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Extrinsic conformal lower bounds of eigenvalue for Dirac operator

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Abstract
In this note, we prove conformal lower bounds for Dirac operators of submanifolds in terms of conformal and extrinsic quantities.

Keywords Eigenvalue estimate · Dirac operator · Geometry of submanifolds

Mathematics Subject Classification 53C27 · 53C40

1 Introduction

The eigenvalues of Dirac operators on spin manifolds are extensively studied. In 1980, Friedrich [7] first derived the lower bound of the first eigenvalues of a Dirac operator D in terms of the scalar curvature S_M and dimension m of the underling manifold M:

$$\lambda^2(D) \geq \frac{m}{4(m-1)} S_M. \tag{1.1}$$

Since then, various kinds of estimates in terms of intrinsic geometric quantities have been proved (see e.g. [8,9] and the references therein). A well known result of Hijazi [11] states that

$$\lambda^2(D) \geq \frac{m}{4(m-1)} \lambda_1(L_M) \tag{1.2}$$

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for $m \geq 3$, where $L_M = -\frac{4(m-1)}{m-2}\Delta + S_M$ is the Yamabe operator of M. If $m = 2$, Bär [2] proved that

$$\lambda^2(D) \geq \frac{4\pi(1-g_M)}{\text{area}(M)}, \tag{1.3}$$

where g_M is the genus of M. The equality in (1.1), (1.2) or (1.3) gives an Einstein metric.

On the other hand, the submanifold theory for Dirac operators was introduced by Bär in [3]. Let $M^m \overset{\iota}{\hookrightarrow} \bar{M}^{m+n}$ be a closed oriented connected spin submanifold isometrically embedded in a Riemannian spin manifold $\bar{M}^{m+n}$ with fixed spin structures. Milnor's Lemma claims that there is a unique spin structure [18] on the normal bundle N of M in $\bar{M}$. Denoted by $\Sigma\bar{M}$, ΣM and ΣN the spinor bundles of $\bar{M}$, M and N respectively. Denoted by $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ the Levi-Civita connections on $\bar{M}$, M, N respectively. Denoted by $\nabla^{\Sigma\bar{M}}$, $\nabla^{\Sigma M}$ and $\nabla^{\Sigma N}$ the Levi-Civita connections on $\Sigma\bar{M}$, ΣM and ΣN respectively. For every $X, Y \in TM$, define

$$\begin{aligned}
\bar{R}(X,Y) &:= [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]},\\
R(X,Y) &:= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},\\
R^{\perp}(X,Y) &:= [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]},\\
R^{\Sigma\bar{M}}(X,Y) &:= [\nabla^{\Sigma\bar{M}}_X, \nabla^{\Sigma\bar{M}}_Y] - \nabla^{\Sigma\bar{M}}_{[X,Y]},\\
R^{\Sigma M}(X,Y) &:= [\nabla^{\Sigma M}_X, \nabla^{\Sigma M}_Y] - \nabla^{\Sigma M}_{[X,Y]},\\
R^{\Sigma N}(X,Y) &:= [\nabla^{\Sigma N}_X, \nabla^{\Sigma N}_Y] - \nabla^{\Sigma N}_{[X,Y]}.
\end{aligned}$$

Denoted by $\bar{\gamma}$, γ, $\gamma^{\perp}$ the Clifford multiplications on $\Sigma\bar{M}$, ΣM and ΣN respectively. Denoted by $\bar{D}$, D, $D^{\perp}$ the Dirac operators on $\Sigma\bar{M}$, ΣM and ΣN respectively. Let A^{μ} be the shape operator of M in $\bar{M}$ with respect to the normal vector field μ, B the second fundamental form of M in $\bar{M}$ and H the normalized mean curvature vector of M in $\bar{M}$. Let $\{\nu_\alpha\}$ be a local orthonormal frame of the normal bundle N, then $A = \sum_{\alpha=1}^{n} A^{\alpha} \otimes \nu_\alpha$ and $H = \sum_{\alpha=1}^{n} H^{\alpha}\nu_\alpha$. Denote by $\mathring{A}$ the trace free part of A, i.e., $\mathring{A} = \sum_{\alpha=1}^{n} \mathring{A}^{\alpha} \otimes \nu_\alpha$ and $\mathring{A}^{\alpha} = A^{\alpha} - H^{\alpha} g$. If M is a hypersurface of $\bar{M}$, we denote A by the shape operator of M in $\bar{M}$ with respect to the unit outward normal vector field. Finally, denote $R(\iota)$ by the normalized trace of the ambient sectional curvature on the tangent space, i.e.,

$$R(\iota) = \frac{1}{m(m-1)} \sum_{i,j=1}^{m} \bar{R}(e_i, e_j, e_i, e_j),$$

where $\{e_i\}$ is a local orthonormal frame of TM.

The lower bounds of hypersurface Dirac operators was studied by Zhang [20–22] and later generalized to submanifold Dirac operators by Hijazi and Zhang [14,15]. In particular, Hijazi and Zhang [14,15] give optimal lower bounds for the submanifold Dirac operator in terms of the mean curvature and other geometric invaraints as the Yamabe number or the energy-momentum tensor under some extra assumptions. Ginoux and Morel [10] also considered the eigenvalue estimates problem for submanifold Dirac operators.

In this paper, we study the eigenvalue estimats problem for Dirac operator $D^{\Sigma N}$ of the twistor bundle $\Sigma M \otimes \Sigma N$ which can be viewed as a Dirac bundle on M [18]. Locally,

$$D^{\Sigma N}(\psi \otimes \theta) := D\psi \otimes \theta + \sum_{i=1}^{m} \gamma(e_i)\psi \otimes \nabla^{\perp}_{e_i}\theta.$$

According to [3], we know that

$$\Sigma \bar{M}|_M = \begin{cases} \Sigma M \otimes \Sigma N, & mn = 0 \mod 2 \\ (\Sigma M \otimes \Sigma N) \oplus (\Sigma M \otimes \Sigma N), & mn = 1 \mod 2. \end{cases}$$

We will prove comformal lower bound estimates for Dirac operator $D^{\Sigma N}$ in terms of conformal and extrinsic quantities.

Theorem 1.1 *Let M^m be a closed oriented submanifold isometrically embedded in a Riemannian spin manifold $\bar{M}^{n+m}$. Suppose $n = 1$ or $\bar{M}$ is locally conformally flat. Then the eigenvalue λ of the Dirac operator $D^{\Sigma N}$ of the twisted bundle $\Sigma M \otimes \Sigma N$ satisfies*

$$\lambda^2 \geq \begin{cases} \dfrac{4\pi(1-g_M)}{\operatorname{area}(M)} - \dfrac{(n-1)\int_M \left|\mathring{A}\right|^2}{2\operatorname{area}(M)}, & m = 2, \\ \dfrac{m}{4(m-1)}\lambda_1(L), & m > 2. \end{cases}$$

Here $\lambda_1(L)$ (if $m > 2$) is the first eigenvalue of the operator L defined by

$$L = -\frac{4(m-1)}{m-2}\Delta + S_M - (n-1)\left|\mathring{A}\right|^2.$$

Moreover, if $\lambda \neq 0$, then the equality implies that the Ricci curvature of M satisfies

$$Ric = (n-1)\sum_{\alpha=1}^{n}\left(\mathring{A}^\alpha\right)^2 + \frac{4(m-1)\lambda^2}{m^2}g.$$

Remark 1.1 1. When $m = 2$,

$$\int_M \left|\mathring{A}\right|^2$$

is invariant under the conformal change of the metric $\bar{g}$. The equality implies that $g_M = 0$ or $g_M = 1$ and $\mathring{A} = 0$, i.e., M is a 2-sphere or a totally umbilici 2-torus.

2. For a Dirac operator D, let λ_i be the eigenvalues. We recall the conformal eigenvalue $\sigma_i(D)$ of D (cf. [1]) given by

$$\sigma_i(D) = \inf_{\tilde{g}\in[g]} |\lambda_i(\tilde{g})| \operatorname{vol}_{M_{\tilde{g}}}^{1/m}.$$

Here $[g]$ stands for the conformal class of g. Similarly, for a second positive self adjoint elliptic operator L, we have the conformal eigenvalue $\lambda_i(L)$ of L by

$$\sigma_i(L) = \inf_{\tilde{g}\in[g]} \lambda_i(\tilde{g}) \operatorname{vol}_{M_{\tilde{g}}}^{2/m}.$$

From Theorem 1.1, we have that

$$\sigma_1^2\left(D^{\Sigma N}\right) \geq \begin{cases} 4\pi(1-g_M) - \dfrac{n-1}{2}\int_M \left|\mathring{A}\right|^2, & m = 2, \\ \dfrac{m}{4(m-1)}\sigma_1(L), & m > 2. \end{cases}$$

3. If M is a hypersurface, i.e., $n = 1$, then ΣN is the trivial complex line bundle and $D^{\Sigma N} = D$ is the classical Dirac operator on M acting on spinors. In this case, Theorem 1.1 is reduced to Hijazi's result [11] for $m \geq 3$ and Bär's result [2] for $m = 2$.

2 Preliminaries

We first compare the Dirac operator on $\bar{M}$ with the one on M. We will use notations in [3]. We also refer the reader to [5,12–15] and the references therein. Basic facts concerning Clifford algebras and spinor representations can be found in classical books [4,18].

2.1 Algebra preliminaries

Let E be an oriented Euclidean vector space. If $\dim E = m$ is even, then the the complex Clifford algebra of E, denoted by $\mathbb{C}l(E)$, has precisely one irreducible module, the spinor module ΣE with dimension $2^{m/2}$. When restricted to the even subalgebra $\mathbb{C}l^0(E)$ the spinor module decomposes into even and odd half-spinors $\Sigma E = \Sigma^+ E \oplus \Sigma^- E$ associated the eigenspaces of the complex volume element $\omega_{\mathbb{C}} = \sqrt{-1}^{m/2}\gamma_E(e_1 \dots e_m)$. On $\Sigma^\pm E$ it acts as ± 1. Here $\{e_i\}$ stand for a positively oriented orthonormal frame of E and $\gamma_E : \mathbb{C}l(E) \longrightarrow \mathrm{End}(E)$ stands for the Clifford multiplication.

If m is odd there are exactly two irreducible modules, $\Sigma^0 E$ and $\Sigma^1 E$, again called spinor modules. In this case $\dim \Sigma^0 E = \dim \Sigma^1 E = 2^{(m-1)/2}$. Also the two modules $\Sigma^0 E$ and $\Sigma^1 E$ can be distinguished by the action of the complex volume element $\omega_{\mathbb{C}} = \sqrt{-1}^{(m+1)/2}\gamma_E(e_1 \cdots e_m)$. On $\Sigma^j E$ it acts as $(-1)^j$, $j = 0, 1$. There exists a vector space isomorphism $\Phi : \Sigma^0 E \longrightarrow \Sigma^1 E$ such that $\Phi \circ \gamma_{E,0} = -\gamma_{E,1} \circ \Phi$, where $\gamma_{E,j} : \mathbb{C}l(E) \longrightarrow \mathrm{End}\, \Sigma^j E$ stand for the Clifford multiplication, $j = 0, 1$.

Let E and F be two oriented Euclidean vector spaces. Let $\dim E = m$ and $\dim F = n$. We will construct the spinor module of $E \oplus F$ from those of E and F.

Case 1. m and n are both even.
Put $\Sigma := \Sigma E \otimes \Sigma F$ and define

$$\begin{aligned}\gamma :&E \oplus F \longrightarrow \mathrm{End}\, \Sigma,\\ \gamma(X \oplus Y)(\sigma \otimes \tau) &= (\gamma_E(X)\sigma) \otimes \tau + (-1)^{\deg \sigma}\sigma \otimes (\gamma_F(Y)\tau)\,.\end{aligned}$$

Here

$$\deg \sigma = \begin{cases} 0, & \sigma \in \Sigma^+ E;\\ 1, & \sigma \in \Sigma^- E.\end{cases}$$

In this case

$$\begin{aligned}\Sigma^+(E \oplus F) &= \left(\Sigma^+ E \otimes \Sigma^+ F\right) \oplus \left(\Sigma^- E \otimes \Sigma^- F\right),\\ \Sigma^-(E \oplus F) &= \left(\Sigma^+ E \otimes \Sigma^- F\right) \oplus \left(\Sigma^- E \otimes \Sigma^+ F\right).\end{aligned}$$

Case 2. m is even and n is odd.
Put $\Sigma^j := \Sigma E \otimes \Sigma^j F$ for $j = 0, 1$. As similar to Case 1, we can define $\gamma_j : E \oplus F \longrightarrow \mathrm{End}\, \Sigma^j$ with obvious modification.

Case 3. m is odd and n is even.
This case is symmetric to the second one. Put $\Sigma^j := \Sigma^j E \otimes \Sigma F$ and define

$$\begin{aligned}\gamma_j :&E \oplus F \longrightarrow \mathrm{End}\, \Sigma^j,\\ \gamma_j(X \oplus Y)(\sigma \otimes \tau) &=(-1)^{\deg \tau}(\gamma_E(X)\sigma) \otimes \tau + \sigma \otimes (\gamma_F(Y)\tau).\end{aligned}$$

Case 4. m and n are both odd.

Set

$$\Sigma^+ := \Sigma^0 E \otimes \Sigma^0 F,$$
$$\Sigma^- := \Sigma^0 E \otimes \Sigma^1 F,$$
$$\Sigma := \Sigma^+ \oplus \Sigma^-.$$

Recall that there exists a vector space isomorphism $\Phi : \Sigma^0 F \longrightarrow \Sigma^1 F$ such that $\Phi \circ \gamma_{F,0} = -\gamma_{F,1} \circ \Phi$. With respect to the splitting $\Sigma = \Sigma^+ \oplus \Sigma^-$, we define

$$\gamma : E \oplus F \longrightarrow \mathrm{End}\, \Sigma,$$
$$\gamma(X \oplus Y) = \begin{pmatrix} 0 & \sqrt{-1}\gamma_{E,0}(X) \otimes \Phi^{-1} + \mathrm{Id} \otimes (\Phi^{-1} \circ \gamma_{F,1}(Y)) \\ -\sqrt{-1}\gamma_{E,0}(X) \otimes \Phi - \mathrm{Id} \otimes (\Phi \circ \gamma_{F,0}(Y)) & 0 \end{pmatrix}.$$

2.2 Geometric preliminaries

With respect to the orthogonal splitting $T\bar{M}|_M = TM \oplus N$, the Gauss formula says

$$\bar{\nabla}_X = \begin{pmatrix} \nabla_X & -B(X, \cdot)^* \\ B(X, \cdot) & \nabla^{\perp}_X \end{pmatrix}.$$

The following equations are well known, i.e., Gauss equations, Codazzi equations and Ricci equations (cf. [19]). For all $X, Y, Z \in TM$, $\mu \in N$,

$$\bar{R}(X, Y)Z = R(X, Y)Z + A^{B(X,Z)}(Y) - A^{B(Y,Z)}(X) + (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z),$$
$$\bar{R}(X, Y)\mu = (\nabla_Y A)^{\mu}(X) - (\nabla_X A)^{\mu}(Y) + R^{\perp}(X, Y)\mu + B(A^{\mu}(X), Y) - B(A^{\mu}(Y), X).$$

From the consideration in the previous subsection we know for the spinor bundles that $\Sigma\bar{M}|_M = \Sigma M \otimes \Sigma N$ unless m and n are both odd in which case $\Sigma\bar{M}|_M = (\Sigma M \otimes \Sigma N) \oplus (\Sigma M \otimes \Sigma N)$. Using a standard formula (cf. [18]), we have

$$\nabla^{\Sigma\bar{M}|_M}_X = \nabla^{\Sigma M}_X \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N}_X + \frac{1}{2}\sum_{\alpha=1}^{n} \bar{\gamma}(A^{\alpha}(X) \cdot \nu_{\alpha}),$$
$$R^{\Sigma\bar{M}|_M}(X, Y) = R^{\Sigma M}(X, Y) \otimes \mathrm{Id} + \mathrm{Id} \otimes R^{\Sigma N}(X, Y) + \frac{1}{4}\sum_{\alpha=1}^{n} \gamma([A^{\alpha}(X), A^{\alpha}(Y)]) \otimes \mathrm{Id}$$
$$+ \frac{1}{4}\sum_{\alpha,\beta=1}^{n} \left(\left\langle A^{\alpha}(X), A^{\beta}(Y)\right\rangle - \left\langle A^{\alpha}(Y), A^{\beta}(X)\right\rangle\right) \mathrm{Id} \otimes \gamma^{\perp}(\nu_{\alpha} \cdot \nu_{\beta})$$
$$+ \frac{1}{2}\sum_{\alpha=1}^{n} \bar{\gamma}\left(((\nabla_X A)^{\alpha}(Y) - (\nabla_Y A)^{\alpha}(X)) \cdot \nu_{\alpha}\right).$$

Define

$$\tilde{D} := \sum_{i=1}^{m} \bar{\gamma}(e_i)\nabla^{\Sigma M \otimes \Sigma N}_{e_i}.$$

Then (cf. [3])

$$\tilde{D}^2 = \begin{cases} \left(D^{\Sigma N}\right)^2, & mn = 0 \mod 2; \\ \left(D^{\Sigma N} \oplus (-D^{\Sigma N})\right)^2, & mn = 1 \mod 2. \end{cases}$$

Recall the Bochner formula (cf. [17,18]),

$$\left(D^{\Sigma N}\right)^2=\left(\nabla^{\Sigma M\otimes\Sigma N}\right)^*\nabla^{\Sigma M\otimes\Sigma N}+\mathcal{R}^{\Sigma N},$$

where

$$\mathcal{R}^{\Sigma N}=\frac{1}{2}\bar{\gamma}(e_i\cdot e_j)R^{\Sigma M\otimes\Sigma N}(e_i,e_j).$$

3 Conformal lower bound estimates

In this section, we will give conformal lower bounds of the first eigenvalue of the Dirac operator on the twisted bundle $\Sigma M\otimes\Sigma N$.

First, we have

Lemma 3.1

$$\begin{aligned}\mathcal{R}^{\Sigma N}&=\frac{m(m-1)}{4}\left(R(\iota)+|H|^2\right)+\frac{1}{4}\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha\right)\right)^2\\&\quad-\frac{1}{8}\bar{W}_{ij\alpha\beta}\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\qquad(3.1)\\&=\frac{S_M-(n-1)\left|\mathring{A}\right|^2}{4}-\frac{n}{4}\sum_{i=1}^{m}\sum_{\beta=1}^{n}\left(\bar{\gamma}\left(\mathring{A}^{\beta}(e_i)\cdot\nu_\beta\right)-\frac{1}{n}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha\right)\right)^2\\&\quad-\frac{1}{8}\bar{W}_{ij\alpha\beta}\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta).\qquad(3.2)\end{aligned}$$

Proof Recall the curvature decomposition of $\bar{R}$. Denoted $\bar{P}$ by the Schouten tensor which is defined by

$$\bar{P}_{AB}:=\frac{1}{n+m-2}\left(\bar{R}ic_{AB}-\frac{\bar{S}}{2(n+m-1)}\bar{g}_{AB}\right),\quad 1\le A,B\le n+m,$$

the Weyl tensor $\bar{W}$ is given by

$$\bar{W}_{ABCD}:=\bar{R}_{ABCD}-\left(\bar{P}_{AC}\bar{g}_{BD}+\bar{P}_{BD}\bar{g}_{AC}-\bar{P}_{AD}\bar{g}_{BC}-\bar{P}_{BC}\bar{g}_{AD}\right).$$

Therefore, for every orthonormal 4-frame $\{e_A,e_B,e_C,e_D\}$, we have

$$\bar{W}_{ABCD}=\bar{R}_{ABCD}.$$

A standard computation (cf. [18]) gives a formula

$$\mathcal{R}^{\Sigma N}=\frac{1}{8}\langle R(e_i,e_j)e_k,e_l\rangle\bar{\gamma}(e_i\cdot e_j\cdot e_k\cdot e_l)+\frac{1}{8}\left\langle R^{\perp}(e_i,e_j)\nu_\alpha,\nu_\beta\right\rangle\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta).\qquad(3.3)$$

The first term in the RHS of (3.3) is

$$\frac{S_M}{4}=\frac{1}{4}\left(\sum_{i,j=1}^{m}\bar{R}(e_i,e_j,e_i,e_j)+m(m-1)|H|^2-\left|\mathring{A}\right|^2\right).\qquad(3.4)$$

According to Ricci equations, we compute the second term of the RHS as follows,

$$\begin{aligned}
&\frac{1}{8}\left\langle R^{\perp}(e_i,e_j)\nu_\alpha,\nu_\beta\right\rangle\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\\
&=\frac{1}{8}\left(\langle\bar{R}(e_i,e_j)\nu_\alpha,\nu_\beta\rangle+\langle A^\alpha(e_j),A^\beta(e_i)\rangle-\langle A^\alpha(e_i),A^\beta(e_j)\rangle\right)\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\\
&=\frac{1}{8}\langle\bar{W}(e_i,e_j)\nu_\alpha,\nu_\beta\rangle\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\\
&\quad+\frac{1}{8}\left(\left\langle\mathring{A}^\alpha(e_j),\mathring{A}^\beta(e_i)\right\rangle-\left\langle\mathring{A}^\alpha(e_i),\mathring{A}^\beta(e_j)\right\rangle\right)\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\\
&=\frac{1}{4}\left(\sum_{i=1}^m\sum_{\alpha,\beta=1}^n\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha\cdot\mathring{A}^\beta(e_i)\cdot\nu_\beta\right)+\left|\mathring{A}\right|^2\right)\\
&\quad+\frac{1}{8}\langle\bar{W}(e_i,e_j)\nu_\alpha,\nu_\beta\rangle\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta),
\end{aligned}\tag{3.5}$$

where we used the fact

$$\bar{W}_{ij\alpha\beta}=\bar{R}_{ij\alpha\beta},\quad\forall i\neq j,\alpha\neq\beta.$$

Thus, the second term in the RHS of (3.3) is

$$\frac{1}{4}\left(\sum_{i=1}^m\sum_{\alpha,\beta=1}^n\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha\cdot\mathring{A}^\beta(e_i)\cdot\nu_\beta\right)+\left|\mathring{A}\right|^2\right)-\frac{1}{8}\bar{W}_{ij\alpha\beta}\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta).\tag{3.6}$$

Now (3.1) follows from (3.3), (3.4) and (3.6).

On the other hand, according to (3.5), for every spinor ψ

$$\begin{aligned}
&\left\langle\frac{1}{8}\left\langle R^{\perp}(e_i,e_j)\nu_\alpha,\nu_\beta\right\rangle\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\psi+\frac{1}{8}\bar{W}_{ij\alpha\beta}\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\psi,\psi\right\rangle\\
&=\frac{1}{4}\left\langle\sum_{i=1}^m\sum_{\alpha,\beta=1}^n\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha\cdot\mathring{A}^\beta(e_i)\cdot\nu_\beta\right)\psi+\left|\mathring{A}\right|^2\psi,\psi\right\rangle\\
&=\frac{1}{4}\left(\sum_{i,\alpha}|\bar{\gamma}(A^\alpha(e_i))\psi|^2-\sum_i\left|\sum_\alpha\bar{\gamma}(A^\alpha(e_i)\cdot\nu_\alpha)\psi\right|^2\right)\\
&=\frac{1}{4}\left(\sum_{i,\alpha}\left|\bar{\gamma}(\mathring{A}^\alpha(e_i))\psi\right|^2-\sum_i\left|\sum_\alpha\bar{\gamma}(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha)\psi\right|^2\right)\\
&=\frac{1}{4}\left(n\sum_{i,\beta}\left|\bar{\gamma}(\mathring{A}^\beta(e_i))\psi-\frac{1}{n}\sum_\alpha\bar{\gamma}(\nu_\beta\cdot\mathring{A}^\alpha(e_i)\cdot\nu_\alpha)\psi\right|^2-(n-1)\left|\mathring{A}\right|^2|\psi|^2\right).
\end{aligned}\tag{3.7}$$

Insert (3.7) into (3.3) and (3.4) to obtain (3.2). □

Remark 3.1 1. If $n=1$,

$$\mathcal{R}^{\Sigma N}=\frac{1}{4}S_M=\frac{m(m-1)}{4}\left(R(\iota)+|H|^2\right)-\frac{1}{4}\left|\mathring{A}\right|^2.$$

2. If $m=2, n=2$,

$$\mathcal{R}^{\Sigma N}|_{\Sigma^\pm} = \frac{1}{2}\kappa_M \pm \frac{1}{2}\kappa_N = \frac{1}{2}\left(\bar{R}(e_1,e_2,e_1,e_2)+|H|^2\right) - \frac{1}{4}\left|\mathring{A}\right|^2 \pm \frac{1}{2}\kappa_N,$$

$$-\frac{1}{4}\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha\right)\right)^2|_{\Sigma^\pm} = \frac{1}{4}\left|\mathring{A}\right|^2 \mp \frac{1}{2}\left(\kappa_N - \bar{R}(e_1,e_2,\nu_1,\nu_2)\right).$$

Here

$$\kappa_N = \left\langle R^\perp(e_1,e_2)\nu_2,\nu_1\right\rangle.$$

A direct consequence is

$$\int_M \left|\mathring{A}\right|^2 \geq 2\left|2\pi\chi(N) - \int_M \bar{R}(e_1,e_2,\nu_1,\nu_2)\right|.$$

Therefore,

$$\chi(M) + \left|\chi(N) - \frac{1}{2\pi}\int_M \bar{R}(e_1,e_2,\nu_1,\nu_2)\right| \leq \frac{1}{2\pi}\left(\int_M \bar{R}(e_1,e_2,e_1,e_2)+|H|^2\right).$$

In particular, if $\bar{M}$ is flat and M is minimal (cf. [16]), then

$$\chi(M) + |\chi(N)| \leq 0.$$

Proof The first remark is obvious. For the first part of the second remark, we refer the reader to H. Iriyeh's paper [16]. For the second part, we have

$$\begin{aligned}
&-\frac{1}{4}\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha\right)\right)^2|_{\Sigma^\pm}\\
&= \frac{1}{4}\sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{2}\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\mathring{A}^\beta(e_i)\cdot\nu_\alpha\cdot\nu_\beta\right)\\
&= \frac{1}{4}\left|\mathring{A}\right|^2 + \frac{1}{4}\sum_{i=1}^{2}\sum_{\alpha\neq\beta}\bar{\gamma}\left(\mathring{A}^\alpha(e_i)\cdot\mathring{A}^\beta(e_i)\cdot\nu_\alpha\cdot\nu_\beta\right)\\
&= \frac{1}{4}\left|\mathring{A}\right|^2 + \frac{1}{4}\sum_{i=1}^{2}\sum_{\alpha\neq\beta}\bar{\gamma}\left(A^\alpha(e_i)\cdot A^\beta(e_i)\cdot\nu_\alpha\cdot\nu_\beta\right)\\
&= \frac{1}{4}\left|\mathring{A}\right|^2 + \frac{1}{4}\sum_{i=1}^{2}\sum_{j\neq k}\sum_{\alpha\neq\beta}\left\langle A^\alpha(e_i),e_j\right\rangle\left\langle A^\beta(e_i),e_k\right\rangle\bar{\gamma}\left(e_j\cdot e_k\cdot\nu_\alpha\cdot\nu_\beta\right)\\
&= \frac{1}{4}\left|\mathring{A}\right|^2 + \frac{1}{2}\sum_{i=1}^{2}\left(\left\langle A^1(e_i),e_1\right\rangle\left\langle A^2(e_i),e_2\right\rangle - \left\langle A^1(e_i),e_2\right\rangle\left\langle A^2(e_i),e_1\right\rangle\right)\bar{\gamma}\left(e_1\cdot e_2\cdot\nu_1\cdot\nu_2\right)\\
&= \frac{1}{4}\left|\mathring{A}\right|^2 \pm \frac{1}{2}\left(\kappa_N - \bar{R}(e_1,e_2,\nu_1,\nu_2)\right).
\end{aligned}$$

The third part follows from the fact

$$-\frac{1}{4}\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha\right)\right)^2\geq 0.$$

Hence,

$$\frac{1}{2}\int_M\left|\mathring{A}\right|^2\geq\int_M\left|\kappa_N-\bar{R}(e_1,e_2,\nu_1,\nu_2)\right|\geq\left|2\pi\chi(N)-\int_M\bar{R}(e_1,e_2,\nu_1,\nu_2)\right|.$$

Finally, according to the Gauss equation,

$$\kappa_M=\bar{R}(e_1,e_2,e_1,e_2)+|H|^2-\frac{1}{2}\left|\mathring{A}\right|^2,$$

we obtain

$$\chi(M)+\left|\chi(N)-\frac{1}{2\pi}\int_M\bar{R}(e_1,e_2,\nu_1,\nu_2)\right|\leq\frac{1}{2\pi}\left(\int_M\bar{R}(e_1,e_2,e_1,e_2)+|H|^2\right).$$

□

Now we are in position to give the proof of our main theorem.

Proof of Theorem 1.1 For every smooth function f on M, we have the following weighted Bochner formula (cf. [6])

$$\begin{aligned}\frac{m-1}{m}&\int_M\exp(f)\left|D^{\Sigma N}\psi\right|^2\\&=\int_M\exp(f)\left(\frac{m-1}{2}\Delta f-\frac{(m-1)(m-2)}{4}|\nabla f|^2+\mathcal{R}_\psi^{\Sigma N}\right)|\psi|^2\\&\quad+\int_M\exp((1-m)f)\left|P^{\Sigma N}\left(\exp\left(\frac{m}{2}f\right)\psi\right)\right|^2,\end{aligned}\tag{3.8}$$

where

$$\mathcal{R}_\psi^{\Sigma N}|\psi|^2=\left(\mathcal{R}^{\Sigma N}\psi,\psi\right),$$

and $P^{\Sigma N}$ is the twistor operator defined by

$$P_X^{\Sigma N}\psi:=\nabla_X^{\Sigma M\otimes\Sigma N}\psi+\frac{1}{m}\underline{\gamma}(X)D^{\Sigma N}\psi,$$

and $\underline{\gamma}=\gamma\otimes\mathrm{Id}$.

According to Lemma 3.1, we have

$$\mathcal{R}_\psi^{\Sigma N}\geq\frac{S_M-(n-1)\left|\mathring{A}\right|^2}{4}-\frac{\bar{W}_{ij\alpha\beta}\left(\bar{\gamma}(e_i\cdot e_j\cdot\nu_\alpha\cdot\nu_\beta)\psi,\psi\right)}{8|\psi|^2}.\tag{3.9}$$

The second term of RHS of the above inequality vanished according to the assumption.

Suppose ψ is an eigenspinor of $D^{\Sigma N}$ associated with λ, i.e.,

$$D^{\Sigma N}\psi=\lambda\psi.$$

Inserting (3.9) into (3.8), we obtain

$$\frac{m-1}{m}\lambda^2\int_M e^f\,|\psi|^2 \geq \int_M e^f\left(\frac{m-1}{2}\Delta f-\frac{(m-1)(m-2)}{4}|\nabla f|^2+\frac{S_M-(n-1)\left|\mathring{A}^2\right|}{4}\right)|\psi|^2. \tag{3.10}$$

We consider two cases.

Case 1. $m=2$.

In this case, we choose $f\in C^\infty(M)$ as the unique solution to the following PDE

$$\Delta f+\kappa_M-\frac{n-1}{2}\mathring{A}^2=\frac{4\pi(1-g_M)}{\text{area}(M)}-\frac{(n-1)\int_M\left|\mathring{A}\right|^2}{2\,\text{area}(M)},\quad \int_M f=0,$$

on M. Therefore, according to (3.10), we get

$$\lambda^2\geq\frac{4\pi(1-g_M)}{\text{area}(M)}-\frac{(n-1)\int_M\left|\mathring{A}\right|^2}{2\,\text{area}(M)}.$$

Case 2. $m>2$.

In this case, (3.10) implies that for every positive function u,

$$\frac{m-1}{m}\lambda^2\int_M u^{1-m/(m-2)}\,|\psi|^2 \geq \int_M u^{-m/(m-2)}\left(-\frac{m-1}{m-2}\Delta u+\frac{S_M-(n-1)\left|\mathring{A}^2\right|}{4}u\right)|\psi|^2. \tag{3.11}$$

Choose $u\in C^\infty(M)$ as a positive eigenfunction of the operator L, i.e.,

$$Lu=-\frac{4(m-1)}{m-2}\Delta u+\left(S_M-(n-1)\left|\mathring{A}\right|^2\right)u=\lambda_1(L)u.$$

Moreover, after rescalling, we can choose u satisfying

$$\int_M u^2=\text{vol}(M).$$

Then the inequality (3.11) implies that

$$\lambda^2\geq\frac{m}{4(m-1)}\lambda_1(L).$$

Next, we will consider the limit case. If we suppose

$$\lambda^2=\frac{4\pi(1-g_M)}{\text{area}(M)}-\frac{(n-1)\int_M\left|\mathring{A}\right|^2}{2\,\text{area}(M)}$$

as $m=2$ is the case and

$$\lambda^2=\frac{m}{4(m-1)}\lambda_1(L)$$

as $m > 2$ is the case. On one hand, since $P^{\Sigma N}\left(\exp\left(\frac{m}{2}f\right)\psi\right) = 0$, i.e.,

$$\begin{aligned} 0 &= P_X^{\Sigma N}\left(\exp\left(\frac{m}{2}f\right)\psi\right) \\ &= \exp\left(\frac{m}{2}f\right)\left[\nabla_X^{\Sigma M\otimes\Sigma N}\psi + \frac{1}{m}\underline{\gamma}(X)D^{\Sigma N}\psi + \frac{m}{2}X(f)\psi + \frac{1}{2}\underline{\gamma}\left(X\cdot\nabla f\right)\psi\right], \quad \forall X \in TM, \end{aligned}$$

we have

$$\nabla_X^{\Sigma M\otimes\Sigma N}\psi + \frac{\lambda}{m}\underline{\gamma}(X)\psi + \frac{m}{2}X(f)\psi + \frac{1}{2}\underline{\gamma}\left(X\cdot\nabla f\right)\psi = 0. \tag{3.12}$$

Consequently, $\nabla\left(e^{(m-1)f/2}|\psi|^2\right) = 0$ which implies that $e^{(m-1)f/2}|\psi|^2$ is a nonzero constant on M. On the other hand, the equality in (3.2) gives

$$\bar{\gamma}(\mathring{A}^\alpha(e_i)\cdot\nu_\alpha)\psi = \bar{\gamma}(\mathring{A}^\beta(e_i)\cdot\nu_\beta)\psi, \quad \forall\, i, \alpha, \beta. \tag{3.13}$$

According to (3.12) and (3.13), a direct computation gives

$$\begin{aligned} \frac{m-1}{m}\left(D^{\Sigma N}\right)^2\psi &= \left(P^{\Sigma N}\right)^* P^{\Sigma N}\psi + \mathcal{R}^{\Sigma N}\psi \\ &= \left[\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}|\nabla f|^2 + \frac{S_M - (n-1)\left|\mathring{A}\right|^2}{4}\right]\psi \\ &\quad - \frac{m-1}{m}\lambda\underline{\gamma}(\nabla f)\psi. \end{aligned}$$

Notice that in the limit case,

$$\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}|\nabla f|^2 + \frac{S_M - (n-1)\left|\mathring{A}\right|^2}{4} = \frac{m-1}{m}\lambda^2.$$

We conclude that

$$\frac{m-1}{m}\lambda\underline{\gamma}(\nabla f)\psi = 0.$$

Since $\lambda \neq 0$ and $\psi \neq 0$ holds everywhere, we know that f is a constant and $f = 0$ according to the normalizing condition. Hence,

$$\nabla_X^{\Sigma M\otimes\Sigma N}\psi + \frac{\lambda}{m}\underline{\gamma}(X)\psi = 0,$$

which implies that

$$\sum_{i=1}^m \bar{\gamma}(e_i)R^{\Sigma M\otimes\Sigma N}(e_i, e_j)\psi = \frac{2(m-1)\lambda^2}{m^2}\bar{\gamma}(e_j)\psi.$$

Applying Gauss equations and Ricci equations, a stand calculation yields (cf. [11,18]),

$$\begin{aligned} &\sum_{i=1}^m \bar{\gamma}(e_i)R^{\Sigma M\otimes\Sigma N}(e_i, e_j)\psi \\ &= \frac{1}{4}\sum_{i,k,l=1}^m \left\langle R(e_i, e_j)e_k, e_l\right\rangle\bar{\gamma}(e_i\cdot e_k\cdot e_l)\psi + \frac{1}{4}\sum_{i=1}^m\sum_{\alpha,\beta=1}^n\left\langle R^\perp(e_i, e_j)\nu_\alpha, \nu_\beta\right\rangle\bar{\gamma}(e_i\cdot\nu_\alpha\cdot\nu_\beta)\psi \\ &= \frac{1}{2}\bar{\gamma}\left(Ric(e_j)\right)\psi - \frac{1}{4}\sum_{i=1}^m\sum_{\alpha=1}^n\bar{\gamma}\left(\mathring{B}(e_j, e_i)\cdot\mathring{A}^\alpha(e_i)\cdot\nu_\alpha + \mathring{A}^\alpha(e_i)\cdot\nu_\alpha\cdot\mathring{B}(e_j, e_i)\right)\psi. \end{aligned}$$

According to (3.13),

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{B}(e_j,e_i)\cdot\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha+\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha\cdot\mathring{B}(e_j,e_i)\right)\psi\\
&=\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\bar{\gamma}\left(2\mathring{B}(e_j,e_i)\cdot\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha-2\left\langle\mathring{A}^{\alpha}(e_i),e_j\right\rangle\mathring{A}^{\alpha}(e_i)\right)\psi\\
&=\bar{\gamma}\left(2\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left\langle\mathring{A}^{\beta}(e_j),e_i\right\rangle\nu_\beta\cdot\mathring{A}^{\alpha}(e_i)\cdot\nu_\alpha-2\sum_{\alpha=1}^{n}\left(\mathring{A}^{\alpha}\right)^2(e_i)\right)\psi\\
&=\bar{\gamma}\left(2n\sum_{i=1}^{m}\sum_{\beta=1}^{n}\left\langle\mathring{A}^{\beta}(e_j),e_i\right\rangle\nu_\beta\cdot\mathring{A}^{\beta}(e_i)\cdot\nu_\beta-2\sum_{\alpha=1}^{n}\left(\mathring{A}^{\alpha}\right)^2(e_i)\right)\psi\\
&=2(n-1)\sum_{\alpha=1}^{n}\bar{\gamma}\left(\left(\mathring{A}^{\alpha}\right)^2(e_i)\right)\psi.
\end{aligned}$$

Thus

$$\sum_{i=1}^{m}\bar{\gamma}(e_i)R^{\Sigma M\otimes\Sigma N}(e_i,e_j)\psi=\frac{1}{2}\bar{\gamma}\left(Ric(e_j)\right)\psi+\frac{1-n}{2}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\left(\mathring{A}^{\alpha}\right)^2(e_j)\right)\psi.$$

Summarize these identities, we get

$$\frac{1}{2}\bar{\gamma}\left(Ric(e_j)\right)\psi+\frac{1-n}{2}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\left(\mathring{A}^{\alpha}\right)^2(e_j)\right)\psi=\frac{2(m-1)\lambda^2}{m^2}\bar{\gamma}(e_j)\psi. \tag{3.14}$$

Since ψ vanish nowhere on M, then (3.14) implies that

$$Ric=(n-1)\sum_{\alpha=1}^{n}\left(\mathring{A}^{\alpha}\right)^2+\frac{4(m-1)\lambda^2}{m^2}g.$$

□

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