabla}_{e_j} J\right) \mathbf{B}\left(e_i, e_k\right),$$

which implies that M is minimal (cf. [12, Theorem 1]). We have the following Simons' identity (cf. [7])

$$\frac{1}{2}\Delta \left|\mathbf{B}\right|^2 = \left|\nabla^{\perp}\mathbf{B}\right|^2 + 3\left|\mathbf{B}\right|^2 - \sum_{\alpha,\beta=1}^{3} \left\langle \mathbf{A}^{\nu_\alpha}, \mathbf{A}^{\nu_\beta}\right\rangle^2 - \sum_{\alpha,\beta=1}^{3} \left|\left[\mathbf{A}^{\nu_\alpha}, \mathbf{A}^{\nu_\beta}\right]\right|^2.$$

Here $\{\nu_\alpha\}$ is a local orthonormal frame of $T^{\perp}M$. Set

$$\sigma_{ijk} = \left\langle \mathbf{B}\left(e_i, e_j\right), Je_k \right\rangle,$$

then σ is a tri-linear symmetric tensor. One can check that

$$\sigma_{ijk,l} = \left\langle \left(\nabla^{\perp}_{e_i}\mathbf{B}\right)\left(e_j, e_k\right), Je_l \right\rangle.$$

Introduce

$$\begin{aligned} u_{ijkl} :=& \frac{1}{4}\left(\sigma_{ijk,l} + \sigma_{jkl,i} + \sigma_{kli,j} + \sigma_{lij,k}\right) \\ =& \sigma_{ijk,l} - \frac{1}{4}\left(\sigma_{jkm}\omega_{lim} + \sigma_{ikm}\omega_{ljm} + \sigma_{ijm}\omega_{lkm}\right). \end{aligned}$$

One can check that u is a four-linear symmetric tensor and $\sum_i u_{iij,k} = 0$. By using the fact $\sigma_{ijk,l} = \sigma_{ijl,k} + \sigma_{ijm}\omega_{lkm}$, a direct calculation yields (cf. [15, emma 4.4])

$$\left|\nabla^{\perp}\mathbf{B}\right|^2 = |u|^2 + \frac{3}{4}\left|\mathbf{B}\right|^2.$$

We therefore obtain

$$\frac{1}{2}\Delta|\sigma|^2 = |u|^2 + \frac{15}{4}|\sigma|^2 - \sum_{i,j=1}^{3}\left\langle \sigma_i, \sigma_j\right\rangle^2 - \sum_{i,j=1}^{3}\left|\left[\sigma_i, \sigma_j\right]\right|^2,$$

where $\sigma_i = \left(\sigma_{ijk}\right)_{1\le j,k\le n}$. Then, similarly as in the proof of Theorem 3.1, we obtain

$$\frac{1}{2}\Delta\left|\mathbf{B}\right|^2 \ge \frac{14}{5}\left(\frac{75}{56} + \frac{10}{7}\Theta^2 - \left|\mathbf{B}\right|^2\right)\left|\mathbf{B}\right|^2.$$

The rest of the proof follows from that in [15]. □

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Authors and Affiliations

Yong Luo[1] · Linlin Sun[2] · Jiabin Yin[3,4]

Yong Luo
yongluo-math@cqut.edu.cn

Jiabin Yin
jiabinyin@126.com

1 Mathematical Science Research Center, Chongqing University of Technology, Chongqing 400054, China

2 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

3 School of Mathematics and Statistics, Guangxi Normal University, Guilin 541001, Guangxi, China

4 Key Laboratory of Mathematical and Statistical Model, Education Department of Guangxi Zhuang Autonomous Region, Guangxi Normal University, Guilin 541001, Guangxi, China