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Dirac-harmonic maps with the trivial index

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Abstract For a homotopy class $[u]$ of maps between a closed Riemannian manifold M and a general manifold N, we want to find a Dirac-harmonic map with the map component in the given homotopy class. Most known results require the index to be nontrivial. When the index is trivial, the few known results are all constructive and produce uncoupled solutions. In this paper, we define a new quantity. As a byproduct of proving the homotopy invariance of this new quantity, we find a new simple proof for the fact that all the Dirac-harmonic spheres in surfaces are uncoupled. More importantly, by using the homotopy invariance of this new quantity, we prove the existence of Dirac-harmonic maps from manifolds in the trivial index case. In particular, when the domain is a closed Riemann surface, we prove the short-time existence of the α-Dirac-harmonic map flow in the trivial index case. Together with the density of the minimal kernel, we get an existence result for Dirac-harmonic maps from closed Riemann surfaces to Kähler manifolds, which extends the previous result of the first and third authors. This establishes a general existence theory for Dirac-harmonic maps in the context of the trivial index.

Keywords Dirac-harmonic map, α-Dirac-harmonic map flow, minimal kernel, existence, Kähler manifolds

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1 Introduction

Motivated by the supersymmetric nonlinear sigma model from quantum field theory (see [6]), Dirac-harmonic maps from Riemann surfaces (with a fixed spin structure) into Riemannian manifolds were introduced in [2]. They generalize harmonic maps and harmonic spinors. From the variational point of view, they are critical points of a conformally invariant action functional. The Euler-Lagrange equation then is an elliptic system coupling a harmonic map-type equation with a Dirac-type equation.

The existence of Dirac-harmonic maps from closed spin manifolds is a very difficult problem. So far, there are only a few results in this direction. Most solutions found so far are uncoupled in the sense that the map part is harmonic. The existence result of [1] for uncoupled solutions depends on the index $I(M,u)$ being non-zero (see Definition 2.2). However, when the domain and target are both closed Riemann surfaces, the index $I(M,u)$ always vanishes. In this case, an existence result about uncoupled

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Dirac-harmonic maps was proved in [3] by the Riemann-Roch formula. Later, this result was generalized to Kähler manifolds in [16]. A general existence result for Dirac-harmonic maps from closed Riemann surfaces to compact manifolds was first established in [9]. This implies the existence of Dirac-harmonic maps when the index $I(M,u)$ is nontrivial.

This then naturally raises the question of the existence of Dirac-harmonic maps in a given homotopy class $[u]$ between manifolds M and N with the trivial index $I(M,u)$. Of course, we should first identify conditions under which this index vanishes. When the domain M is a closed Riemann surface with positive genus and the target N is an odd-dimensional oriented manifold, there is always a spin structure on M such that the index $I(M,u)$ is nontrivial. When, in contrast, M is a closed Riemann surface and N is an even-dimensional spin manifold, the index $I(M,u)$ is always zero. Therefore, here we consider the case where the target manifold N is a Kähler spin manifold. In this case, it is necessary to use a new quantity that can replace the index $I(M,u)$. For this purpose, we introduce a candidate that uses the complex structure of the target manifold. More precisely, we first decompose the twisted Dirac operator as $\not{D}^u = \not{D}^u_{1,0} + \not{D}^u_{0,1}$ according to the decomposition $(u^*TN)^{\mathbb{C}} = u^*T_{1,0}N \oplus u^*T_{1,0}N$. Then we just consider the kernel of one of the two operators, such as $\not{D}_{1,0}$. We define

$$\mathcal{I}(M, u^*T_{1,0}N) := \left[\frac{1}{2}\dim_{\mathbb{C}}\ker\not{D}^u_{1,0}\right]_{\mathbb{Z}_2}$$

for an even-dimensional spin manifold M whenever the complex dimension of the kernel of $\not{D}^u_{1,0}$ is even.

In order to be useful for our purposes, this should be homotopy invariant. Let us first look at an example. Suppose that $M = \mathbb{C}\mathrm{P}^1$ and N is a compact surface. Consider any map $u : M \to N$, the spinor bundle $\Sigma\mathbb{C}\mathrm{P}^1$, and the twisted bundle $\Sigma\mathbb{C}\mathrm{P}^1 \otimes u^*T_{1,0}N$. Let g_N be the genus of N and $c_1(u^*T_{1,0}N) = a\gamma$, $a = 2\deg(u)(1-g_N)$, where γ is the tautological bundle of $\mathbb{C}\mathrm{P}^1$. The unique spin structure of $\mathbb{C}\mathrm{P}^1$ is determined by γ since $\Lambda^{1,0}\mathbb{C}\mathrm{P}^1 = \gamma^2$. Then as a holomorphic bundle, we have

$$\Sigma\mathbb{C}\mathrm{P}^1 \otimes u^*T_{1,0}N = (\gamma \oplus \Lambda^{0,1}\mathbb{C}\mathrm{P}^1 \otimes \gamma) \otimes \gamma^a = \gamma^{a+1} \oplus \Lambda^{0,1}\mathbb{C}\mathrm{P}^1 \otimes \gamma^{a+1}.$$

Since

$$\dim_{\mathbb{C}} H^0(\mathbb{C}\mathrm{P}^1, \gamma^m) = \begin{cases} 0, & m > 0, \\ 1-m, & m \leqslant 0, \end{cases}$$

we conclude that

$$\begin{aligned} \dim_{\mathbb{C}} \ker \not{D}^u_{1,0} &= \dim_{\mathbb{C}} H^0(\mathbb{C}\mathrm{P}^1, \gamma^{a+1}) + \dim_{\mathbb{C}} H^1(\mathbb{C}\mathrm{P}^1, \gamma^{a+1}) \\ &= \dim_{\mathbb{C}} H^0(\mathbb{C}\mathrm{P}^1, \gamma^{a+1}) + \dim_{\mathbb{C}} H^0(\mathbb{C}\mathrm{P}^1, \gamma^{1-a}) \\ &= |a| = 2|\deg(u)(g_N - 1)|. \end{aligned}$$

Therefore, $\dim_{\mathbb{C}} \ker \not{D}^u_{1,0}$ is invariant in the homotopy class $[u]$. This implies the homotopy invariance of $\mathcal{I}(\mathbb{C}\mathrm{P}^1, u^*T_{1,0}N)$, which is equal to $[|\deg(u)(g_N-1)|]_{\mathbb{Z}_2}$. Moreover, the dimension of the kernel of the Dirac operator is a constant in a given homotopy class. Then the following well-known fact follows from the first variational formula.

Proposition 1.1 (See [17]). *There is no coupled Dirac-harmonic map from the* 2*-sphere into a compact Riemann surface.*

In general, we can give two different sufficient conditions to guarantee the homotopy invariance of $\mathcal{I}$.

Theorem 1.2. *Suppose that M is an even-dimensional spin Riemannian manifold and (N,i) is a Kähler manifold. If one of the following holds*:

(1) *the complex spinor bundle $(\Sigma M, i_1)$ over M admits a commuting real structure j, i.e., a real structure $(j^2 = \mathrm{id}_{\Sigma M}, ji_1 = -i_1 j)$ commutes with Clifford multiplication and N is hyperKähler*;

(2) *the complex spinor bundle ΣM over M admits a commuting quaternionic structure j_1 and there exists a parallel real structure j_2 on $T_{1,0}N$, i.e.,*

$$j_2^2 = \mathrm{id}_{T_{1,0}N}, \quad j_2 i = -i j_2, \quad \nabla j_2 = 0,$$

then all the eigenspaces of $\not{D}^u_{1,0}$ *are quaternionic vector spaces for any map* $u : M \to N$ *and* $\mathcal{I}(M, u^*T_{1,0}N)$ *is invariant in the homotopy class* $[u]$.

Moreover, if $\mathcal{I}(M, u^*T_{1,0}N) \neq 0$*, then there is a real vector space of real dimension greater than or equal to* 4 *such that all* $(\tilde{u}, \psi)$*'s are uncoupled* α*-Dirac-harmonic maps as long as there is an* α*-harmonic map* $\tilde{u} \in [u]$ *for* $\alpha \geqslant 1$.

Here, α-Dirac-harmonic maps are the critical points of the following functional:

$$L^\alpha(u, \psi) = \frac{1}{2}\int_M (1 + |du|^2)^\alpha + \frac{1}{2}\int_M \langle \psi, \not{D}^u \psi\rangle_{\Sigma M \otimes u^*TN}, \quad \forall\, \alpha \geqslant 1.$$

They are generalizations of Dirac-harmonic maps (i.e., the case of $\alpha = 1$). As generalizations of harmonic maps, α-harmonic maps are the critical points of the following functional:

$$E_\alpha(u) = \frac{1}{2}\int_M (1 + |du|^2)^\alpha, \quad \forall\, \alpha \geqslant 1,$$

which was introduced by Sacks and Uhlenbeck [14].

By the statement in [5, Theorem 2.2.2], such a commuting real structure in Theorem 1.2 always exists on M if $m = 0, 6, 7 \pmod 8$. In particular, when $m = 0, 6 \pmod 8$, we can get the existence of uncoupled Dirac-harmonic maps.

Corollary 1.3. *Let* m *be the dimension of* M*. Suppose one of the following holds*:

(a) $m = 0, 6 \pmod 8$*,* N *is a hyperKähler manifold, and a homotopy class* $[u]$ *satisfies* $\mathcal{I}(M, u^*T_{1,0}N) \neq 0$.

(b) $m = 2, 4 \pmod 8$*,* N *is a Kähler manifold with a parallel real structure* j_2 *defined in Theorem* 1.2*, and a homotopy class* $[u]$ *satisfies* $\mathcal{I}(M, u^*T_{1,0}N) \neq 0$.

Then there is a real vector space of real dimension greater than or equal to 4 *such that all* $(\tilde{u}, \psi)$*'s are uncoupled* α*-Dirac-harmonic maps as long as there is an* α*-harmonic map* $\tilde{u} \in [u]$ *for* $\alpha \geqslant 1$.

Remark 1.4. Note that the case $m = 6 \pmod 8$ is not included in [1] due to the definition of the index $\alpha(M, u)$. When $m = 0 \pmod 4$, the triviality of the index $\alpha(M, u)$ implies that of the index $\mathrm{ind}(\not{D}^+)$, where $\not{D}^+$ comes from the decomposition of the Dirac operator according to that of the spinor bundle (see Section 2). When $m = 2k$ and k is odd, the triviality of $\mathrm{ind}(\not{D}^+)$ does not imply that of $\mathrm{ind}(\not{D}^+_{1,0})$. Sun [16] used the nontrivial index $\mathrm{ind}(\not{D}^+_{1,0})$ to get an existence result. Our corollary is still valid even if $\mathrm{ind}(\not{D}^+_{1,0}) = 0$. For example, our result applies to the case where the dimensions of the kernels in those four subspaces in the decomposition (2.8) are all equal to one, which is never considered in literature.

When M is a closed Riemann surface, we can prove the short-time existence of the α-Dirac-harmonic map flow into a Kähler manifold, which generalizes the result in [10].

The rest of this paper is organized as follows. In Section 2, we recall some facts about Dirac-harmonic maps as well as the Dirac operator. In Section 3, we prove Theorem 1.2 and end this section by showing the density of the minimal kernel. In Section 4, under the minimality assumption on the kernel of $\not{D}^{u_0}_{1,0}$, we prove the short-time existence of the α-Dirac-harmonic map flow (see Theorem 4.2) and the existence of Dirac-harmonic maps (see Theorem 4.7). In Appendix A, we solve the constraint equation and prove Lipschitz continuity of the solution with respect to the map.

2 Preliminaries

Let (M, g) be a compact Riemann surface with a fixed spin structure χ. On the complex spinor bundle ΣM, we denote the Hermitian inner product by $\langle \cdot, \cdot \rangle_{\Sigma M}$. For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\Sigma M)$, the Clifford multiplication satisfies the following skew-adjointness:

$$\langle X \cdot \xi, \eta\rangle_{\Sigma M} = -\langle \xi, X \cdot \eta\rangle_{\Sigma M}.$$

Let ∇ be the Levi-Civita connection on (M, g). There is a unique connection (also denoted by ∇) on ΣM compatible with $\langle \cdot, \cdot \rangle_{\Sigma M}$. Choosing a local orthonormal basis $\{e_\beta\}_{\beta=1,2}$ on M, we see that the usual

Dirac operator is defined as $\partial\!\!\!/ := e_\beta \cdot \nabla_\beta$, where $\beta = 1, 2$. Here and in the sequel, we use the Einstein summation convention. One can find more about spin geometry in [11].

Let u be a smooth map from M to another compact Riemannian manifold (N, h) of dimension $n \geqslant 2$. Let u^*TN be the pull-back bundle of TN by u and consider the twisted bundle $\Sigma M \otimes_{\mathbb{R}} u^*TN$. On this bundle, there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes u^*TN}$ induced from the metric on ΣM and u^*TN. Also, we have a connection $\tilde{\nabla}$ on this twisted bundle naturally induced from those on ΣM and u^*TN. In local coordinates $\{y^i\}_{i=1,\dots,n}$, the section ψ of $\Sigma M \otimes_{\mathbb{R}} u^*TN$ is written as $\psi = \psi_i \otimes \partial_{y^i}(u)$, where each ψ^i is a usual spinor on M. We also have the following local expression of $\tilde{\nabla}$:

$$\tilde{\nabla}\psi = (\nabla\psi^i + \Gamma^i_{jk}(u)\nabla u^j \psi^k) \otimes \partial_{y^i}(u),$$

where Γ^i_{jk}'s are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map u is defined as

$$D\!\!\!\!/ := e_\alpha \cdot \tilde{\nabla}_{e_\alpha}\psi = (\partial\!\!\!/\psi^i + \Gamma^i_{jk}(u)\nabla_{e_\alpha}u^j(e_\alpha\cdot\psi^k)) \otimes \partial_{y^i}(u), \tag{2.1}$$

which is self-adjoint (see [7]). Sometimes, we use $D\!\!\!\!/^u$ to distinguish the Dirac operators defined on different maps. In [2], Chen et al. introduced the functional

$$\begin{aligned} L(u,\psi) &:= \frac{1}{2}\int_M (|du|^2 + \langle\psi, D\!\!\!\!/\psi\rangle_{\Sigma M\otimes u^*TN}) \\ &= \frac{1}{2}\int_M \left(h_{ij}(u)g^{\alpha\beta}\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta} + h_{ij}(u)\langle\psi^i, D\!\!\!\!/\psi^j\rangle_{\Sigma M}\right). \end{aligned}$$

They computed the Euler-Lagrange equations of L:

$$\begin{cases} \tau^m(u) - \frac{1}{2}R^m_{lij}\langle\psi^i, \nabla u^l\cdot\psi^j\rangle_{\Sigma M} = 0, & (2.2)\\ D\!\!\!\!/\psi^i := \partial\!\!\!/\psi^i + \Gamma^i_{jk}(u)\nabla_{e_\alpha}u^j(e_\alpha\cdot\psi^k) = 0, & (2.3)\end{cases}$$

where $\tau^m(u)$ is the m-th component of the tension field (see [7]) of the map u with respect to the coordinates on N, $\nabla u^l\cdot\psi^j$ denotes the Clifford multiplication of the vector field ∇u^l with the spinor ψ^j, and R^m_{lij} stands for the components of the Riemann curvature tensor of the target manifold N. Define

$$\mathcal{R}(u,\psi) := \frac{1}{2}R^m_{lij}\langle\psi^i, \nabla u^l\cdot\psi^j\rangle_{\Sigma M}\partial_{y^m}.$$

We can write (2.2) and (2.3) in the following global form:

$$\begin{cases} \tau(u) = \mathcal{R}(u,\psi), & (2.4)\\ D\!\!\!\!/\psi = 0, & (2.5)\end{cases}$$

and call the solutions (u,ψ) Dirac-harmonic maps from M to N.

With the aim to get a general existence scheme for Dirac-harmonic maps, the following heat flow for Dirac-harmonic maps was introduced in [4]:

$$\begin{cases} \partial_t u = \tau(u) - \mathcal{R}(u,\psi) & \text{on } (0,T)\times M, \quad (2.6)\\ D\!\!\!\!/\psi = 0 & \text{on } [0,T]\times M. \quad (2.7)\end{cases}$$

When M has a boundary, the short time existence and uniqueness of (2.6)–(2.7) was shown in [4].

For a closed manifold M, the situation is more complicated because one cannot uniquely solve the second equation (2.7) and the kernel of the Dirac operator may jump along the flow. As we stated in Section 1, the short-time existence is only known in the minimal kernel case, i.e., $\dim_{\mathbb{H}}\ker D\!\!\!\!/ = 1$. However, when the target manifold N is an even-dimensional spin manifold, the index $\alpha(M,u)$ always vanishes for any map u between M and N. In order to deal with this case, we utilize the complex structure on N. We denote the complexification of u^*TN by $(u^*TN)^{\mathbb{C}}$. Then we have

$$\Sigma M \otimes_{\mathbb{R}} u^*TN = (\Sigma M \otimes_{\mathbb{C}} \mathbb{C}) \otimes_{\mathbb{R}} u^*TN = \Sigma M \otimes_{\mathbb{C}} (u^*TN)^{\mathbb{C}}.$$

The pull-back metric u^*g on u^*TN could be naturally extended to a Hermitian product on $(u^*TN)^{\mathbb{C}}$. Moreover, there is a natural Hermitian product on $\Sigma M \otimes (u^*TN)^{\mathbb{C}}$ induced from those on ΣM and $(u^*TN)^{\mathbb{C}}$, which is denoted by $\langle\cdot,\cdot\rangle_{\Sigma M\otimes(u^*TN)^{\mathbb{C}}}$.

For a general even-dimensional spin Riemannian manifold M, there is a parallel $\mathbb{Z}_2$-grading $G \in \mathrm{End}(\Sigma M)$ given by $G(\psi) = (\sqrt{-1})^{m/2} e_1 \cdot e_2 \cdots e_m \cdot \psi$ for a positively oriented orthonormal local frame $\{e_1, e_2, \dots, e_m\}$, where $m = \dim M$. Thus the spinor bundle can be decomposed as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where $\Sigma^\pm M$'s are the eigenspaces of G associated with the ± 1, respectively. As G is Hermitian and parallel, the decomposition is orthogonal in the complex sense and parallel. Consequently, we have

$$\begin{aligned}\Sigma M \otimes_{\mathbb{C}} (u^*TN)^{\mathbb{C}} = (\Sigma^+ M \otimes_{\mathbb{C}} u^*T_{1,0}N) \oplus (\Sigma^- M \otimes_{\mathbb{C}} u^*T_{1,0}N)\\ \oplus (\Sigma^+ M \otimes_{\mathbb{C}} u^*T_{0,1}N) \oplus (\Sigma^- M \otimes_{\mathbb{C}} u^*T_{0,1}N),\end{aligned} \tag{2.8}$$

where we used $(u^*TN)^{\mathbb{C}} = u^*T_{1,0}N \oplus u^*T_{1,0}N$. Moreover, we also have the following decomposition for the Dirac operator:

$$\begin{aligned}\not{D} &= \not{D}_{1,0} + \not{D}_{0,1},\\ \not{D}_{1,0} &= \not{D}^+_{1,0} + \not{D}^-_{1,0},\\ \not{D}_{0,1} &= \not{D}^+_{0,1} + \not{D}^-_{0,1},\end{aligned}$$

where $\not{D}^\pm_{1,0}$ (resp. $\not{D}^\pm_{0,1}$) is obtained by restricting $\not{D}$ on $\Sigma^\pm M \otimes_{\mathbb{C}} u^*T_{1,0}N$ (resp. $\Sigma^\pm M \otimes_{\mathbb{C}} u^*T_{0,1}N$).

By [13], we can isometrically embed N into $\mathbb{R}^q$. Then (2.4)–(2.5) is equivalent to the following system:

$$\begin{cases}\Delta_g u = II(du, du) + \mathrm{Re}(P(\mathcal{S}(du(e_\beta), e_\beta \cdot \psi); \psi)),\\ \not{\partial}\psi = \mathcal{S}(du(e_\beta), e_\beta \cdot \psi),\end{cases}$$

where II is the second fundamental form of N in $\mathbb{R}^q$, and

$$\begin{aligned}&\mathcal{S}(du(e_\beta), e_\beta \cdot \psi) := (\nabla u^A \cdot \psi^B) \otimes II(\partial_{z^A}, \partial_{z^B}),\\ &\mathrm{Re}(P(\mathcal{S}(du(e_\beta), e_\beta \cdot \psi); \psi)) := P(S(\partial_{z^C}, \partial_{z^B}); \partial_{z^A})\mathrm{Re}(\langle \psi^A, du^C \cdot \psi^B\rangle).\end{aligned}$$

Here, $P(\xi;\cdot)$ denotes the shape operator, defined by $\langle P(\xi; X), Y\rangle = \langle A(X, Y), \xi\rangle$ for $X, Y \in \Gamma(TN)$ and $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$. Together with the *nearest point projection*:

$$\pi:\ N_\delta \to N,$$

where $N_\delta := \{z \in \mathbb{R}^q \mid d(z, N) \leqslant \delta\}$, we can rewrite the evolution equation (2.6) as an equation in $\mathbb{R}^q$.

Lemma 2.1 (See [4]). *A tuple (u, ψ), where $u : [0, T] \times M \to N$ and $\psi \in \Gamma(\Sigma M \otimes u^*TN)$, is a solution of* (2.6) *if and only if*

$$\partial_t u^A - \Delta u^A = -\pi^A_{BC}(u)\langle \nabla u^B, \nabla u^C\rangle - \pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(\psi^D, \nabla u^E \cdot \psi^F)$$

on $(0, T) \times M$ for $A = 1, \dots, q$. Here, we denote the A-th component function of $u : [0, T] \times M \to N \subset \mathbb{R}^q$ by $u^A : M \to \mathbb{R}$, write $\pi^A_B(z)$ for the B-th partial derivative of the A-th component function of $\pi : \mathbb{R}^q \to \mathbb{R}^q$, and the global sections $\psi^A \in \Gamma(\Sigma M)$ are defined by $\psi = \psi^A \otimes (\partial_A \circ u)$, where $(\partial_A)_{A=1,\dots,q}$ is the standard basis of $T\mathbb{R}^q$. Moreover, ∇ and $\langle\cdot,\cdot\rangle$ denote the gradient and the Riemannian metric on M, respectively.

For future reference, we define

$$F^A_1(u) := -\pi^A_{BC}(u)\langle \nabla u^B, \nabla u^C\rangle, \tag{2.9}$$

$$F^A_2(u, \psi) := -\pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(\psi^D, \nabla u^E \cdot \psi^F). \tag{2.10}$$

Note that for $u \in C^1(M, N)$ and $\psi \in \Gamma(\Sigma M \otimes u^*TN)$, we have

$$II(du_p(e_\alpha), du_p(e_\alpha)) = -F^A_1(u)|_p \partial_A|_{u(p)},$$

$$\mathcal{R}(u,\psi)|_p = -F_2^A(u,\psi)|_p \partial_A|_{u(p)}$$

for all $p \in M$, where $\{e_\alpha\}$ is an orthonormal basis of T_pM.

Next, let us fix some notations, which will be used in Section 4 and Appendix A. For every $T > 0$, we denote by X_T the Banach space of bounded maps:

$$X_T := B([0,T]; C^1(M, \mathbb{R}^q)),$$
$$\|u\|_{X_T} := \max_{A=1,\dots,q} \sup_{t\in[0,T]} (\|u^A(t,\cdot)\|_{C^0(M)} + \|\nabla u^A(t,\cdot)\|_{C^0(M)}).$$

For any map $v \in X_T$, the closed ball with center v and radius R in X_T is defined by

$$B_R^{\mathrm{T}}(v) := \{u \in X_T \mid \|u - v\| \leqslant R\}.$$

We denote by $P^{u_t,v_s} = P^{u_t,v_s}(x)$ the parallel transport of N along the unique shortest geodesic from $\pi(u(x,t))$ to $\pi(v(x,s))$. We also denote by P^{u_t,v_s} the inducing mappings

$$(\pi \circ u_t)^*TN \to (\pi \circ v_s)^*TN,$$
$$\Sigma M \otimes (\pi \circ u_t)^*TN \to \Sigma M \otimes (\pi \circ v_s)^*TN,$$

and

$$\Gamma_{C^1}(\Sigma M \otimes (\pi \circ u_t)^*TN) \to \Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^*TN).$$

We also define

$$\Lambda(u_t) = \sup\{\tilde{\Lambda} \mid \mathrm{spec}(\not{D}^{\pi\circ u_t}) \setminus \{0\} \subset \mathbb{R} \setminus (-\tilde{\Lambda}(u_t), \tilde{\Lambda}(u_t))\}$$

and $\gamma_t(x) : [0, 2\pi] \to \mathbb{C}$ as

$$\gamma_t(x) := \frac{\Lambda(u_t)}{2} \mathrm{e}^{\mathrm{i}x}.$$

In general, we also denote by γ the curve $\gamma(x) : [0, 2\pi] \to \mathbb{C}$ as

$$\gamma(x) := \frac{\Lambda}{2} \mathrm{e}^{\mathrm{i}x} \tag{2.11}$$

for some constant Λ to be determined. Then the orthogonal projection onto $\ker(\not{D}^{\pi\circ u_t})$, which is the mapping

$$\Gamma_{L^2}(\Sigma M \otimes (\pi \circ u_t)^*TN) \to \Gamma_{L^2}(\Sigma M \otimes (\pi \circ u_t)^*TN),$$

can be written via the resolvent by

$$s \mapsto -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_t} R(\lambda, \not{D}^{\pi\circ u_t}) s d\lambda,$$

where $R(\lambda, \not{D}^{\pi\circ u_t}) : \Gamma_{L^2} \to \Gamma_{L^2}$ is the resolvent of $\not{D}^{\pi\circ u_t} : \Gamma_{W^{1,2}} \to \Gamma_{L^2}$.

At the end of this section, we recall the definition of the index.

Definition 2.2. Let $E \to M$ be a Riemannian real vector bundle with a metric connection. Then one can associate the twisted Dirac operator $\not{D}^E : C^\infty(M, \Sigma M \otimes E) \to C^\infty(M, \Sigma M \otimes E)$ with the index $I(M, \chi, E) \in KO_m(\mathrm{pt})$, where

$$KO_m(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & \text{if } m = 0\ (4), \\ \mathbb{Z}_2, & \text{if } m = 1, 2\ (8), \\ 0, & \text{otherwise.} \end{cases}$$

The index $I(M, \chi, E)$ can be determined out of $\ker(\not{D}^E)$ using the following formula:

$$I(M,\chi,E) = \begin{cases} \{\mathrm{ch}(E) \cdot \hat{A}(M)\}[M], & \text{if } m = 0\ (8), \\ [\dim_{\mathbb{C}}(\ker(\not{D}^E))]_{\mathbb{Z}_2}, & \text{if } m = 1\ (8), \\ \left[\dfrac{\dim_{\mathbb{C}}(\ker(\not{D}^E))}{2}\right]_{\mathbb{Z}_2}, & \text{if } m = 2\ (8), \\ \dfrac{1}{2}\{\mathrm{ch}(E) \cdot \hat{A}(M)\}[M], & \text{if } m = 4\ (8). \end{cases}$$

In particular, when $E = u^*TN$ and χ is fixed, we denote $I(M, \chi, E)$ by $I(M, u)$.

3 The quaternionic structure on the twisted bundle

In this section, we prove Theorem 1.2 by constructing a commuting quaternionic structure on the twisted bundle $\Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N$ and show the density of the minimal kernel.

Proof of Theorem 1.2. Let $\rho : \mathbb{C}\mathrm{l}_m \to \mathrm{End}_{\mathbb{C}}(\Sigma_m)$ be an irreducible complex representation of the complex Clifford algebra $\mathbb{C}\mathrm{l}_m$. Suppose that the condition (2) holds. Then every fibre of the complex spinor bundle $\Sigma M = \mathrm{Spin}(M) \times_\rho \Sigma_m$ turns into a quaternionic vector space by defining $[p, v]h := [p, vh]$ for all $p \in \mathrm{Spin}(M)$, $v \in \Sigma_m$, and $h \in \mathbb{H}$.

Since the tensor product of the twisted bundle $\Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N$ is taken over $\mathbb{C}$, there is a natural complex structure I on $\Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N$ defined by

$$I(\psi^k \otimes_{\mathbb{C}} \theta_k) := i(\psi^k) \otimes_{\mathbb{C}} \theta_k = \psi^k \otimes_{\mathbb{C}} i(\theta_k).$$

However, the quaternionic structure on ΣM cannot directly extend to the twisted bundle. To overcome this problem, we need an extra structure on $u^*T_{1,0}N$. By our assumption, we define $J : \Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N \to \Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N$ by

$$J(\psi^i \otimes_{\mathbb{C}} \theta_i) := j_1(\psi^k) \otimes_{\mathbb{C}} j_2(\theta_k).$$

Since both j_1 and j_2 anti-commute with the complex structure i, J is well-defined on $\Sigma M \otimes_{\mathbb{C}} u^*T_{1,0}N$. By the definitions of j_1 and j_2, J anti-commutes with I and $J^2 = -1$. Moreover, J also commutes with the Clifford multiplication and hence the Dirac operator $\not{D}^u_{1,0}$ (see also [5]), i.e.,

$$\not{D}^u_{1,0} \circ J = J \circ \not{D}^u_{1,0}.$$

Therefore, we conclude that all the eigenspaces of $\not{D}^u_{1,0}$ are quaternionic vector spaces with two complex structures I and J, which are anti-commuting with each other.

If the condition (1) holds, i.e., j_1 is a commuting real structure and j_2 is a quaternionic structure, then it follows from the argument above that the conclusion is also true.

When $m \neq 3 \pmod 4$, the eigenvalues are symmetric with respect to the origin (see [5, Remark 2.2.3]). For any two maps in $[u]$, there is a piecewise smooth curve connecting them with the parameter $t \in [0, 1]$. Along this curve, the eigenvalues of the Dirac operator are continuous functions of t. Suppose that there is an eigenvalue $\lambda_1(t)$ that decreases to zero as $t \to T$. By the symmetry of the eigenvalues, there is another eigenvalue $\lambda_{-1}(t)$ such that $\lambda_{-1}(t) = -\lambda_1(t)$. Therefore, the difference in the quaternionic dimension of the kernel of the corresponding Dirac operator is always an even number.

When m is even, we have a parallel $\mathbb{Z}_2$-grading G described in the previous section. From the orthogonality of the splitting, we have

$$\langle \not{D}^u_{1,0}\psi^+, \psi^+ \rangle = \langle \not{D}^u_{1,0}\psi^-, \psi^- \rangle = 0$$

for all $\psi^\pm \in C^\infty(M, \Sigma^\pm M \otimes u^*T_{1,0}N)$. Thus,

$$(\not{D}^u_{1,0}\psi^+, \psi^+)_{L^2} = (\not{D}^u_{1,0}\psi^-, \psi^-)_{L^2} = 0. \tag{3.1}$$

Now, for any smooth variation $(u_s)_{s\in(-\epsilon,\epsilon)}$ of the α-harmonic map $\tilde{u} \in [u]$ with $u_s|_{s=0} = \tilde{u}$, we split the bundle $\Sigma M \otimes u_s^*T_{1,0}N$ into

$$\Sigma M \otimes u_s^*T_{1,0}N = (\Sigma^+ M \otimes u_s^*T_{1,0}N) \oplus (\Sigma^- M \otimes u_s^*T_{1,0}N),$$

which is orthogonal in the complex sense and parallel. Since $\mathcal{I}(M, u^*T_{1,0}N) \neq 0$, there exists a $\Psi \in \ker \not{D}^{\tilde{u}}_{1,0}$ which can be written as $\Psi = \Psi^+ + \Psi^-$, where $\Psi^\pm \in \Gamma(\Sigma^\pm M \otimes \tilde{u}^*T_{1,0}N)$. Then there always exists a variation Ψ_s of Ψ such that $\Psi_s^{\pm} \in \Gamma(\Sigma^{\pm}M \otimes u_s^* T_{1,0}N)$ are smooth variations of $\Psi^{\pm}$, respectively. Moreover, (3.1) implies that

$$\frac{d}{dt}\bigg|_{s=0} (\not{D}^{u_s}\Psi_s^{\pm}, \Psi_s^{\pm})_{L^2} = 0.$$

Therefore, for the α-harmonic map $\tilde{u}$, we have

$$\frac{d}{dt}\bigg|_{s=0} L^{\alpha}(u_s, \Psi_s^{\pm}) = \frac{d}{dt}\bigg|_{s=0} \int_M (1 + |du_s|^2)^{\alpha} = 0.$$

Hence, we get α-Dirac-harmonic maps $(\tilde{u}, \Psi^{\pm})$. □

In the rest of this section, we show the density of the minimal kernel. By the definition of $\mathcal{I}(M, u^*T_{1,0}N)$, we have

$$\dim_{\mathbb{H}}\ker(\not{D}^u_{1,0}) \geqslant \begin{cases} 0, & \text{if } \mathrm{ind}_{u^*T_{1,0}N}(M) = 0, \\ 1, & \text{if } \mathrm{ind}_{u^*T_{1,0}N}(M) \neq 0. \end{cases}$$

If the equality holds above, then we say that $\not{D}^u_{1,0}$ has a minimal kernel. Using the analyticity of N, one can prove the following density result for the minimal kernel.

Lemma 3.1. *If $\not{D}^u_{1,0}$ has the minimal kernel, then $\not{D}^{u'}_{1,0}$ also has the minimal kernel for a generic map $u' \in [u]$.*

Proof. Let $u' \in [u]$ and H be any homotopy between u' and u. More precisely, $H : [0,1] \to C^{\infty}(M,N)$ with $H(0) = u$ and $H(1) = u'$. We can cover the image of H by finitely many balls $\{V_l\}_{l=1}^L$ of radius less than $\frac{1}{2}\mathrm{inj}(N)$ such that

$$V_l \cap V_{l+1} \neq \emptyset \quad \text{for } i = 1, \dots, L-1$$

and

$$u \in V_1, \quad u' \in V_L.$$

We choose $u_1 \in V_1 \cap V_2$ arbitrarily and define a homotopy H_t^1 by

$$H_t^1(x) := \exp_{u(x)}(t \exp_{u(x)}^{-1} u_1(x)),$$

where $x \in M$ and exp is the exponential map on N. We denote by

$$P_t = P_t(x) : T_{1,0}N|_{u(x)} \to T_{1,0}N|_{H_t^1(x)}$$

the parallel transport along the unique shortest geodesic of N connecting $u(x)$ and $H_t^1(x)$ and consider

$$\not{D}_t := P_t^{-1} \circ \not{D}^{H_t^1}_{1,0} \circ P_t.$$

Since $\not{D}_t$ depends analytically on t by the analyticity of N, $\not{D}^{u_t}_{1,0}$ has the minimal kernel for all but finitely many $t \in [0,1]$. Therefore, we can assume that $\not{D}^{u_1}_{1,0}$ has the minimal kernel. Continuing this procedure, we can get $u_{L-1} \in V_{L-1} \cap V_L$ such that $\not{D}^{u_{L-1}}_{1,0}$ also has the minimal kernel and a homotopy H_t^{L-1} between u_{L-1} and u' such that $\not{D}^{H_t^{L-1}}_{1,0}$ has the minimal kernel for all but finitely many $t \in [0,1]$. Hence the set of maps along which the $(1,0)$-part of the Dirac operator has the minimal kernel is C^{∞}-dense in $[u]$. Its C^1-openness directly follows from the continuity of the eigenvalues. □

4 The heat flow for α-Dirac-harmonic maps

In this section, we prove the short-time existence of the heat flow for α-Dirac-harmonic maps. Since we are working on a closed surface M, we cannot uniquely solve the Dirac equation in the following system:

$$\begin{cases} \partial_t u = \dfrac{1}{(1 + |\nabla u|^2)^{\alpha - 1}} \left(\tau^{\alpha}(u) - \dfrac{1}{\alpha}\mathcal{R}(u, \psi) \right), & (4.1) \\ \not{D}^u \psi = 0. & (4.2) \end{cases}$$

The short time existence and its extension are the obstacles. This system (if it converges) leads to an α-Dirac-harmonic map which is a solution of the system

$$\begin{cases}\tau^{\alpha}(u) := \tau((1+|du|^2)^{\alpha}) = \dfrac{1}{\alpha}\mathcal{R}(u,\psi),\\ \not{D}^u\psi = 0,\end{cases}$$

where τ is the tension field.

4.1 Short time existence

As in Section 2, we now embed N into $\mathbb{R}^q$. Let $u : M \to N$ with $u = (u^A)$ and denote the spinor along the map u by $\psi = \psi^A \otimes (\partial_A \circ u)$, where ψ^A's are spinors over M. For any smooth map $\eta \in C_0^{\infty}(M, \mathbb{R}^q)$ and any smooth spinor field $\xi \in C_0^{\infty}(\Sigma M \otimes \mathbb{R}^q)$, we consider the variation

$$u_t = \pi(u + t\eta), \quad \psi_t^A = \pi_B^A(u_t)(\psi^B + t\xi^B), \tag{4.3}$$

where π is the nearest point projection as in Section 2. Then we have the following lemma.

Lemma 4.1 (See [10]). *The Euler-Lagrange equations for L^{α} are*

$$\begin{aligned}\Delta u^A = -2(\alpha-1)\frac{\nabla^2_{\beta\gamma}u^B\nabla_\beta u^B\nabla_\gamma u^A}{1+|\nabla u|^2} + \pi^A_{BC}(u)\langle\nabla u^B, \nabla u^C\rangle\\ + \frac{\pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(u)\langle\psi^D, \nabla u^E\cdot\psi^F\rangle}{\alpha(1+|\nabla u|^2)^{\alpha-1}}\end{aligned}$$

and

$$\partial\!\!\!/\psi^A = \pi^A_{BC}(u)\nabla u^B\cdot\psi^C.$$

Lemma 4.1 implies that (4.1)–(4.2) is equivalent to

$$\begin{cases}\partial_t u^A = \Delta u^A + 2(\alpha-1)\dfrac{\nabla^2_{\beta\gamma}u^B\nabla_\beta u^B\nabla_\gamma u^A}{1+|\nabla u|^2} - \pi^A_{BC}(u)\langle\nabla u^B, \nabla u^C\rangle\\ \qquad\qquad - \dfrac{\pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(u)\langle\psi^D, \nabla u^E\cdot\psi^F\rangle}{\alpha(1+|\nabla u|^2)^{\alpha-1}}, & (4.4)\\ \not{D}^{\pi\circ u}\psi = 0. & (4.5)\end{cases}$$

Now, let us state the main result of this subsection.

Theorem 4.2. *Let M be a closed surface, and N be a closed n-dimensional Riemannian manifold. Let $u_0 \in C^{2+\mu}(M, N)$ for some $0 < \mu < 1$ with $\dim_{\mathbb{H}}\ker(\not{D}^{u_0}_{1,0}) = 1$ and $\psi_0 \in \ker(\not{D}^{u_0}_{1,0})$ with $\|\psi_0\|_{L^2} = 1$. Then there exists an $\epsilon_1 = \epsilon_1(M,N) > 0$ such that for any $\alpha \in (1, 1+\epsilon_1)$, the problem (4.1)–(4.2) has a solution (u,ψ) with*

$$\begin{cases}\|\psi_t\|_{L^2} = 1, \quad \forall\, t \in [0,T],\\ u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0\end{cases} \tag{4.6}$$

satisfying

$$u \in C^{2+\mu, 1+\mu/2}(M\times[0,T], N)$$

and

$$\psi \in C^{\mu,\mu/2}(M\times[0,T], \Sigma M\otimes u^*TN) \cap L^{\infty}([0,T]; C^{1+\mu}(M))$$

for some $T > 0$.

Proof. We prove the theorem in two steps. In Step 1, we find a solution $u : M\times[0,T] \to \mathbb{R}^q$ and $\psi_t : M \to \Sigma M \otimes (\pi\circ u_t)^*TN$ of (4.4)–(4.5) with the initial values (4.6). Since ψ_t takes the value along the projection $\pi\circ u_t$, it remains to show that u takes the value in N, which will be proved in Step 2.

Step 1. Solve (4.4)–(4.5) in $\mathbb{R}^q$.

We first give a solution to (4.5) in a neighborhood of u_0. For any $T > 0$, we can choose ϵ, δ, and R as in Appendix A such that $u(x,t) \in N_\delta$ and

$$d^N((\pi \circ u)(x,t), (\pi \circ v)(x,s)) < \epsilon < \frac{1}{2}\mathrm{inj}(N)$$

for all $u, v \in B_R^{\mathrm{T}} := B_R^{\mathrm{T}}(\bar{u}_0) = \{u \in X_T \mid \|u - \bar{u}_0\|_{X_T} \leqslant R\} \cap \{u|_{t=0} = u_0\}$, $x \in M$, and $t, s \in [0,T]$, where $\bar{u}_0(x,t) = u_0(x)$ for any $t \in [0,T]$. If R is small enough, then by Lemma A.5, we have

$$\dim_{\mathbb{H}} \ker(\not{D}_{1,0}^{\pi \circ u_t}) = 1$$

and there exists a $\Lambda = \frac{1}{2}\Lambda(u_0)$ such that

$$\#\{\mathrm{spec}(\not{D}_{1,0}^{\pi \circ u_t}) \cap [-\Lambda, \Lambda]\} = 1$$

for any $u \in B_R^{\mathrm{T}}$ and $t \in [0,T]$, where $\Lambda(u_0)$ is a constant such that $\mathrm{spec}(\not{D}_{1,0}^{u_0}) \setminus \{0\} \subset \mathbb{R} \setminus [-\Lambda(u_0), \Lambda(u_0)]$. Furthermore, for $\psi_0 \in \ker(\not{D}_{1,0}^{u_0})$ with $\|\psi_0\|_{L^2} = 1$, Lemma A.7 implies that

$$\sqrt{\frac{3}{4}} \leqslant \|\tilde{\psi}_1^{u_t}\|_{L^2} \leqslant 1$$

for any $u \in B_{R_1}^{\mathrm{T}}$ and $t \in [0,T]$, where $\tilde{\psi}^{u_t} = P^{u_0,u_t}\psi = \tilde{\psi}_1^{u_t} + \tilde{\psi}_2^{u_t}$ with respect to the decomposition $\Gamma_{L^2} = \ker(\not{D}_{1,0}^{\pi \circ u_t}) \oplus (\ker(\not{D}_{1,0}^{\pi \circ u_t}))^\perp$ and $R_1 = R_1(R, \epsilon, u_0) > 0$.

Now, for any $T > 0$ and $\kappa > 0$, we define

$$V_\kappa^{\mathrm{T}} := \{v \in C^{1+\mu, \frac{1+\mu}{2}}(M \times [0,T]) \mid \|v\|_{C^{1+\mu, \frac{1+\mu}{2}}} \leqslant \kappa,\ v|_{M \times \{0\}} = 0\}.$$

Then, there exists a $\kappa_{R_1} := \kappa(R_1) > 0$ such that

$$u_0 + v \in B_{R_1}^{\mathrm{T}}, \quad \forall\, v \in V_\kappa^{\mathrm{T}}, \quad \forall\, \kappa \leqslant \kappa_{R_1}.$$

Now, we define $\kappa_0 := \kappa_{R_1}$ and $V^{\mathrm{T}} := V_{\kappa_0}^{\mathrm{T}}$.

For every $v \in V^{\mathrm{T}}$ and $u_0 + v \in B_{R_1}^{\mathrm{T}}$, Lemma A.8 gives us a solution $\psi(v + u_0)$ to the constraint equation. Since $v + u_0 \in C^{1+\mu}(M)$, by L^p regularity and Schauder estimates in [4], we have

$$\|\psi(v + u_0)\|_{C^{1+\mu}(M)} \leqslant C(\mu, M, N, \kappa_0, \|u_0\|_{C^{1+\mu}(M)}). \tag{4.7}$$

For any $0 < t, s < T$, we also have

$$\begin{aligned}
&\not{\partial}(\psi(v+u_0)(t) - \psi(v+u_0)(s)) \\
&\quad = -\Gamma(\pi \circ (v+u_0)(t)) \# \nabla(\pi \circ (v+u_0)(t)) \# \psi(v+u_0)(t) \\
&\qquad + \Gamma(\pi \circ (v+u_0)(s)) \# \nabla(\pi \circ (v+u_0)(s)) \# \psi(v+u_0)(s) \\
&\quad = -\Gamma(\pi \circ (v+u_0)(t)) \# \nabla(\pi \circ (v+u_0)(t)) \# (\psi^v(t) - \psi(v+u_0)(s)) \\
&\qquad - \Gamma(\pi \circ (v+u_0)(t)) \# (\nabla(\pi \circ (v+u_0)(t)) - \nabla(\pi \circ (v+u_0)(s))) \# \psi(v+u_0)(t) \\
&\qquad - (\Gamma(\pi \circ (v+u_0)(t)) - \Gamma(\pi \circ (v+u_0)(s))) \# \nabla(\pi \circ (v+u_0)(s)) \# \psi(v+u_0)(s),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\not{D}^{\pi \circ v(t)}(\psi(v+u_0)(t) - \psi(v+u_0)(s)) \\
&\quad = -\Gamma(\pi \circ (v+u_0)(t)) \# (\nabla(\pi \circ (v+u_0)(t)) - \nabla(\pi \circ (v+u_0)(s))) \# \psi(v+u_0)(t) \\
&\qquad - (\Gamma(\pi \circ (v+u_0)(t)) - \Gamma(\pi \circ (v+u_0)(s))) \# \nabla(\pi \circ (v+u_0)(s)) \# \psi(v+u_0)(s),
\end{aligned}$$

where # denotes a multi-linear map with smooth coefficients. For any $\lambda \in (0,1)$, by the Sobolev embedding, L^p regularity, and Lemma A.8, we have

$$\begin{aligned}
&\|\psi(v+u_0)(t) - \psi(v+u_0)(s)\|_{C^\lambda(M)} \\
&\quad \leqslant C(\lambda, M, N, \kappa_0, \|u_0\|_{C^1(M)})(\|v(t) - v(s)\|_{L^\infty(M)} + \|dv(t) - dv(s)\|_{L^\infty}) \\
&\quad \leqslant C(\lambda, M, N, \kappa_0, \|u_0\|_{C^1(M)})|t-s|^{\mu/2}.
\end{aligned}$$

Therefore,

$$\|\psi(v+u_0)\|_{C^{\mu,\mu/2}(M)} \leqslant C(\mu, M, N, \kappa_0, \|u_0\|_{C^1(M)}). \tag{4.8}$$

Now, when $\alpha - 1$ is sufficiently small, for the (v, ψ^v) above, the standard theory of linear parabolic systems (see [15]) implies that there exists a unique solution $v_1 \in C^{2+\mu,1+\mu/2}(M \times [0,T], \mathbb{R}^q)$ to the following Dirichlet problem

$$\begin{aligned}\partial_t w^A = {} & \Delta_g w^A + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma} w^B \nabla_\beta (v+u_0)^B \nabla_\gamma (v+u_0)^A}{1+|\nabla(v+u_0)|^2} \\ & + \pi^A_{BC}(v+u_0)\langle \nabla(v+u_0)^B, \nabla(v+u_0)^C\rangle \\ & + \frac{(\pi^A_B \pi^C_{BD} \pi^C_{EF})(v+u_0)\langle \psi^D(v+u_0), \nabla(v+u_0)^E \cdot \psi^F(v+u_0)\rangle}{\alpha(1+|\nabla(v+u_0)|^2)^{\alpha-1}} \\ & + \Delta_g u_0^A + 2(\alpha-1)\frac{\nabla^2_{\beta\gamma} u_0^B \nabla_\beta (v+u_0)^B \nabla_\gamma (v+u_0)^A}{1+|\nabla(v+u_0)|^2}, \end{aligned} \tag{4.9}$$

$$w(\cdot,0) = 0, \tag{4.10}$$

satisfying

$$\|v_1\|_{C^{2+\mu,1+\mu/2}(M\times[0,T])} \leqslant C(\mu, M, N)(\|v_1\|_{C^0(M\times[0,T])} + \|u_0\|_{C^{2+\nu}(M)} + \kappa_0).$$

Since $v_1(\cdot,0) = 0$, we have

$$\|v_1\|_{C^0(M\times[0,T])} \leqslant C(\mu, M, N)T(\|v_1\|_{C^0(M\times[0,T])} + \|u_0\|_{C^{2+\nu}(M)} + \kappa_0).$$

By taking $T > 0$ small enough, we get

$$\|v_1\|_{C^0(M\times[0,T])} \leqslant C(\mu, M, N)T(\|u_0\|_{C^{2+\nu}(M)} + \kappa_0).$$

Then the interpolation inequality in [12] implies that $v_1 \in V^{\mathrm{T}}$ for $T > 0$ sufficiently small. For such a v_1, we have $\psi(v_1+u_0)$ satisfying (4.7) and (4.8). Replacing $(v, \psi(v+u_0))$ in (4.9)–(4.10) by $(v_1, \psi(v_1+u_0))$, then we get $v_2 \in V^{\mathrm{T}}$. Iterating this procedure, we get a solution v_{k+1} of (4.9)–(4.10) with $(v, \psi(v+u_0))$ replaced by $(v_k, \psi(v_k+u_0))$, which satisfies

$$\|\psi(v_{k+1}+u_0)\|_{C^{\mu,\mu/2}(M)} \leqslant C(\mu, M, N, \kappa_0, \|u_0\|_{C^1(M)})$$

and

$$\|v_{k+1}\|_{C^{2+\mu,1+\mu/2}(M\times[0,T])} \leqslant C(\mu, M, N)(\|u_0\|_{C^{2+\nu}(M)} + \kappa_0).$$

By passing to a subsequence, we know that v_k converges to some u in $C^{2,1}(M\times[0,T])$ and $\psi^{v_k+u_0}$ converges to some ψ in $C^0(M\times[0,T])$. Then it is easy to see that (u,ψ) is a solution of (4.4)–(4.5) with $u(\cdot,0) = u_0$ and $\psi(\cdot,0) = \psi_0$.

Step 2. $u(x,t)$ takes the value in N for any $(x,t) \in M\times[0,T]$.

Suppose $u \in C^{2,1}(M\times[0,T], \mathbb{R}^q)$ and

$$\psi \in C^{\mu,\mu/2}(M\times[0,T], \Sigma M \otimes (\pi\circ u)^*TN) \cap L^\infty([0,T]; C^{1+\mu}(M))$$

satisfy (4.4)–(4.5). In the following, we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the Euclidean norm and the scalar product, respectively. Similarly, we write $\|\cdot\|_g$ and $\langle\cdot,\cdot\rangle_g$ for the norm and inner product of (M,g), respectively. We define $\rho: \mathbb{R}^q \to \mathbb{R}^q$ by $\rho(z) = z - \pi(z)$ and $\varphi: M\times[0,T] \to \mathbb{R}$ by

$$\varphi(x,t) = \|\rho(u(x,t))\|^2 = \sum_{A=1}^q |\rho^A(u(x,t))|^2.$$

A direct computation yields

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\varphi(x,t) &= -2\sum_{A=1}^{q}\|\nabla(\rho^A\circ u)(x,t)\|_g^2\\ &\quad+2\langle\rho\circ u,-\pi_B^A(u)F_1^B(u)\rangle\\ &\quad+\frac{2}{\alpha(1+|\nabla u|^2)^{\alpha-1}}\langle\rho\circ u,\rho_B^A(u)F_2^B(u,\psi)\rangle\\ &\quad+\frac{4(\alpha-1)}{1+|\nabla u|^2}\langle\rho\circ u,\nabla^2_{\beta\gamma}u^C\nabla_\beta u^C\nabla_\gamma u^B\rho_B^A(u)\rangle,\end{aligned}$$

where F_1^A and F_2^A are defined in (2.9) and (2.10), respectively.

Since $\rho\circ u\in T^{\perp}_{\pi\circ u}N$ and $(d\pi)_u:\mathbb{R}^q\to T_{\pi\circ u}N$, we have

$$\langle\rho\circ u,-\pi_B^A(u)F_1^B\rangle=\langle\rho\circ u,\rho_B^A(u)F_2^B\rangle=0.$$

Together with

$$\begin{aligned}&\frac{4(\alpha-1)}{1+|\nabla u|^2}\langle\rho\circ u,\nabla^2_{\beta\gamma}u^C\nabla_\beta u^C\nabla_\gamma u^B\rho_B^A(u)\rangle\\ &\quad\leqslant 4(\alpha-1)\|u\|_{C^2(M)}\|\rho\circ u\|\|\nabla(\rho\circ u)\|\\ &\quad\leqslant 2(\alpha-1)(\|u\|^2_{C^2(M)}\varphi+\|\nabla(\rho\circ u)\|^2),\end{aligned}$$

we get $(\frac{\partial}{\partial t}-\Delta)\varphi(x,t)\leqslant C\varphi$, where $C=C(\|u\|_{C^{2,1}(M\times[0,T])})$. Since $\varphi(x,t)\geqslant 0$ and $\varphi(x,0)=0$ for any $(x,t)\in M\times[0,T]$, we conclude $\varphi=0$ on $M\times[0,T]$. We have shown that $u(x,t)\in N$ for all $(x,t)\in M\times[0,T]$.

Finally, by using the ϵ-regularity (see Lemma 4.5 below), we conclude that

$$u\in C^{2+\mu,1+\mu/2}(M\times[0,T],N)$$

and

$$\psi\in C^{\mu,\mu/2}(M\times[0,T],\Sigma M\otimes u^*TN)\cap L^\infty([0,T];C^{1+\mu}(M)).$$

This completes the proof. □

4.2 Regularity of the flow

In this subsection, we give some estimates for the regularity of the flow. The proofs can be found in [10] and the references therein. Let us start with the following estimate of the energy of the map part.

Lemma 4.3. *Suppose that* (u,ψ) *is a solution of* (4.1)–(4.2) *with the initial values* (4.6)*. Then*

$$E_\alpha(u(t))+2\alpha\int_0^t\int_M(1+|\nabla u|^2)^{\alpha-1}|\partial_t u|^2=E_\alpha(u_0).$$

Moreover, $E_\alpha(u(t))$ *is absolutely continuous on* $[0,T]$ *and non-increasing.*

Consequently, we can also control the spinor part along the heat flow of the α-Dirac-harmonic map.

Lemma 4.4. *Suppose that* (u,ψ) *is a solution of* (4.1)–(4.2) *with the initial values* (4.6)*. Then for any* $p\in(1,2)$,

$$\|\psi(\cdot,t)\|_{W^{1,p}(M)}\leqslant C,\quad\forall\, t\in[0,T],$$

where $C=C(p,M,N,E_\alpha(u_0))$*.*

To get the convergence of the flow, we also need the following ϵ-regularity.

Lemma 4.5. *Suppose that* (u, ψ) *is a solution of* (4.1)–(4.2) *with the initial values* (4.6). *Given* $\omega_0 = (x_0, t_0) \in M \times (0, T]$, *define*

$$P_R(\omega_0) := B_R(x_0) \times [t_0 - R^2, t_0].$$

Then there exist three constants $\epsilon_2 = \epsilon_2(M, N) > 0$, $\epsilon_3 = \epsilon_3(M, N, u_0) > 0$, *and* $C = C(\mu, R, M, N, E_\alpha(u_0)) > 0$ *such that if*

$$1 < \alpha < 1 + \epsilon_2 \quad \text{and} \quad \sup_{[t_0 - 4R^2, t_0]} E(u(t); B_{2R}(\omega_0)) \leqslant \epsilon_3,$$

then

$$\sqrt{R}\|\psi\|_{L^\infty(P_R(\omega_0))} + R\|\nabla u\|_{L^\infty(P_R(\omega_0))} \leqslant C,$$

and for any $0 < \beta < 1$,

$$\sup_{[t_0 - \frac{R^2}{4}, t_0]} \|\psi(t)\|_{C^{1+\mu}(B_{R/2}(x_0))} + \|\nabla u\|_{C^{\beta,\beta/2}(P_{R/2}(\omega_0))} \leqslant C(\beta).$$

Moreover, if

$$\sup_M \sup_{[t_0 - 4R^2, t_0]} E(u(t); B_{2R}(\omega_0)) \leqslant \epsilon_3,$$

then

$$\|u\|_{C^{2+\mu,1+\mu/2}(M \times [t_0 - \frac{R^2}{8}, t_0])} + \|\psi\|_{C^{\mu,\mu/2}(M \times [t_0 - \frac{R^2}{8}, t_0])} + \sup_{[t_0 - \frac{R^2}{8}, t_0]} \|\psi(t)\|_{C^{1+\mu}(M)} \leqslant C.$$

4.3 Existence of Dirac-harmonic maps

In this subsection, we prove Theorem 4.7 by the short-time existence of the α-Dirac-harmonic map flow. First, we prove the following existence result about the α-Dirac-harmonic maps for $\alpha > 1$. Then, by the compactness, we get a Dirac-harmonic map as the limit of these α-Dirac-harmonic maps. Last, we prove that the bubbles can only be harmonic spheres, and finish the proof of Theorem 4.7.

Theorem 4.6. *Let* M *be a closed spin surface and* (N, h) *be a closed Kähler manifold. Suppose that there exists a map* $u_0 \in C^{2+\mu}(M, N)$ *for some* $\mu \in (0, 1)$ *such that* $\dim_{\mathbb{H}} \ker \not{D}^{u_0}_{1,0} = 1$. *Then for any* $\alpha \in (1, 1 + \epsilon_1)$, *there exists a nontrivial smooth* α*-Dirac-harmonic map* (u_α, ψ_α) *such that the map part* u_α *stays in the same homotopy class as* u_0 *and* $\|\psi_\alpha\|_{L^2} = 1$.

Proof. Let us define

$$m_0^\alpha := \inf\{E_\alpha(u) \mid u \in W^{1,2\alpha}(M, N) \cap [u_0]\},$$

where $[u_0]$ denotes the homotopy class of u_0. If u_0 is a minimizing α-harmonic map, it follows from Lemma 4.3 that (u_0, ψ_0) is an α-Dirac-harmonic map for any $\psi_0 \in \ker \not{D}^{u_0}_{1,0}$. If $E_\alpha(u_0) > m_0^\alpha$, then Theorem 4.2 gives us a solution

$$u \in C^{2+\mu,1+\mu/2}(M \times [0, T), N)$$

and

$$\psi \in C^{\mu,\mu/2}(M \times [0, T), \Sigma M \otimes u^*TN) \cap \bigcap_{0<s<T} L^\infty([0, s]; C^{1+\mu}(M))$$

to the problem (4.1)–(4.2) with the initial values (4.6).

By Lemma 4.3, we know

$$\int_M (1 + |\nabla u|^2)^\alpha \leqslant E_\alpha(u_0).$$

Then it is easy to see that for any $0 < \epsilon < \epsilon_3$, there exists a positive constant $r_0 = r_0(\epsilon, \alpha, E_\alpha(u_0))$ such that for all $(x, t) \in M \times [0, T)$,

$$\int_{B_{r_0}(x)} |\nabla u|^2 \leqslant C E_\alpha(u_0)^{1/\alpha} r_0^{1 - \frac{1}{\alpha}} \leqslant \epsilon.$$

Therefore, by Theorem 4.2 and Lemma 4.5, we know that the singular time can be characterized as

$$Z = \Big\{ T \in \mathbb{R} \Bigm| \lim_{t_i \nearrow T} \dim_{\mathbb{H}} \ker \not{D}_{1,0}^{u(t_i)} > 1 \Big\}$$

and there exists a sequence $\{t_i\} \nearrow T$ such that

$$(u(\cdot, t_i), \psi(\cdot, t_i)) \to (u(\cdot, T), \psi(\cdot, T)) \quad \text{in } C^{2+\mu}(M) \times C^{1+\mu/2}(M)$$

and

$$\|\psi(\cdot, T)\|_{L^2} = 1.$$

If $Z = \emptyset$, then by Theorem 4.2, we can extend the solution (u, ψ) beyond the time T by using $(u(\cdot, T), \psi(\cdot, T))$ as new initial values. Thus, we have the global existence of the flow. For the limit behavior as $t \to \infty$, Lemma 4.3 implies that there exists a sequence $\{t_i\} \to \infty$ such that

$$\int_M |\partial_t u|^2(\cdot, t_i) \to 0. \tag{4.11}$$

Together with Lemma 4.5, there are a subsequence, still denoted by $\{t_i\}$, and an α-Dirac-harmonic map $(u_\alpha, \psi_\alpha) \in C^\infty(M, N) \times C^\infty(M, \Sigma M \otimes (u_\alpha)^* TN)$ such that $(u(\cdot, t_i), \psi(\cdot, t_i))$ converges to (u_α, ψ_α) in $C^2(M) \times C^1(M)$ and $\|\psi_\alpha\|_{L^2} = 1$.

If $Z \neq \emptyset$ and $T \in Z$, let us assume that $E_\alpha(u(\cdot, T)) > m_0^\alpha$ and $(u(\cdot, T), \psi(\cdot, T))$ is not already an α-Dirac-harmonic map. We extend the flow as follows: by Lemma 3.1, there is a map $u_1 \in C^{2+\mu}(M, N)$ such that

$$m_0^\alpha < E_\alpha(u_1) < E_\alpha(u(\cdot, T)) \tag{4.12}$$

and

$$\dim_{\mathbb{H}} \ker \not{D}_{1,0}^{u_1} = 1. \tag{4.13}$$

Thus, picking any $\psi_1 \in \ker \not{D}^{u_1}$ with $\|\psi_1\|_{L^2} = 1$, we can restart the flow from the new initial values (u_1, ψ_1). If there is no singular time along the flow starting from (u_1, ψ_1), then we get an α-Dirac-harmonic map as in the case of $Z = \emptyset$. Otherwise, we use again the procedure above to choose (u_2, ψ_2) as initial values and restart the flow. This procedure will stop in finitely or infinitely many steps.

If infinitely many steps are required, then there exist infinitely many flow pieces $\{u_i(x, t)\}_{i=1,\dots,\infty}$ and $\{T_i\}_{i=1,\dots,\infty}$ such that

$$E_\alpha(u_i(t)) + 2\alpha \int_0^t \int_M (1 + |\nabla u_i|^2)^{\alpha-1} |\partial_t u_i|^2 = E_\alpha(u_i), \quad \forall\, t \in (0, T_i),$$

where $u_i(\cdot, 0) = u_i \in C^{2+\mu}(M, N)$. If T_i's are bounded away from zero, then there is $\{t_i\}$ such that (4.11) holds for $t_i \in (0, T_i)$. Therefore, we have an α-Dirac-harmonic map as before. If $T_i \to 0$, then we look at the limit of $E_\alpha(u_i)$. If the limit is strictly bigger than m_0^α, we again choose another map satisfying (4.12) and (4.13) as a new starting point. If the limit is exactly m_0^α, then we choose $\{t_i\}$ such that $t_i \in (0, T_i)$ for each i. By Lemma 4.5, $u_i(t_i)$ converges in $C^2(M) \times C^1(M)$ to a minimizing α-harmonic map u_α. If $\not{D}_{1,0}^{u_\alpha}$ has the minimal kernel, then for any $\psi \in \ker \not{D}_{1,0}^{u_\alpha}$, (u_α, ψ) is an α-Dirac-harmonic map as we showed at the beginning of the proof. If $\not{D}_{1,0}^{u_\alpha}$ has the non-minimal kernel, by using the $\mathbb{Z}_2$-grading $G \otimes \mathrm{id}$ as in the proof of Theorem 1.2, we get α-Dirac-harmonic maps $(u_\alpha, \psi_\alpha^\pm)$ for any $\ker \not{D}^{u_\alpha} \ni \psi_\alpha = \psi_\alpha^+ + \psi_\alpha^-$. In particular, we can choose ψ_α such that $\|\psi_\alpha^+\|_{L^2} = 1$ or $\|\psi_\alpha^-\|_{L^2} = 1$. By this procedure, we either get an α-Dirac-harmonic map or keep on choosing new maps satisfying (4.12) and (4.13). In the latter case, since the energies of the initial maps are bounded and decreasing, they converge to the minimizing energy m_0^α. (Otherwise, suppose that the constant is $C > m_0^\alpha$. Then one can choose a new map with a lower energy such that the limit is not C.) Therefore, we also get an α-Dirac-harmonic map in the latter case as before.

If it stops in finitely many steps, there exist a sequence $\{t_i\}$ and some $0 < T_k \leqslant +\infty$ such that

$$\lim_{t_i \nearrow T} (u(\cdot, t_i), \psi(\cdot, t_i)) \to (u_\alpha, \psi_\alpha) \quad \text{in } C^2(M) \times C^1(M),$$

where (u_α, ψ_α) either is an α-Dirac-harmonic map or satisfies $E_\alpha(u_\alpha) = m_0^\alpha$. In the latter case, u_α is a minimizing α-harmonic map. Then we can again get a nontrivial α-Dirac-harmonic map as above. □

By Theorem 4.6, for any $\alpha > 1$ sufficiently close to 1, there exists an α-Dirac-harmonic map (u_α, ψ_α) with the properties

$$E_\alpha(u_\alpha) \leqslant E_\alpha(u_0), \quad \|\psi_\alpha\|_{L^2} = 1, \tag{4.14}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \leqslant C(p, M, N, E_\alpha(u_0)) \tag{4.15}$$

for any $1 < p < 2$. Then it is natural to consider the limit behavior when α decreases to 1. Together with the blow-up analysis in [8], we have the following existence result.

Theorem 4.7. *Let M be a closed Riemann surface and N be a complex n-dimensional analytic Kähler manifold and a parallel real structure j_2 be defined in Theorem* 1.2. *Suppose that there exists a map $u_0 \in C^{2+\mu}(M, N)$ for some $\mu \in (0,1)$ such that* $\dim_{\mathbb{H}} \ker \not{D}^{u_0}_{1,0} = 1$. *Then there exists a nontrivial* (*i.e.*, $\Psi \neq 0$) *smooth Dirac-harmonic map* (Φ, Ψ) *with* $\|\Psi\|_{L^2} = 1$. *In particular, if N has nonpositive curvature, then the map Φ stays in the same homotopy class as u_0.*

Proof. By Theorem 4.6, we have a sequence of smooth α-Dirac-harmonic maps $(u_{\alpha_k}, \psi_{\alpha_k})$ with (4.14) and (4.15), where $\alpha_k \searrow 1$ as $k \to \infty$. Then, by the compactness theorem in [8], there are a constant $\epsilon_0 > 0$ and a Dirac-harmonic map

$$(\Phi, \Psi) \in C^\infty(M, N) \times C^\infty(M, \Sigma M \otimes \Phi^* TN)$$

such that

$$(u_{\alpha_k}, \psi_{\alpha_k}) \to (\Phi, \Psi) \quad \text{in } C^2_{\text{loc}}(M \setminus \mathcal{S}) \times C^1_{\text{loc}}(M \setminus \mathcal{S}),$$

where

$$\mathcal{S} := \left\{ x \in M \;\middle|\; \liminf_{\alpha_k \to 1} E(u_{\alpha_k}; B_r(x)) \geqslant \frac{\epsilon_0}{2}, \, \forall\, r > 0 \right\}$$

is a finite set.

Now, taking $x_0 \in \mathcal{S}$, we see that there exist a sequence $x_{\alpha_k} \to x_0$, $\lambda_{\alpha_k} \to 0$, and a nontrivial Dirac-harmonic map $(\phi, \xi) : \mathbb{R}^2 \to N$ such that

$$(u_{\alpha_k}(x_{\alpha_k} + \lambda_{\alpha_k} x), \lambda_{\alpha_k}^{\alpha_k - 1} \sqrt{\lambda_{\alpha_k}} \psi_{\alpha_k}(x_{\alpha_k} + \lambda_{\alpha_k} x)) \to (\phi, \xi) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2),$$

as $\alpha \to 1$. Choose any $p^* > 4$, and by taking $p = \frac{2p^*}{2+p^*}$ in (4.15), we get

$$\|\psi_{\alpha_k}\|_{L^{p^*}(M)} \leqslant C(p^*, M, N, E^{\alpha_k}(u_0))$$

and

$$\begin{aligned} \|\xi\|_{L^4(D_R(0))} &= \lim_{\alpha_k \to 1} \lambda_{\alpha_k}^{\alpha_k - 1} \|\psi_{\alpha_k}\|_{L^4(D_{\lambda_{\alpha_k} R}(x_{\alpha_k}))} \\ &\leqslant \lim_{\alpha_k \to 1} C \|\psi_{\alpha_k}\|_{L^{p^*}(M)} (\lambda_{\alpha_k} R)^{2(\frac{1}{4} - \frac{1}{p^*})} = 0. \end{aligned}$$

Thus, $\xi = 0$ and ϕ can be extended to a nontrivial smooth harmonic sphere. Since $\|\psi_\alpha\|_{L^2} = 1$, the Sobolev embedding implies that

$$\|\Psi\|_{L^2(M)} = \lim_{\alpha_k \to 1} \|\psi_\alpha\|_{L^2(M)} = 1.$$

Therefore, (Φ, Ψ) is nontrivial. Furthermore, if (N, h) does not admit any nontrivial harmonic sphere, then

$$(u_{\alpha_k}, \psi_{\alpha_k}) \to (\Phi, \Psi) \quad \text{in } C^2(M) \times C^1(M).$$

Therefore, Φ is in the same homotopy class as u_0. □

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Appendix A

We use the parallel construction in [10] to construct the solution to the constraint equation for spinors under a different pull-back bundle $u^*T_{1,0}N$. Since the only thing changed is the bundle we twisted, the proofs of those nice properties are parallel to those in [10]. For completeness, we give the details in this appendix.

For every $T > 0$, we consider the space $B_R^{\mathrm{T}}(\bar{u}_0) := \{u \in X_T \mid \|u - \bar{u}_0\|_{X_T} \leqslant R\} \cap \{u|_{t=0} = u_0\}$, where $\bar{u}_0(x,t) = u_0(x)$ for any $t \in [0,T]$. To get the necessary estimate for the solution of the constraint equation, we use the parallel transport along the unique shortest geodesic between $u_0(x)$ and $\pi \circ u_t(x)$ in N. To do this, we need the following lemma which tells us that the distances in N can be locally controlled by the distances in $\mathbb{R}^q$.

Lemma A.1. *Let $N \subset \mathbb{R}^q$ be a closed embedded submanifold of $\mathbb{R}^q$ with the induced Riemannian metric. Denote by A its Weingarten map. Choose $C > 0$ such that $\|A\| \leqslant C$, where*

$$\|A\| := \sup\{\|A_v X\| \mid v \in T_p^{\perp}N,\, X \in T_pN,\, \|v\| = 1,\, \|X\| = 1,\, p \in N\}.$$

Then there exists $0 < \delta_0 < \frac{1}{C}$ such that for all $0 < \delta \leqslant \delta_0$ and for all $p, q \in N$ with $\|p - q\|_2 < \delta$,

$$d^N(p,q) \leqslant \frac{1}{1 - \delta C}\|p - q\|_2,$$

where we denote the Euclidean norm by $\|\cdot\|_2$ in this section.

In the following, we choose δ and R to ensure the existence of the unique shortest geodesics between the projections of any two elements in $B_R^{\mathrm{T}}(\bar{u}_0)$. By the definition of $B_R^{\mathrm{T}}(\bar{u}_0)$, we have

$$\|u(x,t) - \bar{u}_0(x,t)\|_2 = \|u(x,t) - u_0(x)\|_2 \leqslant R$$

for all $(x,t) \in M \times [0,T]$. Then taking any $R \leqslant \delta$, we get

$$d(u(x,t), N) \leqslant \|u(x,t) - u_0(x)\|_2 \leqslant \delta$$

for all $(x,t) \in M \times [0,T]$. Therefore, $u(x,t) \in N_\delta$. In particular, $\pi \circ u$ is N-valued, and

$$\|(\pi \circ u)(x,t) - u_0(x)\|_2 \leqslant \|(\pi \circ u)(x,t) - u(x,t)\|_2 + \|u(x,t) - u_0(x)\|_2 \leqslant 2\delta. \tag{A.1}$$

Now, we choose $\epsilon > 0$ with $2\epsilon < \operatorname{inj}(N)$ and δ such that

$$\delta < \min\left\{\frac{1}{4}\delta_0, \frac{1}{4}\epsilon(1-\delta_0 C)\right\}, \tag{A.2}$$

where $\delta_0, C > 0$ are as in Lemma A.1. From (A.1), we know that for all $u, v \in B_R^{\mathrm{T}}(\bar{u}_0)$,

$$\|(\pi \circ u)(x,t) - (\pi \circ v)(x,s)\|_2 \leqslant 4\delta < \delta_0.$$

Then Lemma A.1 and (A.2) imply that

$$\begin{aligned} d^N((\pi \circ u)(x,t), (\pi \circ v)(x,s)) &\leqslant \frac{1}{1-\delta_0 C}\|(\pi \circ u)(x,t) - (\pi \circ v)(x,s)\|_2 \\ &\leqslant \frac{1}{1-\delta_0 C} 4\delta < \epsilon < \frac{1}{2}\operatorname{inj}(N). \end{aligned} \tag{A.3}$$

To summarize, under the choice of constants as follows:

$$\begin{cases} \epsilon > 0 & \text{s.t. } 2\epsilon < \operatorname{inj}(N), \\ \delta > 0 & \text{s.t. } \delta < \min\left\{\frac{1}{4}\delta_0, \frac{1}{4}\epsilon(1-\delta_0 C)\right\}, \\ R \leqslant \delta, \end{cases} \tag{A.4}$$

we have shown that

$$u(x,t) \in N_\delta \tag{A.5}$$

and

$$d^N((\pi \circ u)(x,t), (\pi \circ v)(x,s)) < \epsilon < \frac{1}{2}\operatorname{inj}(N) \tag{A.6}$$

for all $u, v \in B_R^{\mathrm{T}}(\bar{u}_0)$, $x \in M$, and $t, s \in [0,T]$.

Using the properties (A.5) and (A.6), we can prove two important estimates. One is for the Dirac operators along maps.

Lemma A.2. *Choose ϵ, δ, and R as in* (A.4). *If $\epsilon > 0$ is small enough, then there exists a $C = C(R) > 0$ such that*

$$\|((P^{v_s,u_t})^{-1} \not{D}_{1,0}^{\pi\circ u_t} P^{v_s,u_t} - \not{D}_{1,0}^{\pi\circ v_s})\psi(x)\| \leqslant C\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}\|\psi(x)\|$$

for any $u, v \in B_R^{\mathrm{T}}(\bar{u}_0)$, $\psi \in \Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^* T_{1,0}N)$, $x \in M$, and $t, s \in [0,T]$.

Proof. We write $f_0 := \pi \circ v_s$ and $f_1 := \pi \circ u_t$ and define the C^1 map $F: M \times [0,1] \to N$ by

$$F(x,t) := \exp_{f_0(x)}(t \exp_{f_0(x)}^{-1} f_1(x)),$$

where exp denotes the exponential map of the Riemannian manifold N. Note that $F(\cdot,0) = f_0$, $F(\cdot,1) = f_1$, and $t \mapsto F(x,t)$ is the unique shortest geodesic from $f_0(x)$ to $f_1(x)$. We denote by

$$\mathcal{P}_{t_1,t_2} = \mathcal{P}_{t_1,t_2}(x) : T_{1,0}N|_{F(x,t_1)} \to T_{1,0}N|_{F(x,t_2)}$$

the parallel transport in $F^*T_{1,0}N$ with respect to $\nabla^{F^*T_{1,0}N}$ (pull-back of the connection on $T_{1,0}N$) along the curve $\gamma_x(t) := (x,t)$ from $\gamma_x(t_1)$ to $\gamma_x(t_2)$ with $x \in M$, $t_1, t_2 \in [0,1]$. In particular, $\mathcal{P}_{0,1} = P^{v_s,u_t}$. Let $\psi \in \Gamma_{C^1}(\Sigma M \otimes (f_0)^* T_{1,0}N)$. We have

$$((\mathcal{P}_{0,1})^{-1} \not{D}^{f_1} \mathcal{P}_{0,1} - \not{D}^{f_0})\psi = (e_\alpha \cdot \psi^i) \otimes (((\mathcal{P}_{0,1})^{-1} \nabla_{e_\alpha}^{f_1^* T_{1,0}N} \mathcal{P}_{0,1} - \nabla_{e_\alpha}^{f_0^* T_{1,0}N})(b_i \circ f_0)), \tag{A.7}$$

where $\psi = \psi^i \otimes (b_i \circ f_0)$, $\{b_i\}$ is an orthonormal frame of $T_{1,0}N$, ψ^i's are local C^1 sections of ΣM, and $\{e_\alpha\}$ is an orthonormal frame of TM.

We define local C^1 sections Θ_i of $F^*T_{1,0}N$ by

$$\Theta_i(x,t) := \mathcal{P}_{0,t}(x)(b_i \circ f_0)(x).$$

For each $t \in [0,1]$, we define the functions $T_{ij}(\cdot,t) := T^\alpha_{ij}(\cdot,t)$ by

$$(\mathcal{P}_{0,t})^{-1}((\nabla^{F^*T_{1,0}N}_{e_\alpha}\Theta_i)(x,t)) = \sum_j T^\alpha_{ij}(x,t)(b_j \circ f_0)(x). \tag{A.8}$$

So far, we only know that T_{ij}'s are continuous. In the following, we perform some formal calculations and justify them afterward. By a straightforward computation, we have

$$\begin{aligned}
&\|((\mathcal{P}_{0,1})^{-1}\nabla^{f_1^*T_{1,0}N}_{e_\alpha}\mathcal{P}_{0,1} - \nabla^{f_0^*T_{1,0}N}_{e_\alpha})(b_i\circ f_0)(x)\|^2\\
&\quad= \|(\mathcal{P}_{0,1})^{-1}((\nabla^{F^*T_{1,0}N}_{e_\alpha}\Theta_i)(x,1)) - (\mathcal{P}_{0,0})^{-1}((\nabla^{F^*T_{1,0}N}_{e_\alpha}\Theta_i)(x,0))\|^2\\
&\quad= \left\|\sum_j T_{ij}(x,1)(b_j\circ f_0)(x) - \sum_j T_{ij}(x,0)(b_j\circ f_0)(x)\right\|^2\\
&\quad= \sum_j (T_{ij}(x,1) - T_{ij}(x,0))^2\\
&\quad= \sum_j \left(\int_0^1 \frac{d}{dt}\Big|_{t=r} T_{ij}(x,t)dr\right)^2.
\end{aligned} \tag{A.9}$$

Therefore we want to control the first time-derivative of T_{ij}. Equation (A.8) implies that these time-derivatives are related to the curvature of $F^*T_{1,0}N$. More precisely, for all $X \in \Gamma(TM)$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=r}((\mathcal{P}_{0,t})^{-1}((\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,t)))\\
&\quad= \frac{d}{dt}\Big|_{t=0}((\mathcal{P}_{0,t+r})^{-1}((\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,t+r)))\\
&\quad= \frac{d}{dt}\Big|_{t=0}((\mathcal{P}_{0,r})^{-1}(\mathcal{P}_{r,r+t})^{-1}((\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,t+r)))\\
&\quad= (\mathcal{P}_{0,r})^{-1}\frac{d}{dt}\Big|_{t=0}((\mathcal{P}_{r,r+t})^{-1}((\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,t+r)))\\
&\quad= (\mathcal{P}_{0,r})^{-1}((\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,r)).
\end{aligned} \tag{A.10}$$

Now, let us justify the formal calculations (A.9) and (A.10). Combining the definition of Θ_i as parallel transport and a careful examination of the regularity of F, we deduce that $(\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\nabla^{F^*T_{1,0}N}_X\Theta_i)(x,r)$ exists. Then (A.10) holds. Together with (A.8), we know that T_{ij}'s are differentiable in t. Therefore, (A.9) also holds. We further get

$$\begin{aligned}
&\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\nabla^{F^*T_{1,0}N}_X\Theta_i\\
&\quad= R^{F^*T_{1,0}N}\left(\frac{\partial}{\partial t}, X\right)\Theta_i + \nabla^{F^*T_{1,0}N}_X\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\Theta_i - \nabla^{F^*T_{1,0}N}_{[\frac{\partial}{\partial t},X]}\Theta_i\\
&\quad= R^{F^*T_{1,0}N}\left(\frac{\partial}{\partial t}, X\right)\Theta_i = R^{T_{1,0}N}\left(dF\left(\frac{\partial}{\partial t}\right), dF(X)\right)\Theta_i,
\end{aligned}$$

since $\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\Theta_i = 0$ by the definition of Θ_i and $[\frac{\partial}{\partial t}, X] = 0$.

This implies

$$\sum_j \left(\frac{d}{dt}\bigg|_{t=r} T_{ij}(x,t)\right)^2 = \left\|\frac{d}{dt}\bigg|_{t=r}((\mathcal{P}_{0,t})^{-1}((\nabla^{F^*T_{1,0}N}_{e_\alpha}\Theta_i)(x,t)))\right\|^2$$
$$= \|(\nabla^{F^*T_{1,0}N}_{\frac{\partial}{\partial t}}\nabla^{F^*T_{1,0}N}_{e_\alpha}\Theta_i)(x,r)\|^2$$
$$= \left\|R^{T_{1,0}N}\left(dF_{(x,r)}\left(\frac{\partial}{\partial t}\right), dF_{(x,r)}(e_\alpha)\right)\Theta_i(x,r)\right\|^2$$
$$\leqslant C_1\|dF_{(x,r)}(\partial_t)\|^2\|dF_{(x,r)}(e_\alpha)\|^2,$$

where C_1 only depends on N.

In the following, we estimate $\|dF_{(x,r)}(\partial_t)\|$ and $\|dF_{(x,r)}(e_\alpha)\|$. We have

$$dF_{(x,r)}(\partial_t|_{(x,r)}) = \frac{\partial}{\partial t}\bigg|_{t=r}(\exp_{f_0(x)}(t\exp^{-1}_{f_0(x)} f_1(x))) = c'(r),$$

where $c(t) := \exp_{f_0(x)}(t\exp^{-1}_{f_0(x)} f_1(x))$ is a geodesic in N. In particular, c' is parallel along c and thus $\|c'(r)\| = \|c'(0)\| = \|\exp^{-1}_{f_0(x)} f_1(x)\|$. Therefore, we get

$$\|dF_{(x,r)}(\partial_t)\| = \|\exp^{-1}_{f_0(x)} f_1(x)\| \leqslant d^N(f_0(x), f_1(x)) \leqslant C_2\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)},$$

where we have used Lemma A.1 and the Lipschitz continuity of π. Moreover, there exists a $C_3(R) > 0$ such that $\|dF_{(x,r)}(e_\alpha)\| \leqslant C_3(R)$ for all $(x,r) \in M \times [0,1]$. Thus, we have shown

$$\sum_j \left(\frac{d}{dt}\bigg|_{t=r} T_{ij}(x,t)\right)^2 \leqslant C_1C_2^2C_3(R)^2\|u_t - v_s\|^2_{C^0(M,\mathbb{R}^q)}$$

for all (x,t). Combining this with (A.7) and (A.9), we complete the proof. □

The other one is for the parallel transport.

Lemma A.3. *Choose ϵ, δ, and R as in* (A.4). *If $\epsilon > 0$ is small enough, then there exists a $C = C(\epsilon) > 0$ such that*

$$\|P^{v_s,u_0}P^{u_t,v_s}P^{u_0,u_t}Z - Z\| \leqslant C\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}\|Z\|$$

for all $Z \in T_{1,0}N|_{u_0(x)}$, $u, v \in B^{\mathrm{T}}_R(\bar{u}_0)$, $x \in M$, and $t, s \in [0,T]$.

Consequently, we also have the following lemma.

Lemma A.4. *Choose ϵ, δ, and R as in* (A.4). *For $u, v \in B^{\mathrm{T}}_R(\bar{u}_0)$ and $s, t \in [0,T]$, the operator norm of the isomorphism of Banach spaces*

$$P^{v_s,u_t} : \Gamma_{W^{1,p}}(\Sigma M \otimes (\pi\circ v_s)^*T_{1,0}N) \to \Gamma_{W^{1,p}}(\Sigma M \otimes (\pi\circ u_t)^*T_{1,0}N)$$

is uniformly bounded, i.e., there exists a $C = C(R,p)$ such that $\|P^{v_s,u_t}\|_{L(W^{1,p},W^{1,p})} \leqslant C$ for all $u, v \in B^{\mathrm{T}}_R(\bar{u}_0)$, $x \in M$, and $t, s \in [0,T]$.

The proofs of these two lemmas only depend on the existence of the unique shortest geodesic between any two maps in $B^{\mathrm{T}}_R(\bar{u}_0)$, which has already been shown in (A.6). So we omit them here. Besides, by Lemma A.2, one can immediately prove the following lemma by the min-max principle.

Lemma A.5. *Assume that* $\dim_{\mathbb{H}}\ker(\not{D}^{u_0}_{1,0}) = 1$. *Choose ϵ, δ, and R as in Lemma* A.2. *If R is small enough, then*

$$\dim_{\mathbb{H}}\ker(\not{D}^{\pi\circ u_t}_{1,0}) = 1$$

and there exists a $\Lambda = \frac{1}{2}\Lambda(u_0)$ such that

$$\#\{\mathrm{spec}(\not{D}^{\pi\circ u_t}_{1,0}) \cap [-\Lambda, \Lambda]\} = 1$$

for any $u \in B^{\mathrm{T}}_R(\bar{u}_0)$ and $t \in [0,T]$, where $\Lambda(u_0)$ is a constant such that

$$\mathrm{spec}(\not{D}^{u_0}_{1,0}) \setminus \{0\} \subset \mathbb{R} \setminus (-\Lambda(u_0), \Lambda(u_0)).$$

Once we have the minimality of the kernel in Lemma A.5, we can prove the following uniform bounds for the resolvents, which are important for the Lipschitz continuity of the solution to the Dirac equation.

Lemma A.6. *Assume that we are in the situation of Lemma* A.5. *We consider the resolvent* $R(\lambda, \not{D}^{\pi\circ u_t}_{1,0}) : \Gamma_{L^2} \to \Gamma_{L^2}$ *of* $\not{D}^{\pi\circ u_t}_{1,0} : \Gamma_{W^{1,2}} \to \Gamma_{L^2}$. *By the* L^p *estimate* (*see* [4, *Lemma* 3.3]), *we know the restriction*

$$R(\lambda, \not{D}^{\pi\circ u_t}_{1,0}) : \Gamma_{L^p} \to \Gamma_{W^{1,p}}$$

is well-defined and bounded for any $2 \leqslant p < \infty$. *If* $R > 0$ *is small enough, then there exists a* $C = C(p, R) > 0$ *such that* $\sup_{|\lambda|=\frac{\Lambda}{2}} \|R(\lambda, \not{D}^{\pi\circ u_t}_{1,0})\|_{L(L^p, W^{1,p})} < C$ *for any* $u \in B^{\mathrm{T}}_R(\bar{u}_0)$ *and* $t \in [0, T]$.

Now, by the projector of the Dirac operator, we can construct a solution to the constraint equation whose nontriviality follows from the following lemma.

Lemma A.7. *In the situation of Lemma* A.5, *for any fixed* $u \in B^{\mathrm{T}}_R(\bar{u}_0)$ *and any* $\psi_0 \in \ker(\not{D}^{u_0})$ *with* $\|\psi_0\|_{L^2} = 1$, *we have*

$$\sqrt{\frac{1}{2}} \leqslant \|\tilde{\psi}^{u_t}_1\|_{L^2} \leqslant 1,$$

where $\tilde{\psi}^{u_t} = P^{u_0,u_t}\psi_0 = \tilde{\psi}^{u_t}_1 + \tilde{\psi}^{u_t}_2$ *with respect to the decomposition* $\Gamma_{L^2} = \ker(\not{D}^{\pi\circ u_t}_{1,0}) \oplus (\ker(\not{D}^{\pi\circ u_t}_{1,0}))^{\perp}$.

In Section 3, to show the short-time existence of the heat flow for α-Dirac-harmonic maps, we need the following Lipschitz estimate.

Lemma A.8. *Choose* δ *as in* (A.4), ϵ *as in Lemmas* A.2 *and* A.3, *and* R *as in Lemmas* A.5 *and* A.6. *For any harmonic spinor* $\psi_0 \in \ker(\not{D}^{u_0}_{1,0})$, *we define*

$$\bar{\psi}(u_t) := \tilde{\psi}^{u_t}_1 = -\frac{1}{2\pi \mathrm{i}} \int_\gamma R(\lambda, \not{D}^{\pi\circ u_t}_{1,0}) P^{u_0,u_t}\psi_0 d\lambda$$

for any $u \in B^{\mathrm{T}}_R(\bar{u}_0)$, *where* γ *is defined in the Section* 2 *with* $\Lambda = \frac{1}{2}\Lambda(u_0)$. *In particular,*

$$\bar{\psi}(u_t) \in \ker(\not{D}^{\pi\circ u_t}_{1,0}) \subset \Gamma_{C^0}(\Sigma M \otimes (\pi\circ u_t)^* T_{1,0}N).$$

We write

$$\psi(u_t) := \psi(u(\cdot, t)) = \frac{\bar{\psi}(u_t)}{\|\bar{\psi}(u_t)\|_{L^2}}.$$

Let $\psi^A(u_t)$ *be the sections of* ΣM *such that* $\psi(u_t) = \psi^A(u_t) \otimes (\partial_A \circ \pi \circ u_t)$ *for* $A = 1, \dots, q$. *Then there exists a* $C = C(R, \epsilon, \psi_0) > 0$ *such that*

$$\|P^{u_t,v_s}\bar{\psi}(u_t)(x) - \bar{\psi}(u_t)(x)\| \leqslant C\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)} \tag{A.11}$$

and

$$\|\psi^A(u_t)(x) - \psi^A(v_s)(x)\| \leqslant C\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)} \tag{A.12}$$

for all $u, v \in B^{\mathrm{T}}_R(\bar{u}_0)$, $A = 1, \dots, q$, $x \in M$, *and* $s, t \in [0, T]$.

Proof. Using the following resolvent identity for two operators D_1 and D_2:

$$R(\lambda, D_1) - R(\lambda, D_2) = R(\lambda, D_1) \circ (D_1 - D_2) \circ R(\lambda, D_2),$$

we have

$$\begin{aligned}
&P^{u_t,v_s}\bar{\psi}(u_t) - \bar{\psi}(v_s)\\
&\quad= -\frac{1}{2\pi \mathrm{i}} \int_\gamma R(\lambda, P^{u_t,v_s}\not{D}^{\pi\circ u_t}_{1,0}(P^{u_t,v_s})^{-1})(P^{u_t,v_s}P^{u_0,u_t}\psi_0 - P^{u_0,v_s}\psi_0)\\
&\qquad- \frac{1}{2\pi \mathrm{i}} \int_\gamma (R(\lambda, P^{u_t,v_s}\not{D}^{\pi\circ u_t}_{1,0}(P^{u_t,v_s})^{-1}) - R(\lambda, \not{D}^{\pi\circ v_s}_{1,0}))P^{u_0,v_s}\psi_0\\
&\quad= -\frac{1}{2\pi \mathrm{i}} \int_\gamma R(\lambda, P^{u_t,v_s}\not{D}^{\pi\circ u_t}_{1,0}(P^{u_t,v_s})^{-1})(P^{u_t,v_s}P^{u_0,u_t}\psi_0 - P^{u_0,v_s}\psi_0)
\end{aligned}$$

$$-\frac{1}{2\pi \mathrm{i}}\int_\gamma (R(\lambda, P^{u_t,v_s}\not{D}^{\pi\circ u_t}_{1,0}(P^{u_t,v_s})^{-1})\circ(P^{u_t,v_s}\not{D}^{\pi\circ u_t}_{1,0}(P^{u_t,v_s})^{-1}-\not{D}^{\pi\circ v_s}_{1,0})\circ R(\lambda,\not{D}^{\pi\circ v_s}_{1,0}))P^{u_0,v_s}\psi_0,$$

where γ is defined in (2.11) with $\Lambda=\frac{1}{2}\Lambda(u_0)$. Therefore, for p large enough, we get

$$\begin{aligned}
&\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(v_s)(x)\|\\
&\quad\leqslant C_1\|P^{u_t,v_s}\bar\psi^{u_t}-\bar\psi^{v_s}\|_{W^{1,p}(M)}\\
&\quad\leqslant C_2\bigg\|\int_\gamma R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})(P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0)\bigg\|_{W^{1,p}(M)}\\
&\qquad+C_2\bigg\|\int_\gamma (R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})\circ(P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1}-\not{D}^{\pi\circ v_s})\\
&\qquad\circ R(\lambda,\not{D}^{\pi\circ v_s}))P^{u_0,v_s}\psi_0\bigg\|_{W^{1,p}(M)}\\
&\quad\leqslant C_2\int_\gamma\|R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})(P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0)\|_{W^{1,p}(M)}\\
&\qquad+C_2\int_\gamma\|(R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})\circ(P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1}-\not{D}^{\pi\circ v_s})\\
&\qquad\circ R(\lambda,\not{D}^{\pi\circ v_s}))P^{u_0,v_s}\psi_0\|_{W^{1,p}(M)}\\
&\quad\leqslant C_3\sup_{\mathrm{Im}(\gamma)}\|R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})\|_{L(L^p,W^{1,p})}\|P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0\|_{L^p}\\
&\qquad+C_3\sup_{\mathrm{Im}(\gamma)}\|R(\lambda,P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1})\|_{L(L^p,W^{1,p})}\sup_{\mathrm{Im}(\gamma)}\|R(\lambda,\not{D}^{\pi\circ v_s})\|_{L(L^p,W^{1,p})}\\
&\qquad\times\|P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1}-\not{D}^{\pi\circ v_s}\|_{L(W^{1,p},L^p)}\|P^{u_0,v_s}\psi_0\|_{L^p}.
\end{aligned}$$

Now, we estimate all the terms on the right-hand side of the inequality above. First, by Lemmas A.6 and A.4, we know that all the resolvents above are uniformly bounded. Next, by Lemma A.2, we have

$$\|P^{u_t,v_s}\not{D}^{\pi\circ u_t}(P^{u_t,v_s})^{-1}-\not{D}^{\pi\circ v_s}\|_{L(W^{1,p},L^p)}\leqslant C(R)\|u_t-v_s\|_{C^0(M,\mathbb{R}^q)}.$$

Finally, by Lemma A.3, we obtain

$$\|P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0\|_{L^p}\leqslant C(\epsilon,\psi_0)\|u_t-v_s\|_{C^0(M,\mathbb{R}^q)}.$$

Putting these together, we get (A.11).

Next, we want to show the following estimate which is very close to (A.12):

$$\|\bar\psi^A(u_t)(x)-\bar\psi^A(v_s)(x)\|\leqslant C(R,\epsilon,\psi_0)\|u_t-v_s\|_{C^0(M,\mathbb{R}^q)}. \tag{A.13}$$

In fact, we have

$$\begin{aligned}
&\|\bar\psi^A(u_t)(x)-\bar\psi^A(v_s)(x)\|\\
&\quad\leqslant\|\bar\psi(u_t)(x)-\bar\psi(v_s)(x)\|_{\Sigma_xM\otimes\mathbb{R}^q}\\
&\quad\leqslant\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(v_s)(x)\|_{\Sigma_xM\otimes\mathbb{R}^q}+\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(u_t)(x)\|_{\Sigma_xM\otimes\mathbb{R}^q}\\
&\quad=\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(v_s)(x)\|_{\Sigma_xM\otimes T_{(\pi\circ v_s(x))}N}\\
&\qquad+\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(u_t)(x)\|_{\Sigma_xM\otimes\mathbb{R}^q}\\
&\quad\leqslant C(R,\epsilon,\psi_0)\|u_t-v_s\|_{C^0(M,\mathbb{R}^q)}+\|P^{u_t,v_s}\bar\psi(u_t)(x)-\bar\psi(u_t)(x)\|_{\Sigma_xM\otimes\mathbb{R}^q}.
\end{aligned}$$

It remains to estimate the last term in the inequality above. To that end, let

$$\gamma(r):=\exp_{(\pi\circ u_t)(x)}(r\exp^{-1}_{(\pi\circ u_t)(x)}(\pi\circ v_s(x))),\quad r\in[0,1]$$

be the unique shortest geodesic of N from $(\pi\circ u_t)(x)$ to $(\pi\circ v_s)(x)$. Let $X\in T_{\gamma(0)}N$ be given and denote by $X(r)$ the unique parallel vector field along γ with $X(0)=X$. Then we have

$$P^{u_t,v_s}X-X=X(1)-X(0)=\int_0^1\frac{dX}{dr}\bigg|_{r=\xi}d\xi=\int_0^1 II(\gamma'(r),X(r))dr.$$

Therefore,

$$\|P^{u_t,v_s}X - X\|_{\mathbb{R}^q} \leqslant C_1 \sup_{r\in[0,1]} \|\gamma'(r)\|_N \sup_{r\in[0,1]} \|X(r)\|_N = C_1\|\gamma'(0)\|_N\|X\|_N,$$

where II is the second fundamental form of N in $\mathbb{R}^q$ and C_1 only depends on N. Using (A.3) and the Lipschitz continuity of π, we get

$$\|\gamma'(0)\|_N \leqslant d^N((\pi\circ u_t)(x), (\pi\circ v_s)(x)) \leqslant C_2\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}$$

and

$$\|P^{u_t,v_s}X - X\|_{\mathbb{R}^q} \leqslant C_3\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}\|X\|_N.$$

This implies

$$\|P^{u_t,v_s}\bar{\psi}(u_t)(x) - \bar{\psi}(u_t)(x)\|_{\Sigma_x M\otimes\mathbb{R}^q} \leqslant C(R,\epsilon,\psi_0)\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}.$$

Hence, (A.13) holds.

Now, using (A.11) and (A.13), we get

$$\begin{aligned}
&\|\psi^A(u_t)(x) - \psi^A(v_s)(x)\| \\
&\quad = \left\|\frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(u_t)\|_{L^2}} - \frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(v_s)\|_{L^2}} + \frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(v_s)\|_{L^2}} - \frac{\bar{\psi}^A(v_s)(x)}{\|\bar{\psi}(v_s)\|_{L^2}}\right\| \\
&\quad \leqslant \frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(u_t)\|_{L^2}\|\bar{\psi}(v_s)\|_{L^2}}\big|\|\bar{\psi}(v_s)\|_{L^2} - \|\bar{\psi}(u_t)\|_{L^2}\big| \\
&\qquad + \frac{1}{\|\bar{\psi}(v_s)\|_{L^2}}\|\bar{\psi}^A(u_t)(x) - \bar{\psi}^A(v_s)(x)\| \\
&\quad = \frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(u_t)\|_{L^2}\|\bar{\psi}(v_s)\|_{L^2}}\big|\|\bar{\psi}(v_s)\|_{L^2} - \|P^{u_t,v_s}\bar{\psi}(u_t)\|_{L^2}\big| \\
&\qquad + \frac{1}{\|\bar{\psi}(v_s)\|_{L^2}}\|\bar{\psi}^A(u_t)(x) - \bar{\psi}^A(v_s)(x)\| \\
&\quad \leqslant \frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(u_t)\|_{L^2}\|\bar{\psi}(v_s)\|_{L^2}}\|P^{u_t,v_s}\bar{\psi}(u_t) - \bar{\psi}(v_s)\|_{L^2} \\
&\qquad + \frac{1}{\|\bar{\psi}(v_s)\|_{L^2}}\|\bar{\psi}^A(u_t)(x) - \bar{\psi}^A(v_s)(x)\| \\
&\quad \leqslant \left(\frac{\bar{\psi}^A(u_t)(x)}{\|\bar{\psi}(u_t)\|_{L^2}\|\bar{\psi}(v_s)\|_{L^2}} + \frac{1}{\|\bar{\psi}(v_s)\|_{L^2}}\right)C(R,\epsilon,\psi_0)\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}.
\end{aligned}$$

Then the inequality (A.12) follows from Lemma A.7 and (A.13). This completes the proof. □