Annals of Global Analysis and Geometry (2020) 58:177–189
https://doi.org/10.1007/s10455-020-09719-4

Complete Willmore Legendrian surfaces in $\mathbb{S}^5$ are minimal Legendrian surfaces

Yong Luo[1,2] · **Linlin Sun**[1,2,3]

Received: 21 January 2020 / Accepted: 23 May 2020 / Published online: 4 June 2020

Abstract
In this paper, we continue to consider Willmore Legendrian surfaces and csL Willmore surfaces in $\mathbb{S}^5$, notions introduced by Luo (Calc Var Partial Differ Equ 56, Art. 86, 19, 2017. https://doi.org/10.1007/s00526-017-1183-z). We will prove that every complete Willmore Legendrian surface in $\mathbb{S}^5$ is minimal and find nontrivial examples of csL Willmore surfaces in $\mathbb{S}^5$.

Keywords Willmore Legendrian surface · csL surface · csL Willmore surface

Mathematics Subject Classification 53C24 · 53C42 · 53C44

1 Introduction

Let Σ be a Riemann surface, $(M^n, g) = \mathbb{S}^n$ or $\mathbb{R}^n (n \geq 3)$ the unit sphere or the Euclidean space with standard metrics and f an immersion from Σ to M. Let B be the second fundamental form of f with respect to the induced metric, H the mean curvature vector field of f defined by

$$H = \operatorname{tr} B,$$

κ_M the Gauss curvature of $df(T\Sigma)$ with respect to the ambient metric g and $\mathrm{d}\mu_f$ the area element on $f(\Sigma)$. The Willmore functional of the immersion f is then defined by

$$W(f) = \int_\Sigma \left(\frac{1}{4}|H|^2 + \kappa_M\right) \mathrm{d}\mu_f,$$

✉ Linlin Sun
sunll@whu.edu.cn

Yong Luo
yongluo@whu.edu.cn

1 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

2 Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China

3 Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

For a smooth and compactly supported variation $f : \Sigma \times I \mapsto M$ with $\phi = \partial_t f$, we have the following first variational formula (cf. [22, 23])

$$\frac{\mathrm{d}}{\mathrm{d}t} W(f) = \int_\Sigma \left\langle \overrightarrow{W}(f), \phi \right\rangle \mathrm{d}\mu_f,$$

with $\overrightarrow{W}(f) = \sum_{\alpha=3}^n \overrightarrow{W}(f)^\alpha e_\alpha$, where $\{e_\alpha : 3 \le \alpha \le n\}$ is a local orthonormal frame of the normal bundle of $f(\Sigma)$ in M and

$$\overrightarrow{W}(f)^\alpha = \frac{1}{2}\left(\Delta H^\alpha + \sum_{i,j,\beta} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2|H|^2 H^\alpha \right), \quad 3 \le \alpha \le n,$$

where h_{ij}^α is the component of B and H^α is the trace of $\left(h_{ij}^\alpha\right)$.

A smooth immersion $f : \Sigma \mapsto M$ is called a *Willmore immersion*, if it is a critical point of the Willmore functional W. In other words, f is a Willmore immersion if and only if it satisfies

$$\Delta H^\alpha + \sum_{i,j,\beta} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2|H|^2 H^\alpha = 0, \quad 3 \le \alpha \le n. \tag{1.1}$$

When $(M, g) = \mathbb{R}^3$, Willmore [25] proved that the Willmore energy of closed surfaces is larger than or equal to 4π and equality holds only for round spheres. When Σ is a torus, Willmore conjectured that the minimum is $2\pi^2$ and it is attained only by the Clifford torus, up to a conformal transformation of $\mathbb{R}^3$ [6, 24], which was verified by Marques and Neves in [13]. When $(M, g) = \mathbb{R}^n$, Simon [20], combined with the work of Bauer and Kuwert [1], proved the existence of an embedded surface which minimizes the Willmore functional among closed surfaces of prescribed genus. Motivated by these mentioned papers, Minicozzi [14] proved the existence of an embedded torus which minimizes the Willmore functional in a smaller class of Lagrangian tori in $\mathbb{R}^4$. In the same paper, Minicozzi conjectured that the Clifford torus minimizes the Willmore functional in its Hamiltonian isotropic class, which he verified has a close relationship with Oh's conjecture [17, 18]. We should also mention that before Minicozzi, Castro and Urbano proved that the Whitney sphere in $\mathbb{R}^4$ is the only minimizer for the Willmore functional among closed Lagrangian sphere. This result was further generalized by Castro and Urbano in [4] where they proved that the Whitney sphere is the only closed Willmore Lagrangian sphere (a Lagrangian sphere which is also a Willmore surface) in $\mathbb{R}^4$. Examples of Willmore Lagrangian tori (Lagrangian tori which also are Willmore surfaces) in $\mathbb{R}^4$ were constructed by Pinkall [19] and Castro and Urbano [5]. Motivated by these works, Luo and Wang [11] considered the variation in the Willmore functional among Lagrangian surfaces in $\mathbb{R}^4$ or variation in a Lagrangian surface of the Willmore functional among its Hamiltonian isotropic class in $\mathbb{R}^4$, whose critical points are called LW or HW surfaces, respectively. We should also mention that Willmore-type functional of Lagrangian surfaces in $\mathbb{CP}^2$ were studied by Montiel and Urbano [16] and Ma et al. [12].

Inspired by the study of the Willmore functional for Lagrangian surfaces in $\mathbb{R}^4$, Luo [9] naturally considered the Willmore functional of Legendrian surfaces in $\mathbb{S}^5$.

Definition 1.1 A Willmore and Legendrian surface in $\mathbb{S}^5$ is called **a Willmore Legendrian surface**.

Definition 1.2 A Legendrian surface in $\mathbb{S}^5$ is called **a contact stationary Legendrian Willmore surface** (in short, a csL Willmore surface) if it is a critical point of the Willmore functional under contact deformations.

Luo [9] proved that Willmore Legendrian surfaces in $\mathbb{S}^5$ are csL surfaces (see Definition 2.1). In this paper, we continue to study Willmore Legendrian surfaces and csL Willmore surfaces in $\mathbb{S}^5$. Surprisingly, we will prove that every complete Willmore Legendrian surface in $\mathbb{S}^5$ must be a minimal surface (Theorem 2.5). We also find nontrivial examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$ for the first time, by exploring relationships between them (Proposition 3.1).

The method here we used to find nontrivial csL Willmore surfaces in $\mathbb{S}^5$ in Sect. 3 should also be useful in discovering nontrivial HW surfaces in $\mathbb{R}^4$ introduced by Luo and Wang in [11]. We will consider this problem in the future.

2 Willmore Legendrian surfaces in $\mathbb{S}^5$

In this section, we will prove that every complete Willmore Legendrian surface in $\mathbb{S}^5$ is minimal. Firstly, we briefly record several facts about Legendrian surfaces in $\mathbb{S}^5$. We refer the reader to consult [2] for more materials about the contact geometry.

Let $\mathbb{S}^5$, the five-dimensional unit sphere, be the standard Sasakian Einstein manifold with contact one form α, almost complex structure J, Reed field $\mathbf{R}$ and canonical metric g. Let Σ be a closed surface of $\mathbb{S}^5 \subset \mathbb{C}^3$. We say that Σ is Legendrian if

$$JT\Sigma \subset T^{\nu}\Sigma, \quad JF \in \Gamma(T^{\nu}\Sigma)$$

where $F : \Sigma \longrightarrow \mathbb{S}^5$ is the position vector and $T\Sigma, T^{\nu}\Sigma$ are tangent and normal bundles of Σ, respectively. We say that Σ is a minimal Legendrian surface of $\mathbb{S}^5$ if Σ is a minimal and Legendrian surface of $\mathbb{S}^5$. Define

$$\sigma(X,Y,Z) := \langle \mathbf{B}(X,Y), JZ\rangle, \quad \forall X,Y,Z \in T\Sigma.$$

The Weingarten equation implies that

$$\sigma(X,Y,Z) = \sigma(Y,X,Z).$$

Moreover, by definition, one can check that σ is a three-order symmetric tensor, i.e.,

$$\sigma(X,Y,Z) = \sigma(Y,X,Z) = \sigma(X,Z,Y). \tag{2.1}$$

The Gauss equation, Codazzi equation and Ricci equation become

$$\begin{aligned} R(X,Y,Z,W) &= \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle \\ &\quad + \sigma\big(X,Z,e_i\big)\sigma\big(Y,W,e_i\big) - \sigma\big(X,W,e_i\big)\sigma\big(Y,Z,e_i\big), \\ \big(\nabla_X\sigma\big)(Y,Z,W) &= \big(\nabla_Y\sigma\big)(X,Z,W), \\ R^{\perp}(X,Y,JZ,JW) &= R(X,Y,Z,W), \end{aligned} \tag{2.2}$$

where $\{e_i\}$ is an orthonormal basis of $T\Sigma$. The Codazzi equation implies

$$\big(\nabla_X\sigma\big)(Y,Z,W) = \big(\nabla_Y\sigma\big)(X,Z,W) = \big(\nabla_X\sigma\big)(Z,Y,W) = \big(\nabla_X\sigma\big)(Y,W,Z), \tag{2.3}$$

, i.e., $\nabla\sigma$ is a fourth-order symmetric tensor.

Recall that

Definition 2.1 Σ is a *csL surface* in $\mathbb{S}^5$ if it is a critical point of the volume functional among Legendrian surfaces.

CsL surfaces in $\mathbb{S}^5$ satisfy the following Euler–Lagrange equation [3, 7]:

$$\operatorname{div}(JH) = 0.$$

It is obvious that Σ is csL in $\mathbb{S}^5$ when Σ is minimal. The following observation is very important for the study of csL surfaces.

Lemma 2.1 *Σ is csL in $\mathbb{S}^5$ iff JH is a harmonic vector field.*

By using the Bochner formula for harmonic vector fields (cf. [8]), we get

Lemma 2.2 *If Σ is csL in $\mathbb{S}^5$, then*

$$\frac{1}{2}\Delta|H|^2 = |\nabla(JH)|^2 + Ric(JH, JH).$$

From Lemma 2.2, it is easy to see that we have

Lemma 2.3 *If $\Sigma \subset \mathbb{S}^5$ is csL and non-minimal, then the zero set of H is isolated and*

$$\Delta \log |H| = \kappa$$

provided $H \neq 0$, where κ is the Gauss curvature of Σ.

We then prove that every complete Willmore Legendrian surface in $\mathbb{S}^5$ must be a minimal surface. Firstly, we rewrite the Willmore operator acting on Legendrian surfaces, i.e., we prove the following

Proposition 2.4 *Assume that Σ is a Legendrian surface in $\mathbb{S}^5$, , then its Willmore operator can be written as*

$$\mathbf{W}(\Sigma) = \frac{1}{2}\left\{-J\nabla \operatorname{div}(JH) + B(JH, JH) - \frac{1}{2}|H|^2 H - 2\operatorname{div}(JH)\mathbf{R}\right\}.$$

In particular, the Euler–Lagrange equation of Willmore Legendrian surfaces in $\mathbb{S}^5$ is

$$-J\nabla \operatorname{div}(JH) + B(JH, JH) - \frac{1}{2}|H|^2 H - 2\operatorname{div}(JH)\mathbf{R} = 0. \tag{2.4}$$

Proof Let $\{\nu_1, \nu_2, \mathbf{R}\}$ be a local orthonormal frames of the normal bundle of Σ, then the Willmore equation (1.1) can be rewritten as

$$\Delta^\nu H + \sum_\alpha \langle A^\alpha, A^H\rangle \nu_\alpha - \frac{1}{2}|H|^2 H = 0.$$

Note that by (2.8) in [9], we have

$$\begin{aligned}\nabla^\nu_X(JY) &= (\bar\nabla_X(JY))^\nu\\ &= ((\bar\nabla_X J)Y + J\bar\nabla_X Y)^\nu\\ &= J\nabla_X Y + g(X,Y)\mathbf{R}\end{aligned}$$

for $X, Y \in \Gamma(T\Sigma)$, where $\bar\nabla$ denotes the covariant derivative of $\mathbb{S}^5$. Choose a local orthonormal frame field around p with $\nabla_{e_i}e_j|_p = 0$, then

$$\begin{aligned}&J\nabla_{e_i}(JH)\\ &\quad= \nabla^\nu_{e_i}(J(JH)) - g(e_i, JH)\mathbf{R}\\ &\quad= -\nabla^\nu_{e_i}H - g(e_i, JH)\mathbf{R}\end{aligned}$$

and

$$\begin{aligned}J\nabla_{e_i}(\nabla_{e_i}(JH)) =&\nabla^\nu_{e_i}(J\nabla_{e_i}(JH)) - g(e_i, \nabla_{e_i}JH)\mathbf{R}\\ =&\nabla^\nu_{e_i}(-\nabla^\nu_{e_i}H - g(e_i, JH)\mathbf{R}) - g(e_i, \nabla_{e_i}JH)\mathbf{R}\\ =&-\nabla^\nu_{e_i}\nabla^\nu_{e_i}H - 2g(e_i, \nabla_{e_i}(JH)\mathbf{R} - g(e_i, JH)\big(\bar\nabla_{e_i}\mathbf{R}\big)^\nu\\ =&-\nabla^\nu_{e_i}\nabla^\nu_{e_i}H - 2g(e_i, \nabla_{e_i}(JH)\mathbf{R} - g(H, Je_i)Je_i,\end{aligned}$$

where in the last equality we used (2.7) in [9]. Therefore, we obtain

$$\Delta^\nu H = -J\Delta(JH) - H - 2\,\mathrm{div}\,(JH)\mathbf{R},$$

which implies that Σ satisfies the following equation

$$-J\Delta(JH) + \sum_\alpha \left\langle A^\alpha, A^H\right\rangle \nu_\alpha - \frac{1}{2}\big(2 + |H|^2\big)H - 2\,\mathrm{div}\,(JH)\mathbf{R} = 0.$$

In addition, by [9, Lemma 2.9], the dual one form of JH is closed; thus, by the Ricci identity we have

$$\Delta(JH) = \nabla\,\mathrm{div}\,(JH) + \kappa JH.$$

The proposition is then a consequence of the following Claim together with above two identities. □

Claim

$$\begin{aligned}2\kappa &= 2 + |H|^2 - |B|^2,\\ \sum_\alpha \left\langle A^\alpha, A^H\right\rangle \nu_\alpha - \frac{1}{2}|B|^2 H &= B(JH, JH) - \frac{1}{2}|H|^2 H.\end{aligned}$$

Proof The first equation is obvious by the Gauss equation (2.2). The second equation can be proved by the Gauss equation (2.2) and the tri-symmetry of the tensor σ (see (2.1)). To be precise, for every tangent vector field $Z \in T\Sigma$ we have

$$\begin{aligned}
&\langle B(JH,JH),JZ\rangle - \sum_{\alpha}\langle A^{\alpha},A^{H}\rangle\langle \nu_{\alpha},JZ\rangle\\
&= -\langle B(Z,JH),H\rangle - \sum_{i,j}\langle B(e_i,e_j),JZ\rangle\langle B(e_i,e_j),H\rangle\\
&= \sum_{i,j}\langle B(Z,e_j),Je_i\rangle\langle B(JH,e_j),e_i\rangle - \langle B(Z,JH),H\rangle\\
&= \sum_{j}\langle B(Z,e_j),B(JH,e_j)\rangle - \langle B(Z,JH),H\rangle\\
&= Ric(Z,JH) - \langle Z,JH\rangle\\
&= (\kappa-1)\langle Z,JH\rangle\\
&= \frac{1}{2}\big(|H|^2-|B|^2\big)\langle Z,JH\rangle\\
&= \frac{1}{2}\big(|B|^2-|H|^2\big)\langle H,JZ\rangle.
\end{aligned}$$

This completes the proof of the second equation. □

Now we are in position to prove the following

Theorem 2.5 *Every complete Willmore Legendrian surface in* $\mathbb{S}^5$ *is a minimal surface.*

Proof We prove by a contradiction argument. Assume that Σ is a complete Willmore Legendrian surface in $\mathbb{S}^5$ which is not a minimal surface. If $H \neq 0$, then let $\left\{e_1 = \frac{JH}{|H|}, e_2\right\}$ be a local orthonormal frame field of $T\Sigma$. From (2.4), we have

$$B(e_1,e_1) = -\frac{1}{2}|H|Je_1,$$

which also implies that

$$B(e_2,e_2) = -\frac{1}{2}|H|Je_1, \quad h^2_{11} = 0.$$

Then, by the Gauss equation (2.2) we have

$$\begin{aligned}
\kappa =&1 + \langle B(e_1,e_1),B(e_2,e_2)\rangle - \left|B(e_1,e_2)\right|^2\\
=&1 + \frac{1}{4}|H|^2 - \left|h^1_{12}\right|^2 - \left|h^2_{12}\right|^2\\
=&1 + \frac{1}{4}|H|^2 - \left|h^1_{22}\right|^2\\
=&1.
\end{aligned}$$

Since Σ is a Willmore Legendrian surface, from (2.4) we see that $\operatorname{div}(JH) = 0$. By Lemma 2.3, the minimal points of Σ are discrete and so the Gauss curvature of Σ equals one everywhere on Σ; therefore, Σ is compact by Bonnet-Myers theorem. Apply Lemma 2.2 to obtain that on Σ

$$\frac{1}{2}\Delta|H|^2 = |\nabla(JH)|^2 + |H|^2.$$

Then, the maximum principle implies that $H \equiv 0$, which is a contradiction. Therefore, Σ is a minimal Legendrian surface in $\mathbb{S}^5$. □

3 Examples of csL Willmore surfaces in $\mathbb{S}^5$

From the definition, we see that complete Willmore Legendrian surfaces, which are minimal surfaces by Theorem 2.5 in the last section, are trivial examples of csL Willmore surfaces in $\mathbb{S}^5$. Thus, it is very natural and important to find nonminimal csL Willmore surfaces in $\mathbb{S}^5$. This will be done in this section by analyzing a very close relationship between csL Willmore surfaces and csL surfaces in $\mathbb{S}^5$.

Assume that Σ is a csL Willmore surface in $\mathbb{S}^5$, then since the variation vector field on Σ under Legendrian deformations can be written as $J\nabla u + \frac{1}{2}u\mathbf{R}$ for smooth function u on Σ (cf. [21, Lemma 3.1]), we have

$$\begin{aligned} 0 &= \int_\Sigma \left\langle \overrightarrow{W}(\Sigma), J\nabla u + \frac{1}{2}u\mathbf{R} \right\rangle \mathrm{d}\mu_\Sigma \\ &= \int_\Sigma \langle \overrightarrow{W}(\Sigma), J\nabla u \rangle d\Sigma + \int_\Sigma \left\langle \overrightarrow{W}(\Sigma), \frac{1}{2}u\mathbf{R} \right\rangle \mathrm{d}\mu_\Sigma \\ &= \int_\Sigma \operatorname{div}\left(J\overrightarrow{W}(\Sigma) - 2JH \right) u \,\mathrm{d}\mu_\Sigma, \end{aligned}$$

where in the last equality we used $\left\langle \overrightarrow{W}(\Sigma), \mathbf{R} \right\rangle = -2\operatorname{div}(JH)$, by Proposition 2.4. Therefore, Σ satisfies the following Euler–Lagrange equation:

$$\operatorname{div}\left(J\overrightarrow{W}(\Sigma) - 2JH \right) = 0. \tag{3.1}$$

Remark 3.1 Note that the coefficient of the Euler–Lagrange equation (3.1) for csL Willmore surfaces in $\mathbb{S}^5$ is slightly different with [9, equation (1.7)]. That is because here we use the notation $H = \operatorname{tr} B$, whereas in [9] we defined $H = \frac{1}{2}\operatorname{tr} B$.

Then, by (2.4), Σ satisfies the following equation.

$$\operatorname{div}\left(\nabla \operatorname{div}(JH) + JB(JH, JH) - \frac{1}{2}|H|^2 JH - 4JH \right) = 0.$$

In addition, by the four-symmetric of $\left(\sigma_{ijk,l}\right)$ [see (2.3)], a direct computation shows

$$\operatorname{div}(JB(JH, JH)) = 2\operatorname{tr}\langle B(\cdot, \nabla_\cdot(JH)), H\rangle + \frac{1}{2}\nabla_{JH}|H|^2.$$

Therefore, Σ satisfies the following equation

$$\Delta \operatorname{div}(JH) + 2\operatorname{tr}\langle B(\cdot, \nabla_\cdot(JH)), H\rangle - \frac{1}{2}|H|^2 \operatorname{div}(JH) - 4\operatorname{div}(JH) = 0.$$

Therefore, we have

Proposition 3.1 *Assume that Σ is a csL surface in $\mathbb{S}^5$ and* $\operatorname{tr}\langle B(\cdot, \nabla_{\cdot}(JH)), H\rangle = 0$, *then Σ is a csL Willmore surface.*

With the aid of Proposition 3.1, we can find the following examples of csL Willmore surfaces from csL surfaces in $\mathbb{S}^5$. Firstly, according to Proposition 3.1, all closed Legendrian surfaces with parallel tangent vector field JH, which are exactly minimal surfaces or the Calabi tori (cf. [10, Proposition 3.2]), are csL Willmore surfaces. For reader's convenience, we give some detailed computations as follows.

Example 3.1 (Calabi tori) For every four nonzero real numbers r_1, r_2, r_3, r_4 with $r_1^2 + r_2^2 = r_3^2 + r_4^2 = 1$, the Calabi torus Σ is a csL surface in $\mathbb{S}^5$ defined as follows.

$$
\begin{aligned}
&F : \mathbb{S}^1 \times \mathbb{S}^1 \mapsto \mathbb{S}^5,\\
&(t,s) \mapsto \left(r_1 r_3 \exp\left(\sqrt{-1}\left(\frac{r_2}{r_1}t + \frac{r_4}{r_3}s\right)\right), r_1 r_4 \exp\left(\sqrt{-1}\left(\frac{r_2}{r_1}t - \frac{r_3}{r_4}s\right)\right), r_2 \exp\left(-\sqrt{-1}\frac{r_1}{r_2}t\right)\right).
\end{aligned}
$$

Denote

$$
\phi_1 = \exp\left(\sqrt{-1}\left(\frac{r_2}{r_1}t + \frac{r_4}{r_3}s\right)\right), \quad \phi_2 = \exp\left(\sqrt{-1}\left(\frac{r_2}{r_1}t - \frac{r_3}{r_4}s\right)\right), \quad \phi_3 = \exp\left(-\sqrt{-1}\frac{r_1}{r_2}t\right),
$$

then $F(t,s) = \left(r_1 r_3 \phi_1, r_1 r_4 \phi_2, r_2 \phi_3\right)$. Since

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= \left(\sqrt{-1} r_2 r_3 \phi_1, \sqrt{-1} r_2 r_4 \phi_2, -\sqrt{-1} r_1 \phi_3\right),\\
\frac{\partial F}{\partial s} &= \left(\sqrt{-1} r_1 r_4 \phi_1, -\sqrt{-1} r_1 r_3 \phi_2, 0\right),
\end{aligned}
$$

the induced metric in Σ is given by

$$
g = dt^2 + r_1^2 ds^2.
$$

Let $E_1 = \frac{\partial F}{\partial t}, E_2 = \frac{1}{r_1}\frac{\partial F}{\partial s}$, then $\{E_1, E_2, \nu_1 = \sqrt{-1}E_1, \nu_2 = \sqrt{-1}E_2, \mathbf{R} = -\sqrt{-1}F\}$ is a local orthonormal frame of $\mathbb{S}^5$ such that $\{E_1, E_2\}$ is a local orthonormal tangent frame and $\mathbf{R}$ is the Reeb field. A direct calculation yields

$$\frac{\partial v_1}{\partial t} = \left(-\sqrt{-1}\frac{r_2^2 r_3}{r_1}\phi_1, -\sqrt{-1}\frac{r_2^2 r_4}{r_1}\phi_2, -\sqrt{-1}\frac{r_1^2}{r_2}\phi_3\right),$$
$$\frac{\partial v_1}{\partial s} = \left(-\sqrt{-1}\frac{r_2 r_3^2}{r_4}\phi_1, \sqrt{-1}\frac{r_2 r_4^2}{r_3}\phi_2, 0\right),$$
$$\frac{\partial v_2}{\partial t} = \left(-\sqrt{-1}\frac{r_2 r_4}{r_1}\phi_1, \sqrt{-1}\frac{r_2 r_3}{r_1}\phi_2, 0\right),$$
$$\frac{\partial v_2}{\partial s} = \left(-\sqrt{-1}\frac{r_4^2}{r_3}\phi_1, -\sqrt{-1}\frac{r_3^2}{r_4}\phi_2, 0\right),$$
$$\frac{\partial \mathbf{R}}{\partial t} = \left(r_2 r_3 \phi_1, r_2 r_4 \phi_2, -r_1 \phi_3\right),$$
$$\frac{\partial \mathbf{R}}{\partial s} = \left(r_1 r_4 \phi_1, -r_1 r_3 \phi_2, 0\right).$$

Hence,

$$A^{v_1} = -\Re\langle dF, dv_1\rangle = \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)dt^2 + r_1 r_2 ds^2,$$
$$A^{v_2} = -\Re\langle dF, dv_2\rangle = 2r_2 dtds + r_1\left(\frac{r_4}{r_3} - \frac{r_3}{r_4}\right)ds^2,$$
$$A^{\mathbf{R}} = 0.$$

Thus,

$$H = \left(\frac{2r_2}{r_1} - \frac{r_1}{r_2}\right)v_1 + \frac{1}{r_1}\left(\frac{r_4}{r_3} - \frac{r_3}{r_4}\right)v_2.$$

Moreover, E_1 and E_2 are two parallel tangent vector field. It is obvious that Σ is a csL Willmore surface.

Secondly, we give some examples that JH is not parallel. Mironov [15] constructed the following new csL surfaces in $\mathbb{S}^5$. We will verify that Mironov's examples are in fact csL Willmore surfaces.

Example 3.2 (Mironov's examples [15]) Let $F : \Sigma^2 \mapsto \mathbb{S}^5$ be an immersion. Then, F is a Legendrian immersion iff

$$\langle F_x, F\rangle = \langle F_y, F\rangle = 0.$$

Here, $\{x, y\}$ is a local coordinates of Σ and $\langle , \rangle$ stands for the Hermitian inner product in $\mathbb{C}^3$. Set

$$G = \begin{pmatrix} F \\ F_x \\ F_y \end{pmatrix},$$

then

$$G\bar{G}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle F_x, F_x\rangle & \langle F_x, F_y\rangle \\ 0 & \langle F_y, F_x\rangle & \langle F_y, F_y\rangle \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix},$$

where g is a real positive matrix which is the induce metric of Σ. There is a Hermitian matrix Θ such that

$$G = \begin{pmatrix} 1 & 0 \\ 0 & g^{1/2} \end{pmatrix} e^{\sqrt{-1}\Theta}.$$

We compute

$$\begin{aligned} G\bar{G}_x^T &= \begin{pmatrix} 0 & -\langle F_x, F_x\rangle & -\langle F_x, F_y\rangle \\ \langle F_x, F_x\rangle & \langle F_x, F_{xx}\rangle & \langle F_x, F_{yx}\rangle \\ \langle F_y, F_x\rangle & \langle F_y, F_{xx}\rangle & \langle F_y, F_{yx}\rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_x \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \left(\sqrt{g}\right)_x \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \Re\left(\sqrt{-1}G\bar{G}_x^T\right) &= \Re\sqrt{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \langle F_x, F_{xx}\rangle & \langle F_x, F_{yx}\rangle \\ 0 & \langle F_y, F_{xx}\rangle & \langle F_y, F_{yx}\rangle \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & A^{\sqrt{-1}F_x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix} \Re\left(\sqrt{-1}e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_x\right) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{g} \end{pmatrix}, \end{aligned}$$

which implies

$$\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2}A^{\sqrt{-1}F_x}g^{1/2} \end{pmatrix} = \Re\left(\sqrt{-1}e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_x\right).$$

Similarly,

$$\begin{pmatrix} 0 & 0 \\ 0 & g^{-1/2}A^{\sqrt{-1}F_y}g^{1/2} \end{pmatrix} = \Re\left(\sqrt{-1}e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_y\right).$$

The Lagrangian angle is then given by $\theta = tr\Re\Theta$. The above discussion implies that

$$J\nabla\theta = H.$$

Let a, b, c are three positive constants and consider the following immersion

$$\begin{aligned} &F : \mathbb{S}^1 \times \mathbb{S}^1 \mapsto \mathbb{S}^5, \\ &(x, y) \mapsto \left(\phi(x)e^{\sqrt{-1}ay}, \psi(x)e^{\sqrt{-1}by}, \zeta(x)e^{-\sqrt{-1}cy}\right), \end{aligned}$$

where

$$\phi(x) = \sqrt{\frac{c}{a+c}}\sin x,$$
$$\psi(x) = \sqrt{\frac{c}{b+c}}\cos x,$$
$$\zeta(x) = \sqrt{\frac{a\sin^2 x}{a+c} + \frac{b\cos^2 x}{b+c}} = \sqrt{\frac{ab+u(x)}{(a+c)(b+c)}},$$

where

$$u(x) = \frac{c(a+b+(b-a)\cos(2x))}{2}.$$

One can check that F is a Legendrian immersion. Denote $\Sigma := F\left(\mathbb{S}^1 \times \mathbb{S}^1\right)$. Notice that

$$F_x = \left(\sqrt{\frac{c}{a+c}}\cos x e^{\sqrt{-1}ay}, -\sqrt{\frac{c}{b+c}}\sin x e^{\sqrt{-1}by}, \frac{-c(b-a)\sin(2x)}{2\sqrt{(a+c)(b+c)(ab+u(x))}}e^{-\sqrt{-1}cy}\right),$$
$$F_y = \left(\sqrt{-1}a\phi(x)e^{\sqrt{-1}ay}, \sqrt{-1}b\psi(x)e^{\sqrt{-1}by}, -\sqrt{-1}c\zeta(x)e^{-\sqrt{-1}cy}\right).$$

The induced metric g is given by

$$\begin{aligned} g &= \left[\frac{c\cos^2 x}{a+c} + \frac{c\sin^2 x}{b+c} + \frac{c^2(b-a)^2\sin^2(2x)}{4(a+c)(b+c)(ab+u(x))}\right]dx^2 \\ &\quad + \left[a^2 \times \frac{c\sin^2 x}{a+c} + b^2 \times \frac{c\cos^2 x}{b+c} + c^2\left(\frac{a\sin^2 x}{a+c} + \frac{b\cos^2 x}{b+c}\right)\right]dy^2 \\ &= \frac{u(x)}{ab+u(x)}dx^2 + u(x)dy^2 \\ &:= e^{2p(x)}dx^2 + e^{2q(x)}dy^2. \end{aligned}$$

A strait forward calculation yields that

$$A^{\sqrt{-1}F_x} = \Re\begin{pmatrix} 0 & \sqrt{-1}\langle F_x, F_{xy}\rangle \\ -\sqrt{-1}\langle F_{xy}, F_x\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & c\left(1-e^{2p(x)}\right) \\ c\left(1-e^{2p(x)}\right) & 0 \end{pmatrix},$$
$$A^{\sqrt{-1}F_y} = \Re\begin{pmatrix} \sqrt{-1}\langle F_x, F_{yx}\rangle & 0 \\ 0 & \sqrt{-1}\langle F_y, F_{yy}\rangle \end{pmatrix} = \begin{pmatrix} c\left(1-e^{2p(x)}\right) & 0 \\ 0 & (a+b-c)e^{2q(x)} - abc \end{pmatrix}.$$

We get

$$\Re\left(\sqrt{-1}e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_x\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{abc}{u\sqrt{ab+u}} \\ 0 & \dfrac{abc}{u\sqrt{ab+u}} & 0 \end{pmatrix},$$

$$\Re\left(\sqrt{-1}e^{\sqrt{-1}\Theta}\left(e^{-\sqrt{-1}\Theta}\right)_y\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{abc}{u} & 0 \\ 0 & 0 & (a+b-c) - \dfrac{abc}{u} \end{pmatrix}.$$

Thus,

$$H^{\sqrt{-1}F_x} = 0, \quad H^{\sqrt{-1}F_y} = a+b-c.$$

We get

$$H = \frac{a+b-c}{u(x)}\sqrt{-1}\frac{\partial}{\partial y},$$

and

$$\nabla_{\partial_x}\left(\sqrt{-1}H\right) = \frac{(a+b-c)u_x}{2u^2}\frac{\partial}{\partial y}, \quad \nabla_{\partial_y}\left(\sqrt{-1}H\right) = \frac{(ab+u)(a+b-c)u_x}{2u^2}\frac{\partial}{\partial x}.$$

In particular,

$$\operatorname{div}\left(\sqrt{-1}H\right) = 0.$$

Hence, Σ is csL. Moreover,

$$\sum_{i=1}^{2}\left\langle B\left(e_i, \nabla_{e_i}(JH)\right), H\right\rangle = 0.$$

Therefore, Σ is a csL Willmore surface in $\mathbb{S}^5$.

Acknowledgements Open access funding provided by Projekt DEAL. This work was partially supported by the NSFC of China (Nos. 11501421, 11801420, 11971358) and the Youth Talent Training Program of Wuhan University.

References

1. Bauer, M., Kuwert, E.: Existence of minimizing Willmore surfaces of prescribed genus. Int. Math. Res. Not. (2003). https://doi.org/10.1155/S1073792803208072
2. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds. Volume 203 of Progress in Mathematics, 2nd edn. Birkhäuser, Boston (2010). https://doi.org/10.1007/978-0-8176-4959-3
3. Castro, I., Li, H., Urbano, F.: Hamiltonian-minimal Lagrangian submanifolds in complex space forms. Pacific J. Math. **227**, 43–63 (2006). https://doi.org/10.2140/pjm.2006.227.43
4. Castro, I., Urbano, F.: Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form. Tohoku Math. J. **2**(45), 565–582 (1993). https://doi.org/10.2748/tmj/1178225850
5. Castro, I., Urbano, F.: Willmore surfaces of $\mathbb{R}^4$ and the Whitney sphere. Ann. Global Anal. Geom. **19**, 153–175 (2001). https://doi.org/10.1023/A:1010720100464
6. Chen, B.Y.: On the total curvature of immersed manifolds. VI. Submanifolds of finite type and their applications. Bull. Inst. Math. Acad. Sin. **11**, 309–328 (1983)
7. Iriyeh, H.: Hamiltonian minimal Lagrangian cones in $\mathbb{C}^m$. Tokyo J. Math. **28**, 91–107 (2005). https://doi.org/10.3836/tjm/1244208282
8. Jost, J.: Riemannian Geometry and Geometric Analysis. Universitext, 7nd edn. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-61860-9
9. Luo, Y.: On Willmore Legendrian surfaces in $\mathbb{S}^5$ and the contact stationary Legendrian Willmore surfaces. Calc. Var. Partial Differential Equations **56**, Art. 86, 19 (2017). https://doi.org/10.1007/s00526-017-1183-z
10. Luo, Y., Sun, L.: Rigidity of closed CSL submanifolds in the unit sphere (2018). arXiv e-prints arXiv:1811.02839
11. Luo, Y., Wang, G.: On geometrically constrained variational problems of the Willmore functional I. The Lagrangian–Willmore problem. Comm. Anal. Geom. **23**, 191–223 (2015). https://doi.org/10.4310/CAG.2015.v23.n1.a6
12. Ma, H., Mironov, A.E., Zuo, D.: An energy functional for Lagrangian tori in $\mathbb{CP}^2$. Ann. Global Anal. Geom. **53**, 583–595 (2018). https://doi.org/10.1007/s10455-017-9589-6
13. Marques, F.C., Neves, A.: Min–max theory and the Willmore conjecture. Ann. Math. **2**(179), 683–782 (2014). https://doi.org/10.4007/annals.2014.179.2.6
14. Minicozzi II, W.P.: The Willmore functional on Lagrangian tori: its relation to area and existence of smooth minimizers. J. Amer. Math. Soc. **8**, 761–791 (1995). https://doi.org/10.2307/2152828
15. Mironov, A.E.: New examples of Hamilton-minimal and minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{CP}^n$. Mat. Sb. **195**, 89–102 (2004). https://doi.org/10.1070/SM2004v195n01ABEH000794
16. Montiel, S., Urbano, F.: A Willmore functional for compact surfaces in the complex projective plane. J. Reine Angew. Math. **546**, 139–154 (2002). https://doi.org/10.1515/crll.2002.039
17. Oh, Y.G.: Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds. Invent. Math. **101**, 501–519 (1990). https://doi.org/10.1007/BF01231513
18. Oh, Y.G.: Volume minimization of Lagrangian submanifolds under Hamiltonian deformations. Math. Z. **212**, 175–192 (1993). https://doi.org/10.1007/BF02571651
19. Pinkall, U.: Hopf tori in S^3. Invent. Math. **81**, 379–386 (1985). https://doi.org/10.1007/BF01389060
20. Simon, L.: Existence of surfaces minimizing the Willmore functional. Comm. Anal. Geom. **1**, 281–326 (1993). https://doi.org/10.4310/CAG.1993.v1.n2.a4
21. Smoczyk, K.: Closed Legendre geodesics in Sasaki manifolds. New York J. Math. **9**, 23–47 (2003)
22. Thomsen, G.: Grundlagen der konformen flächentheorie. Abh. Math. Semin. Univ. Hambg. **3**, 31–56 (1924). https://doi.org/10.1007/BF02954615
23. Weiner, J.L.: On a problem of Chen, Willmore, et al. Indiana Univ. Math. J. **27**, 19–35 (1978). https://doi.org/10.1512/iumj.1978.27.27003
24. White, J.H.: A global invariant of conformal mappings in space. Proc. Amer. Math. Soc. **38**, 162–164 (1973). https://doi.org/10.2307/2038790
25. Willmore, T.J.: Note on embedded surfaces. An. Şti. Univ. “Al. I. Cuza” Iaşi Secţ. I a Mat. (N.S.) **11B**, 493–496 (1965)