manuscripta math. 165, 269–282 (2021)

Qing Cui · Linlin Sun

A note on rigidity of Einstein four-manifolds with positive sectional curvature

Received: 28 January 2019 / Accepted: 9 May 2020 / Published online: 6 June 2020

Abstract. In this paper, we first prove a topological obstruction for a four-dimensional manifold carrying an Einstein metric. More precisely, assume (M, g) is a closed Einstein four-manifold with $Ric = \rho g$. Denote by K the sectional curvature of M. If $K \geq \delta \geq \frac{2\rho - \sqrt{5}|\rho|}{6}$ (or $K \leq \delta \leq \frac{2\rho+\sqrt{5}}{6}$) for some constant δ, then the Euler characteristic χ and the signature τ of M satisfy

$$\chi \geq \left(\frac{3}{8\,(1 - 3\delta/\rho)^2} + \frac{3}{2} \right) |\tau| \,.$$

Our second result is a rigidity theorem for closed oriented Einstein four-manifolds with positive sectional curvature. Assume λ_1 is the first eigenvalue of the Laplacian of an oriented closed Einstein four-manifold (M, g) with $Ric = g$. We show that M must be isometric to a round 4-sphere or $\mathbb{CP}^2$ with the (normalized) Fubini-Study metric if the sectional curvature bounded above by $1 - \frac{4}{9\lambda_1+12}$ (or bounded below by $\frac{2}{9\lambda_1+12}$).

1. Introduction

A fundamental problem in Riemannian geometry is to determine whether a four-manifold M carries an Einstein metric. There are several well known topological obstructions for M carrying an Einstein metric. Denote by χ, τ and K the Euler characteristic, signature and sectional curvature of M. Berger [3] proved an Einstein

The first author is partially supported by National Natural Science Foundation of China (Grant No. 11601442). The second author is partially supported by the National Natural Science Foundation of China (Grant Nos. 11971358, 11801420) and the Youth Talent Training Program of Wuhan University. The second author would like to thank the Max Planck Institute for Mathematics in the Sciences for good working conditions when this work carried out. Both of the two authors want to thank the anonymous referee for a very careful reading and for suggesting corrections.

Q. Cui: School of Mathematics, Southwest Jiaotong University, Chengdu 611756, Sichuan, China. e-mail: cuiqing@swjtu.edu.cn

L. Sun (✉): School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China. e-mail: sunll@whu.edu.cn

L. Sun: Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China

Mathematics Subject Classification: 53C25 · 53C24

https://doi.org/10.1007/s00229-020-01217-y

four-manifold must have $\chi \geq 0$, and $\chi \leq 9$ provided $K > 0$. Hitchin [18] showed that if M is a closed Einstein four-manifold, then $\chi \geq \frac{3}{2}|\tau|$, and $\chi \geq \left(\frac{3}{2}\right)^{\frac{3}{2}}|\tau|$ if one further assume $K \geq 0$. Gursky and Lebrun [17] proved that a closed oriented Einstein four-manifold with $K \leq 0$ must have $\chi \geq \frac{15}{8}|\tau|$, and $\chi > \frac{15}{4}|\tau|$ if M is not half-conformally flat and $K \geq 0$. Costa [12] established that an oriented closed Einstein four-manifold with Ricci curvature $\rho < 0$ and $K \geq 2/3\rho$ satisfies $\chi \geq \frac{15}{8}|\tau|$.

Inspired by the above results, we obtain a new obstruction for a four-manifold carrying an Einstein metric.

Theorem 1.1. *Let M be an oriented closed Einstein four-manifold with sectional curvature K and Ricci curvature ρ. If $K_{\min} \geq \delta \geq \frac{2\rho-\sqrt{5}|\rho|}{6}$ (or $K_{\max} \leq \delta \leq \frac{2\rho+\sqrt{5}|\rho|}{6}$), then*

$$\chi \geq \left(\frac{3}{8\,(1-3\delta/\rho)^2} + \frac{3}{2}\right)|\tau|\,.$$

Remark 1.1.

- If $K_{\min} \geq 0$ (or $K_{\max} \leq 0$), then we have $\chi \geq \frac{15}{8}|\tau|$. Thus, we generalize the results in [17,18].
- If $\rho > 0$ and $\delta = \rho/6$, then we have $\chi \geq 3|\tau|$. Note that for $\mathbb{CP}^2$ with the standard Fubini-Study metric, we have $K_{\min} = \rho/6$ and $\chi = 3|\tau|$.
- If $\rho < 0$, our assumption becomes $K_{\min} \geq \frac{\left(2+\sqrt{5}\right)\rho}{6}$ (or $K_{\max} \leq \frac{\left(2-\sqrt{5}\right)\rho}{6}$), which is also weaker than Costa's hypothesis $K_{\min} \geq \frac{2\rho}{3}$.
- If $3\delta = \rho$, M has constant curvature, $\tau = 0$. The equality is trivial.

Another important problem concerning four-manifolds is to classify them when they are Einstein and satisfy additional curvature conditions. This problem was first studied by Berger [2,3]. After that, numerous classification results under different curvature conditions have emerged. For instance, Micallef and Wang [20] proved that an Einstein four-manifold with nonnegative isotropic curvature is locally symmetric. Brendle [5] generalized Micallef and Wang's result to higher dimensions, and he also proved rigidity theorem for an Einstein manifold with positive isotropic curvature. Using similar methods, Brendle and Schoen [6,7] proved the differential sphere theorem. Wu [24] showed that an Einstein metric on a four-manifold with half nonnegative isotropic curvature must be Kähler–Einstein if it is not conformally flat. Xu and Gu [25] obtained rigidity results under the assumption that the scalar curvature is pinched by sectional curvatures. Recently, Diógenes and Ribeiro Jr. [15] showed a closed oriented four-manifold M with sectional curvature 0.16139-pinched must be definite, and very recently, Diógenes, Ribeiro Jr. and Rufino [16] showed M must be homeomorphic to S^4 or $\mathbb{CP}^2$ if it is half conformally flat and has sectional curvature 0.049-pinched.

If we restrict our attention to oriented Einstein four-manifolds with positive sectional curvatures, we have much fewer examples. Actually, the only known examples are the standard 4-sphere S^4 and the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric up to rescaling. Therefore, it is natural to conjecture:

Conjecture. [26] *The only oriented Einstein four-manifolds with positive sectional curvature are the standard four-sphere S^4 and the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric up to rescaling.*

In 2000, Yang [26] gave a partial answer to the above conjecture. He proved that oriented Einstein four-manifolds with $Ric = g$ and sectional curvature $K \geq \frac{1}{120}\left(\sqrt{1249} - 23\right) \approx 0.102843$ are half-conformally flat, and thus (cf. [4]) are isometric to S^4 or $\mathbb{CP}^2$ up to rescaling. Later, Costa [12] relaxed Yang's condition to $K \geq \frac{2-\sqrt{2}}{6} \approx 0.0976$. Recently, Wu [22,23] and Ribeiro Jr. [21] independently further relaxed this condition to $K > \frac{1}{12} \approx 0.0833$. We introduce the first eigenvalue of Laplacian λ_1 and obtain the following rigidity theorem.

Theorem 1.2. *Assume (M, g) is a closed oriented Einstein four-manifold with $Ric = g$, and λ_1 is the first eigenvalue of Laplacian. Then M must be a round 4-sphere S^4 or $\mathbb{CP}^2$ with the standard Fubini-Study metric if one of the following conditions holds:*

(i) $|x| < \left(1 + \frac{3\lambda_1}{2}\right)\left(\frac{2}{3} - a\right)$;
(ii) *the maximum of sectional curvature* $K_{\max} < 1 - \frac{4}{9\lambda_1+12}$;
(iii) *the minimum of sectional curvature* $K_{\min} > \frac{2}{9\lambda_1+12}$.

Here $a = W_{1212}$, $x = W_{1234}$ are Weyl curvatures defined in (2.1).

Remark 1.2.
- Since for a closed four-manifold (M, g), $Ric \geq g$ implies $\lambda_1 \geq 4/3$ (cf. [19]), and moreover, if g is Einstein, $\lambda_1 = 4/3$ if and only if M is isometric to S^4 (with sectional curvature $\equiv 1/3$). Therefore, if M is not isometric to S^4, then $\lambda_1 > 4/3$. Our lower bound $\frac{2}{9\lambda_1+12}$ in condition (iii) is strictly less than $1/12$. Thus, the above theorem generalizes the previous results in [12,22,23,26], and gets closer to the conjecture.
- When M has harmonic Weyl tensor (without the Einstein condition), Ribeiro Jr. [21] obtained some rigidity results with the assumption $K^{\perp} \geq \dfrac{s^2}{24(3\lambda_1 + s)}$, where $K^{\perp}$ denotes the mean sectional curvature (see the definition in [21]) and s denotes the scalar curvature of M.

The following result is an easy consequence of Theorem 1.2.

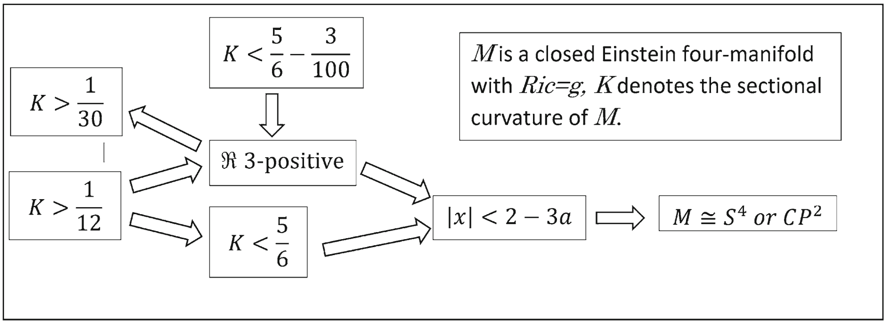
Corollary 1.3. *Assume (M, g) is a closed oriented Einstein four-manifold with $Ric = g$. Then M must be a round sphere or $\mathbb{CP}^2$ with (normalized) Fubini-Study metric if one of the following conditions holds:*

(i) $|x| < 2 - 3a$;
(ii) *the maximum of sectional curvature* $K_{\max} < \frac{5}{6}$;
(iii) *the minimum of sectional curvature* $K_{\min} > \frac{1}{12}$;
(iv) *M has 3-positive curvature operator.*

For the above corollary, it is worth remarking that

Remark 1.3. • The conclusion of Corollary 1.3 under condition (iii) and (iv) was proved by Ribeiro Jr.[21] and Wu [22] (see also [11]). But when $Ric = g$, $K_{\min} > \frac{1}{12}$ implies $K_{\max} < \frac{5}{6}$, and 3-positive curvature operator implies (will be showed in Sect. 4) $|x| < 2 - 3a$. Therefore, condition (ii) is weaker than condition (iii) and condition (i) is weaker than condition (iv).

- Cao and Tran [9] showed if $K_{max} < \dfrac{14-\sqrt{19}}{12} \approx 0.8034$, then M has rigidity, while in our condition (ii), we improve the upper bound to $5/6$.
- Wu [22] also showed $K_{\max} < \frac{5}{6} - \frac{3}{100}$ implies 3-positive curvature operator, and 3-positive curvature operator implies $K_{\min} > \frac{1}{30}$. But it is still unknown whether $K_{\max} < \frac{5}{6}$ implies (or even equivalent to) 3-positive curvature operator. We show the relations of the above 4 conditions in the following diagram.



This paper is arranged as follows. In Sect. 2, we set notations and review some known formulas and facts about Einstein four-manifolds. Section 3 is devoted to the proof of the topological obstruction theorem Theorem 1.1. Finally, in Sect. 4, we will prove the rigidity theorem Theorem 1.2 and Corollary 1.3.

Remark 1.4. This paper was finished in May 2018. Very recently, we are informed that a part of our results was also obtained by Cao and Tran [10] in September 2018 independently.

2. Preliminary

Let (M^4, g) be a four-dimensional Riemannian manifold. Let $\{e_1, e_2, e_3, e_4\}$ be an oriented local orthonormal frame of TM. Denote by $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ the dual frame of $\{e_1, e_2, e_3, e_4\}$. Set $R_{ijkl} := R(e_i, e_j, e_k, e_l)$. Notice that the Hodge star operator $* : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M$ satisfies $*^2 = 1$, and so we get a decomposition $\Lambda^2 T^*M = \Lambda^+ M \oplus \Lambda^- M$ associated with the eigenvalue $+1$ and -1 of $*$ respectively. It is easy to check that

$$\omega_1^{\pm} := \frac{\sqrt{2}}{2}\left(\eta^1 \wedge \eta^2 \pm \eta^3 \wedge \eta^4\right) \in \Lambda^{\pm} M,$$

$$\omega_2^{\pm} := \frac{\sqrt{2}}{2}\left(\eta^1 \wedge \eta^3 \pm \eta^4 \wedge \eta^2\right) \in \Lambda^{\pm} M,$$

$$\omega_3^{\pm} := \frac{\sqrt{2}}{2}\left(\eta^1 \wedge \eta^4 \pm \eta^2 \wedge \eta^3\right) \in \Lambda^{\pm} M.$$

Moreover, $\left\{\omega_1^{\pm}, \omega_2^{\pm}, \omega_3^{\pm}\right\}$ forms a local orthonormal frame of $\Lambda^{\pm} M$. Now we get the following decomposition,

$$\mathcal{R} = \begin{pmatrix} \frac{S}{12}\,\mathrm{Id} + W^+ & B \\ B^T & \frac{S}{12}\,\mathrm{Id} + W^- \end{pmatrix},$$

where S is the scalar curvature,

$$B = \frac{1}{2}\begin{pmatrix} \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{22} & \overset{\circ}{Ric}_{23} - \overset{\circ}{Ric}_{14} & \overset{\circ}{Ric}_{24} - \overset{\circ}{Ric}_{13} \\ \overset{\circ}{Ric}_{23} + \overset{\circ}{Ric}_{14} & \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{33} & \overset{\circ}{Ric}_{34} - \overset{\circ}{Ric}_{12} \\ \overset{\circ}{Ric}_{24} + \overset{\circ}{Ric}_{13} & \overset{\circ}{Ric}_{34} + \overset{\circ}{Ric}_{12} & \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{44} \end{pmatrix},$$

and

$$W^{\pm} = \begin{pmatrix} W_{1212} \pm W_{1234} & W_{1213} \pm W_{1242} & W_{1214} \pm W_{1223} \\ W_{1213} \pm W_{1242} & W_{1313} \pm W_{1342} & W_{1314} \pm W_{1323} \\ W_{1214} \pm W_{1223} & W_{1314} \pm W_{1323} & W_{1414} \pm W_{1423} \end{pmatrix}.$$

By Berger's curvature decomposition [2], without loss of generality, we may assume

$$W^{\pm} = \begin{pmatrix} W_{1212} \pm W_{1234} & 0 & 0 \\ 0 & W_{1313} \pm W_{1342} & 0 \\ 0 & 0 & W_{1414} \pm W_{1423} \end{pmatrix},$$

and $W_{1212} \pm W_{1234} \geq W_{1313} \pm W_{1342} \geq W_{1414} \pm W_{1423}$. Thus

$$B = \frac{1}{2}\begin{pmatrix} \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{22} & 0 & 0 \\ 0 & \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{33} & 0 \\ 0 & 0 & \overset{\circ}{Ric}_{11} + \overset{\circ}{Ric}_{44} \end{pmatrix}.$$

For simplicity, write

$$a = W_{1212}, \quad b = W_{1313}, \quad c = W_{1414}, \quad x = W_{1234}, \quad y = W_{1342}, \quad z = W_{1423}. \tag{2.1}$$

By the Bianchi identity and the tracelessness of W, we have

$$a + b + c = 0, \quad x + y + z = 0.$$

For closed Einstein four-manifold M (with $Ric \equiv \rho g$), $B \equiv 0$. Let $\chi(M)$ and $\tau(M)$ be the Euler characteristic and signature of M. We have the Gauss–Bonnet–Chern formula [1],

$$8\pi^2 \chi(M) = \int_M \left(\frac{2\rho^2}{3} + |W^+|^2 + |W^-|^2\right),$$

the signature formula [4],

$$12\pi^2\tau(M) = \int_M \left(\left|W^+\right|^2 - \left|W^-\right|^2\right),$$

and Bochner formula [13],

$$\frac{1}{2}\Delta\left|W^\pm\right|^2 = \left|\nabla W^\pm\right|^2 + 2\rho\left|W^\pm\right|^2 - 18\det W^\pm.$$

If we write $\alpha = (a, b, c)$, $\beta = (x, y, z)$, then we have

$$\left|W^+\right|^2 + \left|W^-\right|^2 = 2\left(|\alpha|^2 + |\beta|^2\right), \qquad \left|W^+\right|^2 - \left|W^-\right|^2 = 4\langle\alpha, \beta\rangle.$$

Therefore, the Gauss–Bonnet–Chern formula and signature formula become

$$\int_M \left(\frac{\rho^2}{3} + |\alpha|^2 + |\beta|^2\right) = 4\pi^2\chi(M), \tag{2.2}$$

$$\int_M \langle\alpha, \beta\rangle = 3\pi^2\tau(M). \tag{2.3}$$

3. A topological obstruction

In this section, we will give the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose the sectional curvature is bounded from below (or above) by δ, and write

$$\tilde{\alpha} = (a + \rho/3 - \delta, b + \rho/3 - \delta, c + \rho/3 - \delta).$$

We view $\tilde{\alpha}$, β as two vectors in Euclidean 3-space. If we put initial points of $\tilde{\alpha}$ and β to the origin, then the end points of $\tilde{\alpha}$, β lie in the planes

$$\{x + y + z = \rho + 3\delta\} \quad \text{and} \quad \{x + y + z = 0\}.$$

Moreover, note that all the components of $\tilde{\alpha}$ have the same sign. An elementary geometric calculation gives

$$|\langle\alpha, \beta\rangle| = |\langle\tilde{\alpha}, \beta\rangle| \leq \sqrt{\frac{2}{3}}\,|\tilde{\alpha}| \cdot |\beta|.$$

Combined with (2.3), this gives

$$3\pi^2|\tau(M)| \leq \sqrt{\frac{2}{3}}\int_M |\tilde{\alpha}| \cdot |\beta|. \tag{3.1}$$

On the other hand,

$$|\tilde{\alpha}|^2 = |\alpha|^2 + 3\left(\frac{\rho}{3} - \delta\right)^2.$$

Therefore, (2.2) yields,

$$\int_M \left(|\tilde{\alpha}|^2 + |\beta|^2\right) + \delta(2\rho - 3\delta)Vol(M) = 4\pi^2\chi(M). \tag{3.2}$$

If $\rho = 3\delta$, M has constant curvature, which implies $\tau(M) = 0$. Otherwise, suppose $\rho \neq 3\delta$, and define a constant $\varepsilon = \dfrac{\rho^2 + 4(\rho - 3\delta)^2}{4(\rho - 3\delta)^2}$. It is easy to check that $\varepsilon \geq \dfrac{6}{5}$ with our assumptions. The inequalities

$$|\beta|^2 \leq |\alpha|^2 \leq 6\left(\frac{\rho}{3} - \delta\right)^2,$$

together with (3.1) and (3.2) yield,

$$\begin{aligned}
4\pi^2\chi(M) &= \delta(2\rho - 3\delta)\mathrm{Vol}(M) + \varepsilon \int_M \left(\frac{2}{3}|\tilde{\alpha}|^2 + |\beta|^2\right) \\
&\quad + \int_M \left[\left(1 - \frac{2\varepsilon}{3}\right)|\tilde{\alpha}|^2 + (1-\varepsilon)|\beta|^2\right] \\
&= \left(\frac{\rho^2}{3} - \frac{2\varepsilon(\rho - 3\delta)^2}{9}\right)\mathrm{Vol}(M) + \varepsilon \int_M \left(\frac{2}{3}|\tilde{\alpha}|^2 + |\beta|^2\right) \\
&\quad + \int_M \left[\left(1 - \frac{2\varepsilon}{3}\right)|\alpha|^2 + (1-\varepsilon)|\beta|^2\right] \\
&\geq \left(\frac{\rho^2}{3} - \frac{2\varepsilon(\rho - 3\delta)^2}{9}\right)\mathrm{Vol}(M) \\
&\quad + \frac{4\varepsilon}{\sqrt{6}} \int_M |\tilde{\alpha}| \cdot |\beta| + \left(2 - \frac{5\varepsilon}{3}\right)\int_M |\alpha|^2 \\
&\geq \frac{3\varepsilon}{2}\pi^2 |\tau(M)| + \left(\frac{\rho^2}{3} - \frac{2\varepsilon(\rho - 3\delta)^2}{9}\right)\mathrm{Vol}(M) \\
&\quad + \frac{5}{3}\left(\frac{6}{5} - \varepsilon\right)\int_M |\alpha|^2 \\
&\geq \frac{3\varepsilon}{2}\pi^2 |\tau(M)| + \left(\frac{\rho^2}{3} - \frac{2\varepsilon(\rho - 3\delta)^2}{9}\right)\mathrm{Vol}(M) \\
&\quad + \left(\frac{4}{3} - \frac{10\varepsilon}{9}\right)(\rho - 3\delta)^2\mathrm{Vol}(M) \\
&= \frac{3\varepsilon}{2}\pi^2 |\tau(M)| = \frac{3\left(\rho^2 + 4(\rho - 3\delta)^2\right)}{8(\rho - 3\delta)^2} \times 4\pi^2 |\tau(M)|.
\end{aligned}$$

The conclusion follows immediately. □

Remark 3.1. If $\rho > 0$ and $W^{\pm} \neq 0$, Costa [12] showed that

$$K_{\min} < \frac{2 - \sqrt{2}}{6}\rho, \quad K_{\max} > \frac{2 + \sqrt{2}}{6}\rho,$$

and in this case,

$$\chi(M) > \frac{3\left(1+4\left(1-\frac{3\delta}{\rho}\right)^2\right)}{4\left(2\left(1-\frac{3\delta}{\rho}\right)^2-1\right)}\left|\tau(M)\right|,$$

where $\delta = K_{\min}$ or $\delta = K_{\max}$.

4. Rigidity result

Recall that in Sect. 2, by using Berger's curvature decomposition, we obtained

$$W_{1212} \pm W_{1234} \geq W_{1313} \pm W_{1342} \geq W_{1414} \pm W_{1423}. \tag{4.1}$$

With the notations in (2.1), the following inequalities can be easily derived from (4.1):

$$a \geq b \geq c, \quad |x| \leq a.$$

We first prove the following lemma which is actually more general than Theorem 1.2.

Lemma 4.1. *Assume (M, g) is a closed oriented Einstein four-manifold with $Ric = g$, λ_1 is the first positive eigenvalue of the Laplacian. If*

$$3\left(a^2 - x^2\right) - (3\lambda_1 + 4)\,a + 2\lambda_1 + \frac{4}{3} > 0, \quad \text{and} \quad a \leq \frac{1}{3}\left(1 + \frac{3\lambda_1}{4}\right), \tag{4.2}$$

then M must be a round sphere or $\mathbb{CP}^2$ with the standard Fubini–Study metric.

Proof. One can use the trick of evaluating $\Delta\left(\left|W^{\pm}\right| + \varepsilon\right)^{\alpha}$ ($\varepsilon > 0$ constant) instead of $\Delta\left|W^{\pm}\right|^{\alpha}$ if $\alpha < 2$. At the end, one lets $\varepsilon \to 0$ and get the desired result. Thus, we will omit ε and calculate directly. Here we outline the proof.

Suppose M is not half conformally flat. We have

$$\int_M \left|W^+\right|^{1/3} > 0, \quad \int_M \left|W^-\right|^{1/3} > 0.$$

Choose $r > 0$ such that

$$\int_M \left|W^+\right|^{1/3} = r\int_M \left|W^-\right|^{1/3}.$$

By the Bochner formula and refined Kato's inequality (cf. [8])

$$\left|\nabla W^{\pm}\right| \geq \sqrt{\frac{5}{3}}\left|\nabla\left|W^{\pm}\right|\right|$$

we obtain,

$$\begin{aligned}\Delta\left|W^{\pm}\right|^{1/3} &= \frac{1}{3}\left|W^{\pm}\right|^{-5/3}\left(\left|W^{\pm}\right|\Delta\left|W^{\pm}\right| - \frac{2}{3}\left|\nabla\left|W^{\pm}\right|\right|^{2}\right)\\ &= \frac{1}{3}\left|W^{\pm}\right|^{-5/3}\left(\frac{1}{2}\Delta\left|W^{\pm}\right|^{2} - \frac{5}{3}\left|\nabla\left|W^{\pm}\right|\right|^{2}\right)\\ &\geq \frac{2}{3}\left|W^{\pm}\right|^{1/3}\left(1 - 9\left|W^{\pm}\right|^{-2}\det W^{\pm}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= \frac{1}{2}\int_M \Delta\left(\left|W^{+}\right|^{2/3} + r\left|W^{-}\right|^{2/3}\right)\\ &\geq \int_M\left[\left|\nabla\left|W^{+}\right|^{1/3}\right|^{2} + \frac{2}{3}\left(1 - 9\left|W^{+}\right|^{-2}\det W^{+}\right)\left|W^{+}\right|^{2/3}\right]\\ &\quad + \int_M\left[r^{2}\left|\nabla\left|W^{-}\right|^{1/3}\right|^{2}\right.\\ &\quad \left. + \frac{2}{3}r^{2}\left(1 - 9\left|W^{+}\right|^{-2}\det W^{+}\right)\left|W^{+}\right|^{2/3}\right]\\ &= \frac{1}{2}\int_M\left[\left|\nabla\left(\left|W^{+}\right|^{1/3} + r\left|W^{-}\right|^{1/3}\right)\right|^{2}\right.\\ &\quad \left. + \left|\nabla\left(\left|W^{+}\right|^{1/3} - r\left|W^{-}\right|^{1/3}\right)\right|^{2}\right]\\ &\quad + \frac{2}{3}\int_M\left[\left(1 - 9\left|W^{+}\right|^{-2}\det W^{+}\right)\left|W^{+}\right|^{2/3}\right.\\ &\quad \left. + r^{2}\left(1 - 9\left|W^{-}\right|^{-2}\det W^{-}\right)\left|W^{-}\right|^{2/3}\right]\\ &\geq \frac{1}{2}\int_M\left[\left|\nabla\left(\left|W^{+}\right|^{1/3} + r\left|W^{-}\right|^{1/3}\right)\right|^{2} + \lambda_1\left|\left|W^{+}\right|^{1/3} - r\left|W^{-}\right|^{1/3}\right|^{2}\right]\\ &\quad + \frac{2}{3}\int_M\left[\left(1 - 9\left|W^{+}\right|^{-2}\det W^{+}\right)\left|W^{+}\right|^{2/3}\right.\\ &\quad \left. + r^{2}\left(1 - 9\left|W^{-}\right|^{-2}\det W^{-}\right)\left|W^{-}\right|^{2/3}\right]\\ &\geq \frac{\lambda_1}{2}\int_M\left[\left|W^{+}\right|^{2/3} - 2r\left|W^{-}\right|^{1/3}\left|W^{+}\right|^{1/3} + r^{2}\left|W^{-}\right|^{2/3}\right]\\ &\quad + \frac{2}{3}\int_M\left[\left(1 - 9\left|W^{+}\right|^{-2}\det W^{+}\right)\left|W^{+}\right|^{2/3}\right.\\ &\quad \left. + r^{2}\left(1 - 9\left|W^{-}\right|^{-2}\det W^{-}\right)\left|W^{-}\right|^{2/3}\right]\\ &= \frac{2}{3}\int_M\left[\left|W^{-}\right|^{-\frac{4}{3}}\left(\left(1 + \frac{3\lambda_1}{4}\right)\left|W^{-}\right|^{2} - 9\det W^{-}\right)r^{2}\right.\\ &\quad - \frac{3}{2}\lambda_1\left|W^{+}\right|^{\frac{1}{3}}\left|W^{-}\right|^{\frac{1}{3}}r\\ &\quad \left. + \left|W^{+}\right|^{-\frac{4}{3}}\left(\left(1 + \frac{3\lambda_1}{4}\right)\left|W^{+}\right|^{2} - 9\det W^{+}\right)\right].\end{aligned}$$

Now we claim that

$$\left(1+\frac{3\lambda_1}{4}\right)\left|W^{\pm}\right|^2-9\det W^{\pm}\geq 0.$$

Without loss of generality, assume $W^{\pm}\neq 0$ and $\det W^{\pm}\geq 0$. Then it is sufficient to prove

$$\frac{\det W^{\pm}}{\left|W^{\pm}\right|^2}\leq\frac{1}{9}\left(1+\frac{3\lambda_1}{4}\right).$$

On the one hand, assume

$$b+y=(t-1)(a+x),\quad c+z=-t(a+x),\quad t\in[1/2,1].$$

Then

$$\begin{aligned}\frac{\det W^{+}}{\left|W^{+}\right|^2}&=\frac{t(1-t)}{2(t^2-t+1)}\cdot(a+x)\\&\leq\max_{1/2\leq t\leq 1}\frac{t(1-t)}{2(t^2-t+1)}\cdot(a+x)\\&=\frac{a+x}{6}.\end{aligned}$$

On the other hand, the Lichnerowicz estimate [19] for the first eigenvalue of Laplacian on a closed four-manifold gives $\lambda_1\geq\frac{4}{3}$. Therefore,

$$\frac{\det W^{+}}{\left|W^{+}\right|^2}\leq\frac{a+x}{6}\leq\frac{a}{3}\leq\frac{1}{9}\left(1+\frac{3\lambda_1}{4}\right).$$

Similarly, we get

$$\frac{\det W^{-}}{\left|W^{-}\right|^2}\leq\frac{a-x}{6}\leq\frac{1}{9}\left(1+\frac{3\lambda_1}{4}\right).$$

Thus, we obtain

$$\begin{aligned}&\left|W^{-}\right|^{-\frac{4}{3}}\left(\left(1+\frac{3\lambda_1}{4}\right)\left|W^{-}\right|^2-9\det W^{-}\right)r^2-\frac{3}{2}\lambda_1\left|W^{+}\right|^{\frac{1}{3}}\left|W^{-}\right|^{\frac{1}{3}}r\\&\quad+\left|W^{+}\right|^{-\frac{4}{3}}\left(\left(1+\frac{3\lambda_1}{4}\right)\left|W^{+}\right|^2-9\det W^{+}\right)\\&\geq\left|W^{+}\right|^{-2/3}\left|W^{-}\right|^{-2/3}\Bigg[2\left(\left(1+\frac{3\lambda_1}{4}\right)\left|W^{-}\right|^2-9\det W^{-}\right)^{1/2}\\&\quad\left(\left(1+\frac{3\lambda_1}{4}\right)\left|W^{+}\right|^2-9\det W^{+}\right)^{1/2}-\frac{3}{2}\lambda_1\left|W^{+}\right|\left|W^{-}\right|\Bigg]r\end{aligned}$$

$$\geq \left|W^+\right|^{1/3}\left|W^-\right|^{1/3}\left[2\left(\left(1+\frac{3\lambda_1}{4}\right)-\frac{3(a-x)}{2}\right)^{1/2}\right.$$
$$\left.\left(\left(1+\frac{3\lambda_1}{4}\right)-\frac{3(a+x)}{2}\right)^{1/2}-\frac{3}{2}\lambda_1\right]r$$
$$=2\left|W^+\right|^{1/3}\left|W^-\right|^{1/3}\left[\left(\left(1-\frac{3a}{2}+\frac{3\lambda_1}{4}\right)^2-\frac{9x^2}{4}\right)^{1/2}-\frac{3\lambda_1}{4}\right]r.$$

Since $r>0$ is constant, we have

$$\int_M \left|W^+\right|^{1/3}\left|W^-\right|^{1/3}\left[\left(\left(1-\frac{3a}{2}+\frac{3\lambda_1}{4}\right)^2-\frac{9x^2}{4}\right)^{1/2}-\frac{3\lambda_1}{4}\right]\leq 0.$$

However, (4.2) implies

$$\left(\left(1-\frac{3a}{2}+\frac{3\lambda_1}{4}\right)^2-\frac{9x^2}{4}\right)^{1/2}-\frac{3\lambda_1}{4}>0.$$

Hence, $\left|W^+\right|^{1/3}\left|W^-\right|^{1/3}\equiv 0$. Because any Einstein metric is analytic [14], we must have $\left|W^+\right|\equiv 0$ or $\left|W^-\right|\equiv 0$, which is a contradiction. □

The geometric meaning of the condition "$3\left(a^2-x^2\right)-(3\lambda_1+4)\,a+2\lambda_1+\frac{4}{3}>0$" is not clear, but this condition can be derived from the three conditions in Theorem 1.2, and now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first show the relation

$$(iii)\Longrightarrow(ii)\Longrightarrow(i).$$

Since $Ric=g$, it is easy to see (iii) implies (ii). If $K_{\max}<1-\frac{4}{9\lambda_1+12}$, we have

$$a\leq K_{\max}-\frac{1}{3}<\frac{6\lambda_1+4}{9\lambda_1+12}.$$

Therefore,

$$|x|\leq a<\left(1+\frac{3\lambda_1}{2}\right)\left(\frac{2}{3}-a\right).$$

That is, (ii) implies (i). Thus, we only need to prove our conclusion under condition (i). By Lemma 4.1, it is sufficient to show $(i)\Longrightarrow$ (4.2).

By condition (i), since $\lambda_1\geq\frac{4}{3}$, we have

$$a<\frac{2}{3}\leq\frac{1}{3}\left(1+\frac{3\lambda_1}{4}\right).$$

Then if $a < \frac{6\lambda_1+4}{9\lambda_1+12}$, we have

$$3\left(a^2 - x^2\right) - (3\lambda_1 + 4)\,a + 2\lambda_1 + \frac{4}{3} \geq -\,(3\lambda_1 + 4)\,a + 2\lambda_1 + \frac{4}{3} > 0.$$

If $a \geq \frac{6\lambda_1+4}{9\lambda_1+12}$, by condition (i), we also have

$$\begin{aligned}
&3\left(a^2 - x^2\right) - (3\lambda_1 + 4)\,a + 2\lambda_1 + \frac{4}{3} \\
&> 3\left(a^2 - \left(\left(1 + \frac{3\lambda_1}{2}\right)\left(\frac{2}{3} - a\right)\right)^2\right) - (3\lambda_1 + 4)\,a + 2\lambda_1 + \frac{4}{3} \\
&= \frac{9\lambda_1(3\lambda_1 + 4)}{4}\left(\frac{2}{3} - a\right)\left(a - \frac{6\lambda_1 + 4}{9\lambda_1 + 12}\right) > 0.
\end{aligned}$$

This completes the proof. □

At last, we give the proof of Corollary 1.3 .

Proof of Corollary 1.3. Since $\lambda_1 \geq \frac{4}{3}$, conditions (i),(ii), (iii) are stronger than conditions (i), (ii), (iii) in Theorem 1.2. It remains to prove our conclusion under condition (iv). It is sufficient to show (iv) implies (i).

In our notations, 3-positive curvature operator equivalent to

$$1 + 2c + b - |y| > 0 \Longleftrightarrow 1 - 2a - b - |y| > 0. \tag{4.3}$$

Then

- if $b + |y| \geq 0$, we have by (4.3), $a < \frac{1}{2}$. Therefore
$$|x| \leq a < 2 - 3a.$$
- if $b + |y| \leq 0$, then $b \pm y \leq 0$. Thus, $-(b \pm y) \leq \frac{a \pm x}{2}$. Consequently, $-(b + |y|) \leq \frac{a+|x|}{2}$. Thus by (4.3) again, we have
$$1 - 2a + \frac{a + |x|}{2} \geq 1 - 2a - (b + |y|) > 0,$$
which implies $|x| < 2 - 3a$, i.e., condition (i). □

References

[1] Avez, A.: Applications de la formule de Gauss–Bonnet–Chern aux variétés à quatre dimensions. C. R. Acad. Sci. Paris **256**, 5488–5490 (1963)

[2] Berger, M.: Sur les variétés à opérateur de courbure positif. C. R. Acad. Sci. Paris **253**, 2832–2834 (1961)

[3] Berger, M.: Sur quleques variétés riemanniennes compactes d'Einstein. C. R. Acad. Sci. Paris **260**, 1554–1557 (1965)

[4] Besse, A.: Einstein manifolds. volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer, Berlin (1987). https://doi.org/10.1007/978-3-540-74311-8

[5] Brendle, S.: Einstein manifolds with nonnegative isotropic curvature are locally symmetric. Duke Math. J. **151**, 1–21 (2010). https://doi.org/10.1215/00127094-2009-061

[6] Brendle, S., Schoen, R.: Classification of manifolds with weakly 1/4-pinched curvatures. Acta Math. **200**, 1–13 (2008). https://doi.org/10.1007/s11511-008-0022-7

[7] Brendle, S., Schoen, R.: Manifolds with 1/4-pinched curvature are space forms. J. Am. Math. Soc. **22**, 287–307 (2009). https://doi.org/10.1090/S0894-0347-08-00613-9

[8] Calderbank, D., Gauduchon, P., Herzlich, M.: Refined Kato inequalities and conformal weights in Riemannian geometry. J. Funct. Anal. **173**, 214–255 (2000). https://doi.org/10.1006/jfan.2000.3563

[9] Cao, X., Tran, H.: Einstein four-manifolds of pinched sectional curvature. Adv. Math. **335**, 322–342 (2018a). https://doi.org/10.1016/j.aim.2018.07.003

[10] Cao, X., Tran, H.: Four-manifolds of pinched sectional curvature. (2018b). arXiv:1809.05158

[11] Cao, X., Wu, P.: Einstein four-manifolds of three-nonnegative curvature operator (2014). http://pi.math.cornell.edu/~cao/3nonnegative.pdf

[12] Costa, E.: On Einstein four-manifolds. J. Geom. Phys. **51**, 244–255 (2004). https://doi.org/10.1016/j.geomphys.2003.10.013

[13] Derdziński, A.: (1983) Self-dual Kähler manifolds and Einstein manifolds of dimension four. Compositio Math. 49, 405–433. http://www.numdam.org/item?id=CM_1983__49_3_405_0

[14] DeTurck, D., Kazdan, J.: Some regularity theorems in Riemannian geometry. Ann. Sci. École Norm. Sup. (4) 14, 249–260 (1981). http://www.numdam.org/item?id=ASENS_1981_4_14_3_249_0

[15] Diógenes, R., Ribeiro, E.: Four-dimensional manifolds with pinched positive sectional curvature. Dedicata Geom. (2018). https://doi.org/10.1007/s10711-018-0373-y

[16] Diógenes, R., Ribeiro, E., Rufino, E.: Four-manifolds with positive curvature (2018). arXiv:1809.06150

[17] Gursky, M., Lebrun, C.: On Einstein manifolds of positive sectional curvature. Ann. Global Anal. Geom. **17**, 315–328 (1999). https://doi.org/10.1023/A:1006597912184

[18] Hitchin, N.: Compact four-dimensional Einstein manifolds. J. Differ. Geome. **9**, 435–441 (1974). http://projecteuclid.org/euclid.jdg/1214432419

[19] Lichnerowicz, A.: Géométrie des groupes de transformations. Dunod, Paris, Travaux et Recherches Mathématiques, III. Dunod, Paris (1958)

[20] Micallef, M., Wang, M.: Metrics with nonnegative isotropic curvature. Duke Math. J. **72**, 649–672 (1993). https://doi.org/10.1215/S0012-7094-93-07224-9

[21] Ribeiro, E.: Rigidity of four-dimensional compact manifolds with harmonic Weyl tensor. Ann. Mat. Pura Appl. **4**(195), 2171–2181 (2016). https://doi.org/10.1007/s10231-016-0557-8

[22] Wu, P.: Curvature decompositions on Einstein four-manifolds. New York J. Math. **23**, 1739–1749 (2017a). http://nyjm.albany.edu:8000/j/2017/23_1739.html

[23] Wu, P.: A note on Einstein four-manifolds with positive curvature. J. Geom. Phys. **114**, 19–22 (2017b). https://doi.org/10.1016/j.geomphys.2016.11.017

[24] Wu, P.: A Weitzenböck formula for canonical metrics on four-manifolds. Trans. Am. Math. Soc. **369**, 1079–1096 (2017c). https://doi.org/10.1090/tran/6964

[25] Xu, H., Gu, J.: Rigidity of Einstein manifolds with positive scalar curvature. Math. Ann. **358**, 169–193 (2014). https://doi.org/10.1007/s00208-013-0957-7

[26] Yang, D.: Rigidity of Einstein 4-manifolds with positive curvature. Invent. Math. **142**, 435–450 (2000). https://doi.org/10.1007/PL00005792