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Calculus of Variations

A note on the nonexistence of quasi-harmonic spheres

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Abstract In this paper we study the properties of quasi-harmonic spheres from $\mathbb{R}^m$, $m > 2$. We show that if the universal covering $\tilde{N}$ of N admits a nonnegative strictly convex function ρ with the exponential growth condition $\rho(y) \leq C\exp\left(\frac{1}{4}\tilde{d}(y)^{2/m}\right)$ where $\tilde{d}(y)$ is the distance function on $\tilde{N}$, then N does not admit a quasi-harmonic sphere, which generalize Li-Zhu's result (Calc Var Partial Diff Equ 37(3–4):441–460, 2010). We also show that if u is a quasi-harmonic sphere, then the property that u is of finite energy ($\int_{\mathbb{R}^m} e(u)e^{-|x|^2/4}\mathrm{d}x < \infty$) is equivalent to the property that u satisfies the large energy condition ($\lim_{R\to\infty} R^m e^{-R^2/4}\int_{B_R(0)} e(u)e^{-|x|^2/4}\,\mathrm{d}x = 0$).

Mathematics Subject Classification 58E20 · 53C43

1 Introduction

Let M^m, N^n be two compact Riemannian manifolds of dimension m and n respectively. Let $u \in W^{1,2}(M, N)$, the energy of u is defined by

$$E(u) = \frac{1}{2}\int_M |\mathrm{d}u|^2 \,\mathrm{d}\,\mathrm{Vol}_M\,.$$

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The critical points of the energy functional are called harmonic maps. Eells and Sampson [4] introduce the heat flow and prove that, the heat flow has a global solution which subconverges strongly to a harmonic map at time infinity if the sectional curvature of the target manifold is non-positive. This result was generalized by Ding and Lin [3] to the case that the universal covering of N admits a nonnegative strictly convex function with quadratic growth.

However, in general, the heat flow may produce singularities at a finite time (e.g. [1,2]). Struwe divided singularities of the heat flow into two different types. One of this type is associated to quasi-harmonic spheres (c.f. [9]).

Definition 1.1 Let $m > 2$. A quasi-harmonic sphere is a non-constant harmonic map from $\left(\mathbb{R}^m, \exp\left(-\frac{|x|^2}{2(m-2)}\right) g_0\right)$ to a Riemannian manifold, where g_0 is the Euclidean metric in $\mathbb{R}^m$, i.e.,

$$\tau(u) = \frac{1}{2} x \cdot \mathrm{d}u, \tag{1.1}$$

with finite energy

$$\int_{\mathbb{R}^m} e(u) e^{-|x|^2/4} \mathrm{d}x < \infty, \tag{1.2}$$

where

$$e(u) = \frac{1}{2} |\mathrm{d}u|^2 .$$

Based on the work of Lin and Wang [9], we know that Liouville theorems for harmonic spheres (harmonic maps from spheres) and quasi-harmonic spheres imply the global existence of the heat flows. Li and Wang [6] proved that there are no non-constant quasi-harmonic spheres with images in a regular ball. Li and Zhu [8] proved that, if the heat flow has a global solution and there is no harmonic map from S^l to N for $2 \leq l \leq m-1$, then this flow subconverges in C^2 norm to a smooth harmonic map at infinity. Moreover, in the same paper, they also proved that the heat flow exists globally provided that the universal covering $\tilde{N}$ of N admits a strictly convex positive function ρ with polynomial growth, i.e.,

$$\tilde{\nabla}^2 \rho > 0, \quad 0 < \rho(y) < C(1 + \tilde{d}(y, y_0))^P, \quad \forall y \in \tilde{N},$$

for some $y_0 \in \tilde{N}$ and some positive constants C, P. Here $\tilde{d}$ is the distance function on $\tilde{N}$. Li and Yang [7] generalized these results to the case of "quasi-harmonic sphere with large energy condition" under the same assumption on ρ. The large energy condition is defined by

$$\lim_{R \longrightarrow \infty} R^m e^{-R^2/4} \int_{B_R(0)} e(u) e^{-|x|^2/4} \, \mathrm{d}x = 0. \tag{1.3}$$

Our first main result is as follows.

Theorem 1.1 *Suppose u satisfies (1.1), then the following three conditions are equivalent to each other.*

1. *The large energy condition holds, i.e., (1.3) holds.*
2. $$\int_{\mathbb{R}^m} |u_r|^2 |x|^{4-m} \, \mathrm{d}x < \infty.$$
3. *The total energy is finite, i.e., (1.2) holds.*

Remark 1.1 Li and Zhu [8] stated the following estimate for quasi-harmonic sphere,

$$\int_{B_R(0)} |\mathrm{d}u|^2 \,\mathrm{d}x \le CR^{m-2}, \quad \forall R > 0, \tag{1.4}$$

where C is a constant independent of R. As a consequence, this condition (1.4)[1] is equivalent to (1.2) and is also equivalent to the following condition

$$\int_{\mathbb{R}^m} |\mathrm{d}u|^2 \, |x|^{2-m-\delta} \,\mathrm{d}x < \infty$$

for some or every $\delta > 0$. In fact, one can get more, see Corollary 2.5.

Our second main result is that, Li-Zhu's result holds, if the universal covering $\tilde{N}$ of N admits a nonnegative strictly convex function ρ with the following exponential growth condition: for some constant C,

$$\rho(y) \le C \exp\left(\frac{1}{4}\tilde{d}(y)^{2/m}\right), \quad \forall y \in \tilde{N}. \tag{1.5}$$

Here $\tilde{d}(y) = \tilde{d}(y, y_0)$ is the distance function on $\tilde{N}$ from some fixed point $y_0 \in \tilde{N}$. It is easy to check that this assumption is weaker than the one in [8]. In appendix, we constructed a strictly convex positive function on $\mathbb{R}^3$ which is of exponential growth.

Theorem 1.2 *Suppose $m \ge 3$ and there is a nonnegative strictly convex function ρ on the universal covering of the target manifold N such that (1.5) holds. Then there is no non-constant quasi-harmonic sphere u from $\mathbb{R}^m$ to N.*

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2 Proof of Theorem 1.1

In this section, we derive some estimates and prove Theorem 1.1. Introduce

$$H(r) := \int_{\mathrm{S}^{m-1}} \left(|u_r|^2 - e(u)\right) \mathrm{d}\theta, \quad \forall r > 0.$$

We begin with the following Lemma.

Lemma 2.1 *Suppose u satisfies (1.1). Then*

1. *either*

$$-R^{-2}(m-2) \int_{B_{\sqrt{2(m-2)}}} r^{2-m} |u_r|^2 \,\mathrm{d}x \le H(R) \le 0, \quad \forall R > 0, \tag{2.1}$$

2. *or there exists $R_0 \ge \sqrt{2(m-2)}$ such that*

$$H(R) \ge R^{2-2m} e^{R^2/2} R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0, \quad \forall\, R > R_0. \tag{2.2}$$

Here S^{m-1} stands for the unit sphere in $\mathbb{R}^m$ centering at 0 and $B_R = B_R(0)$.

[1] We thank ZHU Xiangrong for pointing out this equivalent condition.

Proof A direct computation gives (c.f. Lemma 3.3 in [8])

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta-\int_{\mathbb{S}^{m-1}}\left(\frac{2}{r}e(u)+\left(\frac{r}{2}-\frac{m}{r}\right)|u_r|^2\right)\mathrm{d}\theta=0,\quad \forall r>0. \tag{2.3}$$

According to this identity, we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta+\frac{2}{r}\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta=\left(\frac{r}{2}-\frac{m-2}{r}\right)\int_{\mathbb{S}^{m-1}}|u_r|^2\,\mathrm{d}\theta.$$

From this formula, we know

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2H(r)\right)=r^2\left(\frac{r}{2}-\frac{m-2}{r}\right)\int_{\mathbb{S}^{m-1}}|u_r|^2\,\mathrm{d}\theta. \tag{2.4}$$

Thus, $r^2H(r)$ increases from $\sqrt{2(m-2)}$ to infinity, and decreases from 0 to $\sqrt{2(m-2)}$. Setting $C_0:=\sqrt{2(m-2)}$, we get

$$r^2H(r)\geq C_0^2H(C_0),\quad \forall r>0.$$

Again according to (2.3) we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta+\left(\frac{2(m-1)}{r}-r\right)\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta\\&=\left(r-\frac{2(m-2)}{r}\right)\int_{\mathbb{S}^{m-1}}\left(e(u)-\frac{1}{2}|u_r|^2\right)\mathrm{d}\theta,\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{2m-2}e^{-r^2/2}H(r)\right)=r^{2m-2}e^{-r^2/2}\left(r-\frac{2m-4}{r}\right)\int_{\mathbb{S}^{m-1}}\left(e(u)-\frac{1}{2}|u_r|^2\right)\mathrm{d}\theta. \tag{2.5}$$

Hence, $r^{2m-2}e^{-r^2/2}H(r)$ is increase from $\sqrt{2(m-2)}$ to infinity, and is decrease from 0 to $\sqrt{2(m-2)}$. It is obvious that

$$r^{2m-2}e^{-r^2/2}\int_{\mathbb{S}^{m-1}}\left(|u_r|^2-e(u)\right)\mathrm{d}\theta\to 0,\quad \text{as } r\to 0.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2H(r)\right)\geq -(m-2)r\int_{\mathbb{S}^{m-1}}|u_r|^2\,\mathrm{d}\theta,$$

which yields

$$R^2H(R)\geq -(m-2)\int_{B_R}r^{2-m}|u_r|^2\,\mathrm{d}x,\quad \forall R>0.$$

Here we have used the fact

$$\lim_{r\to 0}r^2H(r)=0.$$

Therefore,

$$r^2H(r)\geq C_0^2H(C_0)\geq -(m-2)\int_{B_{C_0}}r^{2-m}|u_r|^2\,\mathrm{d}x,\quad \forall r>0.$$

Now we can finish the proof of this Lemma. If we do not have (2.1), then there exists $R_0 \geq \sqrt{2(m-2)}$, such that

$$\int_{\{R_0\}\times \mathbb{S}^{m-1}} \left(|u_r|^2 - e(u)\right) \mathrm{d}\theta > 0,$$

then for every $r > R_0$,

$$r^{2m-2} e^{-r^2/2} H(r) \geq R_0^{2m-2} e^{-R_0^2/2} H(R_0) > 0,$$

which means that (2.2) holds. □

Remark 2.1 Suppose u satisfies (1.1), then

$$-R^2 H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} r^{2-m} |u_r|^2 \, \mathrm{d}x, \tag{2.6}$$

$$-R^{2m-2} e^{-R^2/2} H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} r^{m-2} e^{-r^2/2} \frac{|u_\theta|^2}{r^2} \, \mathrm{d}x, \tag{2.7}$$

$$-R^m e^{-R^2/4} H(R) \leq (m-2) \int_{B_{\sqrt{2(m-2)}}} e^{-r^2/4} e(u) \, \mathrm{d}x, \tag{2.8}$$

holds for all $R > 0$.

Proof The proof of (2.6) and (2.7) can be found in the proof of Lemma 2.1. The proof of (2.8) can be proved similarly since (2.3) implies the following formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^m e^{-r^2/4} H(r)\right) = \left(\frac{r}{2} - \frac{m-2}{r}\right) r^m e^{-r^2/4} \int_{\mathbb{S}^{m-1}} e(u) \mathrm{d}\theta, \quad \forall r \in (0, \infty).$$

□

Lemma 2.2 *Suppose u satisfies (1.1) and*

$$\liminf_{R\to\infty} R^{2m-2} e^{-R^2/2} \int_{\{R\}\times \mathbb{S}^{m-1}} \left(|u_r|^2 - e(u)\right) \mathrm{d}\theta > 0,$$

then

$$\liminf_{R\to\infty} R^m e^{-R^2/4} \int_{B_R} \left(|u_r|^2 - e(u)\right) e^{-r^2/4} \, \mathrm{d}x > 0.$$

Proof A direct computation. □

Next, we prove the following energy estimate.

Proposition 2.3 *Suppose u satisfies (1.1), then there is a constant C_1 depending only on m such that for every $0 \leq \delta \leq 2$, we have*

$$\int_{B_R} r^{4-m-\delta} |u_r|^2 \, \mathrm{d}x \leq C_1 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, \mathrm{d}x + 4R^2 H(R)^+, \quad \forall R > 0.$$

Here $f^+ = \max\{f, 0\}$.

Proof It suffices to consider the case $R > 2\sqrt{(m-2)}$ and we start with the formula (2.4), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 H(r)\right) = r^2\left(\frac{r}{2} - \frac{m-2}{r}\right)\int_{\mathbb{S}^{m-1}} |u_r|^2 \,\mathrm{d}\theta.$$

For every $0 < \rho < R$, we have

$$\begin{aligned} R^2 H(R) - \rho^2 H(\rho) &= \int_\rho^R r^2\left(\frac{r}{2} - \frac{m-2}{r}\right)\int_{\mathbb{S}^{m-1}} |u_r|^2 \,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_{B_R\setminus B_\rho}\left(\frac{r}{2} - \frac{m-2}{r}\right) r^{3-m}\,|u_r|^2 \,\mathrm{d}x. \end{aligned}$$

For $\sqrt{4(m-2)} \le \rho < R$, we have

$$\int_{B_R\setminus B_\rho} r^{4-m}\,|u_r|^2 \,\mathrm{d}x \le 4R^2 H(R)^+ - 4\rho^2 H(\rho),$$

which implies

$$\begin{aligned} \int_{B_R\setminus B_{2\sqrt{m-2}}} r^{4-m}\,|u_r|^2 \,\mathrm{d}x \le& 4R^2 H(R)^+ - 4\left(2\sqrt{m-2}\right)^2 H\left(2\sqrt{m-2}\right) \\ \le& 4R^2 H(R)^+ + 4(m-2)\int_{B_{2\sqrt{m-2}}} r^{2-m}\,|u_r|^2 \,\mathrm{d}x. \end{aligned}$$

Here we have used (2.6). In particular, we get the desired estimate for $\delta = 0$. In general $0 \le \delta \le 2$,

$$\begin{aligned} \int_{B_R} r^{4-m-\delta}\,|u_r|^2 \,\mathrm{d}x &= \int_{B_R\setminus B_{2\sqrt{m-2}}} r^{4-m-\delta}\,|u_r|^2 \,\mathrm{d}x + \int_{B_{2\sqrt{m-2}}} r^{4-m-\delta}\,|u_r|^2 \,\mathrm{d}x \\ &\le \int_{B_R\setminus B_{2\sqrt{m-2}}} r^{4-m}\,|u_r|^2 \,\mathrm{d}x + \left(2\sqrt{m-2}\right)^{2-\delta}\int_{B_{2\sqrt{m-2}}} r^{2-m}\,|u_r|^2 \,\mathrm{d}x \\ &\le 8(m-2)\int_{B_{2\sqrt{m-2}}} r^{2-m}\,|u_r|^2 \,\mathrm{d}x + 4R^2 H(R)^+. \end{aligned}$$

□

As a consequence, we have

Corollary 2.4 *Suppose u satisfies (1.1). Then there is a constant C_2 such that for every $0 < \delta < 1$,*

$$\delta R^{-\delta}\int_{B_R} r^{2-m+\delta} e(u)\,\mathrm{d}x \le C_2\int_{B_{2\sqrt{m-2}}} r^{2-m}\,|u_r|^2 \,\mathrm{d}x + 4R^2 H(R)^+, \quad \forall R > 0.$$

In particular,

$$R^{2-m}\int_{B_R} e(u)\,\mathrm{d}x \le C_2\int_{B_{2\sqrt{m-2}}} r^{2-m}\,|u_r|^2 \,\mathrm{d}x + 4R^2 H(R)^+, \quad \forall R > 0. \tag{2.9}$$

Proof Since

$$
\begin{aligned}
\int_{B_R} r^{2-m+\delta} e(u)\,\mathrm{d}x &= -\int_0^R r^{1+\delta} H(r)\,\mathrm{d}r + \int_{B_R} r^{2-m+\delta} |u_r|^2\,\mathrm{d}x \\
&\le \sup_{0<r<R} \left(-r^2 H(r)\right) \times \int_0^R r^{\delta-1}\,\mathrm{d}r + R^\delta \int_{B_R} r^{2-m} |u_r|^2\,\mathrm{d}x \\
&= \sup_{0<r<R} \left(-r^2 H(r)\right) \times \frac{R^\delta}{\delta} + R^\delta \int_{B_R} r^{2-m} |u_r|^2\,\mathrm{d}x.
\end{aligned}
$$

Now applying Lemma 2.1 and Proposition 2.3, there exists a constant C_2 depending only on m such that

$$
\delta R^{-\delta} \int_{B_R} r^{2-m+\delta} e(u)\,\mathrm{d}x \le C_2 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2\,\mathrm{d}x + 4R^2 H(R)^+.
$$

□

Also, we can prove the following

Corollary 2.5 *Suppose u satisfies* (1.1), *then there is a constant C_3 depending only on m such that for every $0 < \delta < 1$,*

$$
\delta \int_{B_R} r^{2-m-\delta} e(u)\,\mathrm{d}x \le C_3 \int_{B_{2\sqrt{m-2}}} r^{1-m} e(u)\,\mathrm{d}x + 4R^2 H(R)^+, \quad \forall R > 0.
$$

Proof Similar to the proof of Corollary 2.4, for $0 < \delta < 1$ and $R > 2\sqrt{m-2}$,

$$
\begin{aligned}
\int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} e(u)\,\mathrm{d}x &= -\int_{2\sqrt{m-2}}^R r^{1-\delta} H(r)\,\mathrm{d}r + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m-\delta} |u_r|^2\,\mathrm{d}x \\
&\le \sup_{2\sqrt{m-2}<r<R} \left(-r^2 H(r)\right) \times \int_{2\sqrt{m-2}}^R r^{\delta-1}\,\mathrm{d}r \\
&\quad + \int_{B_R \setminus B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2\,\mathrm{d}x \\
&\le \sup_{2\sqrt{m-2}<r<R} \left(-r^2 H(r)\right) \times \frac{2\sqrt{m-2}}{\delta} + \int_{B_R} r^{2-m} |u_r|^2\,\mathrm{d}x.
\end{aligned}
$$

Then Lemma 2.1 and Proposition 2.3 gives the desired estimate. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1 Suppose the large energy condition holds, i.e., the claim (1) is true. Then according to Lemmas 2.1 and 2.2 (or c.f. [7]), we know that $H(r) \le 0$ for every $r > 0$. Now the claim (2) follows from Proposition 2.3.

From the claim (2) to the claim (3), we need only to prove that

$$
\int_{\mathbb{R}^m} r^{2-m-\delta} |\mathrm{d}u|^2\,\mathrm{d}x < \infty.
$$

holds for some $\delta > 0$. According to Corollary 2.5, we need only to claim that $\liminf_{R\to\infty} R^2 H(R)^+ \le 0$. This is true because

$$
\liminf_{R\to\infty} R^2 H(R)^+ \le \liminf_{R\to\infty} \int_{\{R\}\times\mathbb{S}^{m-1}} |u_r|^2\,\mathrm{d}\theta
$$

and the claim (2) implies the righthand is zero.

From the claim (3) to the claim (1) is obvious. □

3 Proof of Theorem 1.2

The following Lemma is proved in [8].

Lemma 3.1 *Suppose f is a non-constant nonnegative smooth function satisfying*

$$\Delta f \geq \frac{1}{2} r f_r,$$

then there exists a constant $C > 0$ such that for r large enough,

$$\int_{\mathbb{S}^{m-1}} f(r,\theta)\,\mathrm{d}\theta > C r^{-m} e^{r^2/4}.$$

Let $d(x) = \mathrm{dist}(u(x), u(0))$, then we have the following

Lemma 3.2 (Refined energy estimate) *Suppose u is a quasi-harmonic sphere, then there is a constant C_m depending only on m such that for all $R > 0$,*

$$\int_{B_R} d^2\,\mathrm{d}x \leq C_m R^m \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2\,\mathrm{d}x,$$

$$\int_{B_R} |\nabla d|^2\,\mathrm{d}x \leq C_m R^{m-2} \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2\,\mathrm{d}x.$$

Remark 3.1 1. Denoted $E_R(u)$ by the energy of u on B_R, i.e.,

$$E_R(u) = \frac{1}{2}\int_{B_R} |\mathrm{d}u|^2 e^{-x^2/4}\,\mathrm{d}x.$$

Then apply Corollary 2.5 to this Lemma to obtain the following estimate

$$\int_{B_R} d^2\,\mathrm{d}x \leq C_m R^m E_R(u),$$

$$\int_{B_R} |\nabla d|^2\,\mathrm{d}x \leq C_m R^{m-2} E_R(u).$$

2. Li and Zhu (c.f. Lemma 3.2 in [8]) obtained a similar result with constant $C_{m,u}$ depending only on m and the total energy of u such that

$$\int_{B_R} d^2\,\mathrm{d}x \leq C_{m,u} R^m,$$

$$\int_{B_R} |\nabla d|^2\,\mathrm{d}x \leq C_{m,u} R^{m-2}.$$

Proof of Lemma 3.2 It is clear that

$$d(r,\theta) \leq \int_0^r |u_s(s,\theta)|\,\mathrm{d}s, \quad |\nabla d| \leq |\mathrm{d}u|.$$

Since the total energy of u is finite, by Lemma 2.2, we have

$$\int_{\mathbb{S}^{m-1}} \left(|u_r|^2 - e(u)\right) \, \mathrm{d}\theta \leq 0, \quad r > 0.$$

Applying (2.9), we obtain

$$\int_{B_R} |\nabla d|^2 \leq 2C_2 R^{m-2} \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, \mathrm{d}x, \quad R > 0.$$

Next, we show

$$\int_{\mathbb{S}^{m-1}} \left(\int_0^r |u_s(s,\theta)| \, \mathrm{d}s \right)^2 \mathrm{d}\theta \leq C_m \int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 \, \mathrm{d}x, \quad \forall r > 0.$$

Then the first part of the this Lemma follows from this inequality. Without loss of generality, assume $r > 1$. Applying Proposition 2.3 and taking $\delta = 1/2$, we get

$$\int_{B_R} r^{7/2-m} |u_r|^2 \, \mathrm{d}x \leq C_1 \int_{B_{2\sqrt{m-2}}} r^{2-m} |u_r|^2 \, \mathrm{d}x, \quad R > 0.$$

Using Minkowski's inequality, we get

$$\begin{aligned}
\left(\int_{\mathbb{S}^{m-1}} \left(\int_0^r |u_s(s,\theta)| \, \mathrm{d}s \right)^2 \mathrm{d}\theta \right)^{1/2} &\leq \int_0^r \left(\int_{\mathbb{S}^{m-1}} |u_s(s,\theta)|^2 \, \mathrm{d}\theta \right)^{1/2} \mathrm{d}s \\
&\leq \int_0^1 \left(\int_{\mathbb{S}^{m-1}} |u_s(s,\theta)|^2 \, \mathrm{d}\theta \right)^{1/2} \mathrm{d}s \\
&\quad + \int_1^r \left(\int_{\mathbb{S}^{m-1}} |u_s(s,\theta)|^2 \, \mathrm{d}\theta \right)^{1/2} \mathrm{d}s \\
&\leq \left(\int_0^1 \int_{\mathbb{S}^{m-1}} |u_s(s,\theta)|^2 \, \mathrm{d}\theta \, \mathrm{d}s \right)^{1/2} \\
&\quad + \left(\int_1^r s^{5/2} \int_{\mathbb{S}^{m-1}} |u_s|^2 \, \mathrm{d}\theta \, \mathrm{d}s \right)^{1/2} \left(\int_1^r s^{-5/2} \, \mathrm{d}s \right)^{1/2} \\
&\leq C_m \left(\int_{B_{2\sqrt{m-2}}} r^{1-m} |u_r|^2 \, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

□

Lemma 3.3 *Suppose u is a quasi-harmonic sphere, then there is a constant C_m depending only on m such that*

$$⨍_{B_r} \exp\left(C_m^{-1} E_r(u)^{-1/2} r^{2-m} d \right) \mathrm{d}x \leq C_m, \quad \forall r > 1.$$

Proof By the energy estimate Corollary 2.5, using an argument similar to the one used in the proof of Lemma 3.5 in [8], we can prove that the BMO subnorm $[d]_{*,B_{2r}}$ of d over B_{2r} satisfies

$$[d]_{*,B_{2r}} := \sup_{x \in Q \subset B_{2r}} ⨍ \left| d(y) - d_Q \right| \mathrm{d}y \leq C_m \sqrt{E_{2r}(u)} (1+r)^{m-2}, \tag{3.1}$$

where the supermum is taken over all cubes $x \in Q \subset B_{2r}$. The John-Nirenberg theorem (c.f. Lemma 1 in [5]) claims that there is two constants C_5, C_6 depends only on m such that for all cubes $Q \subset B_{2r}$,

$$\left|\left\{x \in Q : \left|d(x) - d_Q\right| > s\right\}\right| \le C_5 \exp\left(-\frac{C_6 s}{[d]_{*,B_{2r}}}\right)|Q|,$$

which implies

$$\fint_{B_r} \exp\left(\frac{C_6\left|d - d_{B_r}\right|}{2[d]_{*,B_r}}\right) \mathrm{d}x \le C_5, \quad \forall r > 0.$$

Since we have the estimate (3.1), as a consequence, there is a constant C_7 which depends only on m such that

$$\fint_{B_r} \exp\left(C_7^{-1} E_r(u)^{-1/2} r^{2-m}\left|d - d_{B_r}\right|\right) \mathrm{d}x \le C_7, \quad \forall r > 1.$$

Finally, according to Lemma 3.2, we can find a constant C_8 depending only m such that

$$d_{B_r} := \fint_{B_r} d \,\mathrm{d}x \le C_8 E_r(u)^{1/2}.$$

Therefore, we get the desired estimate. □

Remark 3.2 Checking the proof of Lemma 3.5 in [8] step by step, and using the argument mentioned above, one can prove the following refined estimate,

$$\fint_{B_r} \exp\left(C_m^{-1} \tilde{E}_{2\sqrt{m-2}}(u)^{-1/2} r^{2-m} d\right) \mathrm{d}x \le C_m, \quad \forall r > 1.$$

Here

$$\tilde{E}_R(u) = \int_{B_R} r^{1-m} |u_r|^2 \,\mathrm{d}x.$$

In fact, checking the proof (c.f. page 455 in [8]), the constants come from either Lemma 3.2 or $\tilde{E}_{3m}(u)$ which can be controlled by $\tilde{E}_{2\sqrt{m-2}}(u)$ thanks to Corollary 2.5. Hence one can prove the required refined BMO estimate (3.1).

Now we give a poof of Theorem 1.2.

Proof of Theorem 1.2 Let $\tilde{N}$ be the universal covering of N. Let $\tilde{u} : \mathbb{R}^m \longrightarrow \tilde{N}$ be a lift of u with $\tilde{u} = u \circ \pi$ where $\pi : \tilde{N} \longrightarrow N$ is the covering map. It is easy to see that

$$\int_{\mathbb{R}^m} e(\tilde{u}) e^{-|x|^2/4} \,\mathrm{d}x < \infty.$$

Set $f = \rho \circ \tilde{u}$, then

$$\Delta f - \frac{1}{2} r \partial_r f = \tilde{\nabla}^2 \rho(\tilde{u})(\mathrm{d}\tilde{u}, \mathrm{d}\tilde{u}) \ge 0.$$

Fixed $p > 0$. Notice that there is a constant $C > 0$ such that

$$\int_{B_{2R}} f^p \,\mathrm{d}x = \int_{B_{2R}} (\rho \circ \tilde{u})^p \,\mathrm{d}x \le C^p \int_{B_{2R}} e^{\frac{p}{4}\tilde{d}^{2/m}} \,\mathrm{d}x, \quad R > 0. \tag{3.2}$$

Applying Young's inequality,

$$A + B \geq (PA)^{1/P} (QB)^{1/Q}, \quad A, B > 0, \quad P, Q \geq 1, \quad 1/P + 1/Q = 1,$$

we obtain that for $\tilde{\delta} = p/(2m)$,

$$\begin{aligned}
\tilde{\delta} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \tilde{\delta}\right) r^2 &= \frac{p}{2m} r^{2-m} \tilde{d} + \left(\frac{p}{4} - \frac{p}{2m}\right) r^2 \\
&= \frac{p}{4} \left(\frac{2}{m} r^{2-m} \tilde{d} + \frac{m-2}{m} r^2\right) \\
&\geq \frac{p}{4} \left(r^{2-m} \tilde{d}\right)^{2/m} \left(r^2\right)^{(m-2)/m} \\
&= \frac{p}{4} \tilde{d}^{2/m}.
\end{aligned}$$

Therefore, according to (3.2), for $R > 0$, we have

$$\begin{aligned}
&\int_{B_{2R}} f^p \, \mathrm{d}x \leq C^p \int_{B_{2R}} e^{\tilde{\delta} R^{2-m} \tilde{d}(\tilde{u}, y_0)} e^{(p/4 - \tilde{\delta}) R^2} \, \mathrm{d}x \\
&= C^p \int_{B_{2R}} e^{2^{m-2} \tilde{\delta} (2R)^{2-m} \tilde{d}(\tilde{u}, y_0)} e^{(p/4 - \tilde{\delta}) R^2} \, \mathrm{d}x.
\end{aligned} \tag{3.3}$$

We can choose $p > 0$ sufficiently small so that

$$2^{m-2} \tilde{\delta} = 2^{m-3} m^{-1} p \leq C_m^{-1} E^{-1/2},$$

which is equivalent to

$$E \leq \frac{m^2}{4^{m-3} C_m^2 p^2}.$$

According to Lemma 3.3 and (3.3), we can see that

$$\begin{aligned}
\int_{B_{2R}} f^p \, \mathrm{d}x &\leq C^p e^{(p/4 - \tilde{\delta}) R^2} \int_{B_{2R}} \exp\left(C_m^{-1} E^{-1/2} (2R)^{2-m} \tilde{d}(\tilde{u}, y_0)\right) \mathrm{d}x \\
&\leq C^p C_m (2R)^m e^{(p/4 - p/(2m)) R^2}
\end{aligned}$$

holds for R large enough.

If f is not a constant, applying Lemma 3.1 we obtain that for R large enough,

$$\int_{B_R} f \, \mathrm{d}x \geq C_u R^{-2} e^{R^2/4}.$$

Here $C_u > 0$ is a constant which is independent of R. Since $f \geq 0$ satisfies

$$\operatorname{div}\left(e^{-|x|^2/4} \nabla f\right) \geq 0,$$

applying maximum principle and Moser's iteration (c.f. page 167 in [7]), for every $p > 0$, there is a constant $C_p > 0$ depending only on p, m such that for some $x^* \in \partial B_R$,

$$\begin{aligned}
\fint_{B_R} f \,\mathrm{d}x &\leq \sup_{B_R} f \\
&\leq f(x^*) \\
&\leq \sup_{B_{1/R}(x^*)} f \\
&\leq C_p \left(\fint_{B_{2/R}(x^*)} f^p \,\mathrm{d}x \right)^{1/p} \\
&\leq C_p R^{m/p} \left(\int_{B_{2R}} f^p \,\mathrm{d}x \right)^{1/p}
\end{aligned}$$

holds for R large enough. Here we used maximum principle for f in the second inequality. Consequently, for R large enough

$$\int_{B_{2R}} f^p \,\mathrm{d}x \geq C_p^{-p} C_u^p R^{-(m+2)p-m} e^{pR^2/4}. \tag{3.4}$$

Together with (3.3) and (3.4), we know that

$$0 < C_p^{-p} C_u^p \leq C^p C_m 2^m R^{2m+(m+2)p} e^{-pR^2/(2m)} \to 0, \quad \text{as } R \to \infty.$$

This contradiction means that f is a constant. Moreover, since ρ is a strictly convex function, we get that $\mathrm{d}\tilde{u} = 0$, i.e., $\tilde{u}$ is a constant. As a consequence, u is a constant. □

Appendix A: Example

Example A.1 We will construct a strictly convex function which is of exponential growth. Consider a metric g on $\mathbb{R}^3$ given by

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + \phi(e^{2x} + e^{2y})\mathrm{d}z^2.$$

Here $\phi : [0, \infty) \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{aligned}
\phi' &> 0, \quad \forall t \\
(\sqrt{\phi})''(t)t + (\sqrt{\phi})'(t) &< 0, \quad t > 1.
\end{aligned}$$

Taking

$$\sqrt{\phi}(s) = 1 + \int_1^s \frac{1}{t(1+t)} \,\mathrm{d}t, \quad s > 1.$$

for example. Let $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, $y = r\sin\theta$, then we can rewrite

$$g = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \phi(e^{2r\cos\theta} + e^{2r\sin\theta})\mathrm{d}z^2.$$

Now the hessian of r is given by

$$\mathrm{Hess}^g\, r = \frac{1}{2} L_{\nabla^g r}\, g = r\mathrm{d}\theta^2 + (e^{2r\cos\theta}\cos\theta + e^{2r\sin\theta}\sin\theta)\phi'\mathrm{d}z^2.$$

On the one hand, $|\nabla^g r| = 1$ and $\nabla^g_{\nabla^g r}\nabla^g r = 0$, hence the curve $r \mapsto (r, \theta, z)$ is a geodesic for every fixed θ, z. Then it is easy to check that

$$\rho((x, y, z)) := \mathrm{dist}_g((x, y, z), (0, 0, z)) = \sqrt{x^2 + y^2}.$$

On the other hand, one can check that

$$\frac{1}{2}\,\mathrm{Hess}^g\, r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + (e^{2x}x + e^{2y}y)\phi'\mathrm{d}z^2.$$

As a consequence, r^2 is not convex.

Now for every $\alpha > 0$, choose $u = e^{\alpha x} + e^{\alpha y}$, we have

$$\mathrm{Hess}^g\, u = \alpha^2(e^{\alpha x} + e^{\alpha y})(\mathrm{d}x^2 + \mathrm{d}y^2) + \alpha(e^{(2+\alpha)x} + e^{(2+\alpha)y})\phi'\mathrm{d}z^2 > 0.$$

Moreover, u is of exponential growth, i.e.,

$$u < e^{\alpha\rho}.$$

By the way, the Riemannian curvature satisfies

$$\begin{aligned} R^g(\nabla^g x, \nabla^g y, \nabla^g x, \nabla^g y) &= 0,\\ R^g(\nabla^g x, \nabla^g z, \nabla^g x, \nabla^g z) &= -\sqrt{\phi}(\sqrt{\phi})_{xx},\\ R^g(\nabla^g z, \nabla^g y, \nabla^g z, \nabla^g y) &= -\sqrt{\phi}(\sqrt{\phi})_{yy}. \end{aligned}$$

By the assumption of ϕ, we know that the Riemannian curvature can not be nonpositive.

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