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Calculus of Variations

Some differentiable sphere theorems

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Abstract

In this paper, we obtain several new intrinsic and extrinsic differentiable sphere theorems via Ricci flow. For intrinsic case, we show that a closed simply connected $n(\geq 4)$-dimensional Riemannian manifold M is diffeomorphic to $\mathbb{S}^n$ if one of the following conditions holds pointwisely:

$$(i)\ R_0 > \left(1 - \frac{24(\sqrt{10}-3)}{n(n-1)}\right) K_{max}; \quad (ii)\ \frac{Ric^{[4]}}{4(n-1)} > \left(1 - \frac{6(\sqrt{10}-3)}{n-1}\right) K_{max}.$$

Here K_{max}, $Ric^{[k]}$ and R_0 stand for the maximal sectional curvature, the k-th weak Ricci curvature and the normalized scalar curvature. For extrinsic case, i.e., when M is a closed simply connected $n(\geq 4)$-dimensional submanifold immersed in $\bar{M}$. We prove that M is diffeomorphic to $\mathbb{S}^n$ if it satisfies some curvature pinching conditions. The only involved extrinsic quantities in our pinching conditions are the maximal sectional curvature $\bar{K}_{max}$ and the squared norm of mean curvature vector $|H|^2$. More precisely, we show that M is diffeomorphic to $\mathbb{S}^n$ if one of the following conditions holds:

(1) $R_0 \geq \left(1 - \frac{2}{n(n-1)}\right) \bar{K}_{max} + \frac{n(n-2)}{(n-1)^2} |H|^2$, and strict inequality is achieved at some point;
(2) $\frac{Ric^{[2]}}{2} \geq (n-2)\bar{K}_{max} + \frac{n^2}{8} |H|^2$, and strict inequality is achieved at some point;
(3) $\frac{Ric^{[2]}}{2} \geq \frac{n(n-3)}{n-2} \left(\bar{K}_{max} + |H|^2\right)$, and strict inequality is achieved at some point.

It is worth pointing out that, in the proof of extrinsic case, we apply suitable complex orthonormal frame and simplify the calculations considerably. We also emphasize that both of the pinching constants in (2) and (3) are optimal for $n = 4$.

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1 Introduction

It is a basic problem in Riemannian geometry to classify closed Riamannian manifolds in the category of either topology, diffeomorphism, or isometry under some curvature conditions. Among a huge literature on this problem, the uniqueness of sphere under pinched curvatures accounts for a large proportion. One of the reasons for studying uniqueness of sphere is the simpleness of its topology. These uniqueness results are usually called topological sphere theorems (in the homeomorphism sense), differentiable sphere theorems (in the diffeomorphism sense), and isometric (or rigidity) sphere theorems (in the isometry sense).

Suppose M is a closed n-dimensional Riemannian manifold. If $n = 2$ and M has positive Gaussian curvature, then one can easily see from Gauss-Bonnet formula that M must be a topological sphere. Since the differential structure is unique on a 2-sphere, M must be diffeomorphic to a standard 2-sphere $\mathbb{S}^2$. When $n = 3$, the Riemannian curvature tensor is uniquely determined by the Ricci tensor. Hamilton [15] showed that if a closed 3-dimensional manifold has a metric with positive Ricci curvature, then it must be diffeomorphic to a spherical space form. Moreover, if M is simply connected, M must be diffeomorphic to $\mathbb{S}^3$. Hamilton [16] classified all closed 3-dimensional Riemannian manifold with nonnegative Ricci curvature. Therefore, in this paper, we focus our attention on the dimension $n \geq 4$ and study sphere theorems with pinched curvatures.

The study of sphere theorems under pinched sectional curvatures goes back to a question of Hopf. In 1951, Rauch [27] showed that a closed simply connected Riemannian manifold with globally δ-pinched ($\delta \approx 0.75$) sectional curvature is homeomorphic to a sphere. Rauch also proposed a question of what the optimal pinching constant should be. Berger [2] and Killingenberg [20] proved that $\delta = \frac{1}{4}$ is the optimal pinching constant. Since on a sphere of arbitrary dimension, the differential structure is not necessarily unique, it is natural to ask that if $\frac{1}{4}$-pinched sectional curvature is necessary for a differentiable sphere? This question was finally answered by Brendle and Schoen [8] via the Ricci flow.

Another important differentiable sphere theorem via Ricci flow is due to Böhm and Wilking [3]. They proved that closed manifolds with 2-positive curvature operator are spherical space forms. Moreover, Berger [2] classified all manifolds with weakly 1/4-pinched curvatures in the homeomorphism sense. Brendle and Schoen [7] provided a classification, up to a diffeomorphism, of all manifolds with weakly 1/4-pinched curvatures. For more sphere theorems under pinched sectional curvatures, we refer the reader to a good survey book of Brendle [6].

It is well known that the complex projective space $\mathbb{CP}^n$ with Fubini-Study metric has exactly pointwise $\frac{1}{4}$-pinched sectional curvature (see also Example 3.3). Therefore, Brendle-Schoen's theorem is optimal for even dimension. It is natural to study sphere theorems under other pinched curvature conditions. In 1990's, Yau collected some open problems and he wrote in Problem 12 (cf. [34, page 404]):

> The famous pinching problem says that on a compact simply connected manifold if $K_{min} > \frac{1}{4}K_{max}$, then the manifold is homeomorphic to a sphere. If we replace K_{max} by normalized scalar curvature, can we deduce similar pinching results?

However, classical examples (see [13, Example 1], see also Example 3.3 in this paper) show that the pinching constant is at least $\frac{n-1}{n+2}$. Therefore, the revised version of Yau's problem should be considered as a new conjecture which was formulated by Gu-Xu ([13]):

Conjecture *Let (M^n, g) be a closed simply connected Riemannian manifold. Denote by R_0 the normalized scalar curvature of M^n. If $K_{min} > \frac{n-1}{n+2}R_0$, then M^n is diffeomorphic to a standard sphere $\mathbb{S}^n$.*

If $K_{min} > \left(1 - \frac{6}{n^2-n+6}\right) R_0,\ n \geq 4$, Gu and Xu [13] proved M must be diffeomorphic to a standard sphere, which partially answered Yau's problem. Moreover, if M is an Einstein manifold, Gu and Xu [32] proved the pinching constant $\frac{n-1}{n+2}$ is optimal and gave an isometric sphere theorem. When the dimension $n = 4$, Costa and Ribeiro Jr. [10] proved Yau's conjecture. They actually used a weaker assumption by replacing sectional curvature by biorthogonal curvature condition. We can prove when $K_{min} > \left(1 - \frac{12}{n^2-n+12}\right) R_0$, M must be diffeomorphic to $\mathbb{S}^n$. However, when we finish this paper, we know from Professor Hong-Wei Xu that he and his collaborators obtained the same result [14] independently. We would like to thank Professor Hong-Wei Xu for sending their manuscript [14]. For readers' convenience, we still give a complete proof of this result in Sect. 3 (see Theorem 3.2).

It is also interesting to study sphere theorems with normalized scalar curvature pinched by K_{max}. Gu and Xu [13, Theorem 1] showed that if $R_0 > \frac{12}{5n(n-1)}K_{max}$, $n \geq 4$, then M is diffeomorphic to a spherical space form. Based on an example of $\mathbb{OP}^2$, the authors also posed a Conjecture (see [13, Conjecture 1]):

Conjecture *Let $M^n (n \geq 4)$ be a closed and simply connected Riemannian manifold. If $R_0 > \frac{3}{5}K_{max}$, then M is diffeomorphic to $\mathbb{S}^n$.*

We also get a new differentiable sphere theorem in this direction:

Theorem 1.1 *Let $M^n (n \geq 4)$ be a closed and simply connected Riemannian manifold. If*

$$R_0 > \left(1 - \frac{24(\sqrt{10}-3)}{n(n-1)}\right) K_{max},$$

then M is diffeomorphic to $\mathbb{S}^n$.

Remark 1.1 Under the assumtion

$$R_0 > \left(1 - \frac{6}{n(n-1)}\right) K_{max},$$

we can prove M has positive isotropic curvature, see Remark 3.1. Gu-Xu-Zhao [14] also obtained this result independently.

For pinched Ricci curvature and sectional curvature, we also have the following sphere theorem.

Theorem 1.2 *Let $M^n (n \geq 4)$ be a closed and simply connected Riemannian manifold. If*

$$\frac{Ric_M^{[4]}}{4(n-1)} > \left(1 - \frac{6(\sqrt{10}-3)}{n-1}\right) K_{max},$$

then M is diffeomorphic to $\mathbb{S}^n$.

Remark 1.2 Gu-Xu-Zhao [14] actually proved M is diffeomorphic to $\mathbb{S}^n$ when M satisfies

$$\frac{Ric_M}{n-1} > \left(1 - \frac{3}{2(n-1)}\right) K_{max}.$$

It is also of interest to study sphere theorems for submanifolds. In recent years, many authors investigated related problems and plenty of works were obtained (e.g. [1,13,17, 18,22,30–33] and therein). We also get sphere theorems for submanifolds corresponding to Theorem 1.1 and Theorem 1.2, see Theorem 4.2, Theorem 4.1 and Theorem 4.3. Besides these results, we use complex orthonormal frames to obtain the following new sphere theorems. The assumptions of these theorems only involve R_0, $Ric^{[2]}$, $\bar{K}_{max}$ and $|H|^2$.

We prove the following three theorems which are generalizations of Gu-Xu's results [13, Theorem 3, Theorem 4], Xu-Gu's result [30, Theorem 1.1], Anderws-Baker's result [1, Theorem 1], Liu-Xu-Ye-Zhao's result [22, Corollary 1.2] and Xu-Tian's result [33, Theorem 1.1].

Theorem 1.3 *Suppose $M^n(n \geq 4)$ is a closed and simply connected submanifold of $\bar{M}^N$ satisfying*

$$R_0 \geq \left(1 - \frac{2}{n(n-1)}\right) \bar{K}_{max} + \frac{n(n-2)}{(n-1)^2} |H|^2,$$

with strict inequality at some point, then M is diffeomorphic to $\mathbb{S}^n$.

Theorem 1.4 *Suppose $M^n(n \geq 4)$ is a closed and simply connected submanifold of $\bar{M}^N$ satisfying*

$$\frac{Ric^{[2]}}{2} \geq (n-2)\bar{K}_{max} + \frac{n^2}{8} |H|^2,$$

with strict inequality at some point, then M is diffeomorphic to $\mathbb{S}^n$.

The pinching condition in Theorem 1.4 is optimal. In fact, when $\bar{M}$ is the space form $F^N(c), c > 0$, Ejiri [11] obtained a rigidity theorem for minimal submanifolds under the pinching condition

$$Ric_M > (n-2)c.$$

Xu-Gu [31] obtained an extension of Ejiri's results for constant mean curvature submanifolds in the space form $F^N(c)$ under the condition

$$Ric_M > (n-2)\left(c + |H|^2\right) > 0.$$

They also obtained a topological sphere theorem for general submanifolds in the space form $F^N(c), c \geq 0$ under the same pinching condition mentioned above by using Lawson-Simons theory for stable integral currents [21,29]. Motivated by these facts, the authors posed the following Conjecture (cf., [31, Conjecture A]):

Conjecture *Let $M^n(n \geq 4)$ be a closed and simply connected orientated submanifold in the space form $F^N(c)$. If $Ric_M > (n-2)\left(c + |H|^2\right) > 0$, then M is diffeomorphic to $\mathbb{S}^n$.*

Here is a generalization of Gu-Xu's result [31, Theorem 4.2].

Theorem 1.5 *Suppose $M^n(n \geq 4)$ is a closed and simply connected submanifold of $\bar{M}^N$ satisfying*

$$\frac{Ric^{[2]}}{2} \geq \frac{n(n-3)}{n-2} \left(\bar{K}_{max} + |H|^2\right),$$

with strict inequality at some point, then M is diffeomorphic to $\mathbb{S}^n$.

Remark 1.3 The Bonnet-Myers theorem [25] states that every complete Riemannian manifold with Ricci curvature bounded from below by a positive constant is compact. For complete noncompact Riemannian manifold with quasi-positive sectional curvature, the soul theorem [9,12,26] says that such manifold is diffeomorphic to the Euclidean space. Thus, one can consider the sphere theorems for complete Riemannian manifolds with similar curvature pinching conditions in the above theorems.

This paper is organized as follows. In Sect. 2, we list some notations and known facts. In Sect. 3, we prove some intrinsic differentiable sphere theorems with pinched normalized scalar curvatures and pinched Ricci curvatures. In Sect. 4, we study a Riemannian manifold immersed into another and give several new extrinsic topological sphere theorems and differentiable sphere theorems.

2 Preliminaries

In this section, we will fix some notations and list several known facts which will be used in next two sections.

Let $(M^n, \langle\ ,\ \rangle)$ be a Riemannian manifold, ∇ be the Levi-Civita connection related to $\langle\ ,\ \rangle$ and R be the Riemannian curvature tensor defined by

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \forall X, Y \in TM.$$

Denote

$$R(X, Y, Z, W) := \langle R(X, Y)W, Z\rangle .$$

Define

$$K(X, Y) := R(X, Y, X, Y), \quad \forall X, Y \in TM.$$

Denote $K(X, Y)$ by $K(\pi)$ if X, Y are orthonormal and $\pi = span\{X, Y\}$. By the linearity and symmetry of R, it is easy to check the following identities.

Lemma 2.1 *For all $X, Y, Z, W \in TM$ and $a, b \in \mathbb{R}$, we have*

$$\begin{aligned} K(X+Y, X-Y) &= 4K(X, Y), \\ K(X, Y+Z) + K(X, Y-Z) &= 2(K(X, Y) + K(X, Z)), \\ K(aX, bY) &= a^2b^2K(X, Y), \\ 4R(X, Y, X, Z) &= K(X, Y+Z) - K(X, Y-Z), \qquad (2.1) \\ 24R(X, Y, Z, W) &= K(X+Z, Y+W) + K(X-Z, Y-W) \\ &\quad + K(Y+Z, X-W) \\ &\quad + K(Y-Z, X+W) - K(X+Z, Y-W) \\ &\quad - K(X-Z, Y+W) \\ &\quad - K(Y+Z, X+W) - K(Y-Z, X-W). \qquad (2.2) \end{aligned}$$

Identities (2.1) and (2.2) actually were first used by Karcher [19] to give a short proof of Berger's curvature tensor estimate.

Let $\left(\bar{M}^N, \bar{g}\right)$ $(N \geq n)$ be another Riemannian manifold such that there exists an isometric immersion

$$f : \left(M^n, \langle\ ,\ \rangle\right) \to \left(\bar{M}^N, \bar{g}\right).$$

When we do calculation on the submanifold, we always omit f and also write $\bar{g}$ as $\langle\ ,\ \rangle$. Let $\{e_1, \ldots, e_N\}$ be a local orthonormal frame on $\bar{M}$ such that $\{e_1, \ldots, e_n\}$ form a local orthonormal frame of M. Let $\left\{\omega^1, \ldots, \omega^n\right\}$ be the coframe of $\{e_1, \ldots, e_n\}$. Define $\bar{R}$ and $\bar{K}$ on $\bar{M}$ similarly as those on M. In what follows, without special explanation, i, j, k, l will always range from 1 to n and α, β, γ will always range from $n+1$ to N. The second fundamental form is defined to be

$$B = h_{ij}^{\alpha}\omega^i \otimes \omega^j \otimes e_\alpha.$$

The squared norm of B is $|B|^2 = \sum_{i,j,\alpha}\left(h_{ij}^{\alpha}\right)^2$. Write $H^\alpha = \frac{1}{n}\sum_i h_{ii}^{\alpha}$, the mean curvature vector is given by $\mathbf{H} = H^\alpha e_\alpha$, and the (normalized) mean curvature is $H = \sqrt{\sum_\alpha (H^\alpha)^2}$.

The Gauss equation can be written as

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_\alpha \left(h_{ik}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jk}^{\alpha}\right),$$

where $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and $\bar{R}_{ijkl} = \bar{R}(e_i, e_j, e_k, e_l)$. In tensor language, Gauss equation also can be written as

$$R = \bar{R}^T + \frac{1}{2}\sum_\alpha h^\alpha \owedge h^\alpha := \bar{R}^T + \frac{1}{2}B \owedge B, \tag{2.3}$$

where $\bar{R}^T$ means the restriction of $\bar{R}$ on TM, $\owedge$ denotes the Kulkarni–Nomizu product of two symmetric (0,2)-tensor a and b which defined in local coordinates by

$$(a \owedge b)_{ijkl} := a_{ik}b_{jl} - a_{il}b_{jk} - a_{jk}b_{il} + a_{jl}b_{ik}.$$

Fix $p \in M$, $X, Y \in T_pM$, the following notations will be used throughout this paper:

$$K_{min}(p) = \min_{\pi \subset T_pM} K(\pi), \qquad K_{max}(p) = \max_{\pi \subset T_pM} K(\pi),$$

$$Ric(X, Y) = \sum_i R(X, e_i, Y, e_i), \quad Ric_{jj} = Ric(e_j, e_j), \quad R_0 = \frac{\sum_{i,j} R_{ijij}}{n(n-1)},$$

$$\left[e_{i_1}, \ldots, e_{i_k}\right] = span\left\{e_{i_1}, \ldots, e_{i_k}\right\}, \qquad \forall 1 \le i_1 < i_2 < \ldots < i_k \le n,$$

$$Ric^{[k]}\left[e_{i_1}, \ldots, e_{i_k}\right] = \sum_{j=1}^{k} Ric_{i_j i_j}, \qquad Ric_{min}^{[k]}(p) = \min_{\left[e_{i_1}, \ldots, e_{i_k}\right] \subset T_pM} Ric^{[k]}\left[e_{i_1}, \ldots, e_{i_k}\right](p),$$

where $Ric^{[k]}\left[e_{i_1}, \ldots, e_{i_k}\right]$ is called *k-th weak Ricci curvature* of $\left[e_{i_1}, \ldots, e_{i_k}\right]$ which was first introduced by Gu-Xu in [13]. One can also give similar notations as above on $\bar{M}$. Since all our calculations is local (at p), we will always omit the letter "p" in what follows.

Complexify TM to $T^{\mathbb{C}}M$ and assume $\varepsilon_1, \ldots, \varepsilon_n$ is a local orthonormal frame of $T^{\mathbb{C}}M$. Extend R, $\bar{R}$, B and $\langle\ ,\ \rangle$ $\mathbb{C}$-linearly and denote by

$$h_{i\bar{j}}^{\alpha} = \left\langle B(\varepsilon_i, \bar{\varepsilon}_j), e_\alpha\right\rangle, \quad R_{ij\bar{i}\bar{j}} = R(\varepsilon_i, \varepsilon_j, \bar{\varepsilon}_i, \bar{\varepsilon}_j), \quad Ric_{i\bar{i}} = \sum_{j=1}^{n} R_{ij\bar{i}\bar{j}}.$$

It is easy to check

$$h^{\alpha}_{i\bar{i}} \in \mathbb{R}, \quad h^{\alpha}_{i\bar{j}} = \overline{h^{\alpha}_{\bar{i}j}}, \quad R_{ij\bar{i}\bar{j}} \in \mathbb{R}, \quad \sum_{i,j=1}^{n} R_{ij\bar{i}\bar{j}} = n(n-1)R_0.$$

A direct computation via the complex linearity gives the following complex Gauss equation, for $i \neq j$,

$$\begin{aligned} R_{ij\bar{i}\bar{j}} =& \bar{R}_{ij\bar{i}\bar{j}} + \sum_{\alpha} \left(h^{\alpha}_{i\bar{i}} h^{\alpha}_{j\bar{j}} - h^{\alpha}_{i\bar{j}} h^{\alpha}_{\bar{i}j} \right) \\ =& \bar{R}_{ij\bar{i}\bar{j}} + |H|^2 + \sum_{\alpha} \left(H^{\alpha} \left(\mathring{h}^{\alpha}_{i\bar{i}} + \mathring{h}^{\alpha}_{j\bar{j}} \right) + \mathring{h}^{\alpha}_{i\bar{i}} \mathring{h}^{\alpha}_{j\bar{j}} - \left| \mathring{h}^{\alpha}_{i\bar{j}} \right|^2 \right), \end{aligned} \tag{2.4}$$

where $\mathring{h}^{\alpha}_{i\bar{j}} = h^{\alpha}_{i\bar{j}} - H^{\alpha}\delta_{i\bar{j}}$. Therefore, the complex Ricci curvature is given by

$$Ric_{i\bar{i}} = \sum_{j=1}^{n} \bar{R}_{ij\bar{i}\bar{j}} + (n-1)|H|^2 + \sum_{\alpha} \left((n-2) H^{\alpha} \mathring{h}^{\alpha}_{i\bar{i}} - \sum_{k=1}^{n} \left| \mathring{h}^{\alpha}_{i\bar{K}} \right|^2 \right). \tag{2.5}$$

The curvature operator $\mathcal{R} : \Lambda^2 TM \longrightarrow \Lambda^2 TM$ is defined as follows:

$$\langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle := R(X, Y, Z, W).$$

A linear subspace $V \in T^{\mathbb{C}}M$ is called *totally isotropic* if $g(v, v) = 0$, for all $v \in V$. In other words, for all $v = X + \sqrt{-1}Y \in V$,

$$|X|^2 - |Y|^2 = \langle X, Y \rangle = 0.$$

To each complex 2-plane $\sigma \in \Lambda^2 T^{\mathbb{C}}M$ the complex sectional curvature $K(\sigma)$ is defined to be

$$K(\sigma) := \frac{\langle \mathcal{R}(z \wedge w), \bar{z} \wedge \bar{w} \rangle}{|z \wedge w|^2},$$

where $\sigma = span_{\mathbb{C}}\{z, w\}$. It is obvious that $K(\sigma) \in \mathbb{R}$. $K(\sigma)$ is called *isotropic curvature* if σ is totally isotropic. The concept of isotropic curvature was first introduced by Micallef and Moore [24].

It is easy to check that, for every totally isotropic 2-plane, there exists an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, such that

$$\sigma = span_{\mathbb{C}} \left\{ e_1 + \sqrt{-1}e_2, e_3 + \sqrt{-1}e_4 \right\}.$$

Moreover, by $\mathbb{C}$-linearity of $\mathcal{R}$ and $\langle\ ,\ \rangle$, we have

$$\begin{aligned} &4K(\sigma) \\ &= \left\langle \mathcal{R}\left((e_1 + \sqrt{-1}e_2) \wedge (e_3 + \sqrt{-1}e_4) \right), (e_1 - \sqrt{-1}e_2) \wedge (e_3 - \sqrt{-1}e_4) \right\rangle \\ &= \left\langle \mathcal{R}\left(e_1 \wedge e_3 - e_2 \wedge e_4 + \sqrt{-1}(e_1 \wedge e_4 + e_2 \wedge e_3) \right) \right. \\ &\qquad \left. e_1 \wedge e_3 - e_2 \wedge e_4 - \sqrt{-1}(e_1 \wedge e_4 + e_2 \wedge e_3) \right\rangle \\ &= \langle \mathcal{R}(e_1 \wedge e_3 - e_2 \wedge e_4), e_1 \wedge e_3 - e_2 \wedge e_4 \rangle \\ &\quad + \langle \mathcal{R}(e_1 \wedge e_4 + e_2 \wedge e_3), e_1 \wedge e_4 + e_2 \wedge e_3 \rangle \end{aligned}$$

$$= R_{1313} + R_{2424} - 2R_{1324} + R_{1414} + R_{2323} + 2R_{1423}$$
$$= R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234},$$

where we have used Bianchi identity in the last equality. When M has positive isotropic curvature, Micallef and Moore proved the following theorem.

Theorem A (Micallef-Moore [23]) *Let M be a closed $n(\geq 4)$-dimensional Riemannian manifold. Assume for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, the following inequality holds*

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0.$$

Then $\pi_k(M) = 0$ for $2 \leq k \leq \left[\frac{n}{2}\right]$. In particular, if M is simply connected, then M is homeomorphic to a sphere.

When $M \times \mathbb{R}$ has positive isotropic curvature, i.e., (cf. [4])

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0 \tag{2.6}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$, we have the following differentiable sphere theorem.

Theorem B (Brendle [4]) *Let (M, g_0) be a closed Riemannian manifold of dimension $n \geq 4$ such that $M \times \mathbb{R}$ has positive isotropic curvature. Then the normalized Ricci flow with initial metric g_0 exists for all time and converges to a constant curvature metric as $t \to \infty$.*

Remark 2.1 Theorem B is also true if one can verify inequality (2.6) for $\lambda \in [0, 1]$. Actually, if inequality (2.6) holds for $\lambda \in [0, 1]$, then for $\mu \in [-1, 0]$, consider orthonormal four-frame $\{e_1, e_2, e_3, -e_4\}$, we have

$$R_{1313} + \mu^2 R_{1414} + R_{2323} + \mu^2 R_{2424} - 2\mu R_{1234}$$
$$= R_{1313} + \mu^2 R_{1414} + R_{2323} + \mu^2 R_{2424} - 2(-\mu) R(e_1, e_2, e_3, -e_4) > 0.$$

Seshadri [28] studied the classification of closed Riemannian manifolds with nonnegative isotropic curvature. When $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, i.e., (cf. [8])

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0, \tag{2.7}$$

for all points $p \in M$, all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_p M$, and all $\lambda, \mu \in [-1, 1]$, or equivalently M has nonnegative complex sectional curvature (cf. [24, Remark 3.3] or [6, Proposition 7.18]), we have the following classification theorem.

Theorem C (Brendle-Schoen [7]) *Let M be a closed, locally irreducible Riemannian manifold of dimension $n \geq 4$. If $M \times \mathbb{R}^2$ has nonnegative isotropic curvature, then one of the following statements holds:*

(i) *M is diffeomorphic to a spherical space form;*
(ii) *$n = 2m$ and the universal cover of M is a Kähler manifold biholomorphic to $\mathbb{C}P^m$;*
(iii) *the universal cover of M is isometric to a compact symmetric space.*

Remark 2.2 Similar to the remark after Theorem B, this classification theorem is true if we can verify the condition (2.7) for all four-frame $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [0, 1]$.

3 Sphere theorems for pinched curvatures

In this section, we will prove the intrinsic sphere theorems listed in the introduction. Before we prove these theorems, we give a useful lemma.

Lemma 3.1 *Let $\{e_1, e_2, e_3, e_4\}$ be any orthonormal four-frame, then we have*

$$\begin{aligned}12R_{1234} =&4\sum_{1\le i<j\le 4} R_{ijij} - 2\left(R_{1313}+R_{1414}+R_{2323}+R_{2424}\right)\\&-\left(K(e_1+e_3, e_2-e_4)+K(e_1-e_3, e_2+e_4)\right.\\&\left.+K(e_2+e_3, e_1+e_4)+K(e_2-e_3, e_1-e_4)\right).\end{aligned}$$

Proof First note that

$$\left\{\frac{e_1+e_3}{\sqrt{2}}, \frac{e_1-e_3}{\sqrt{2}}, \frac{e_2+e_4}{\sqrt{2}}, \frac{e_2-e_4}{\sqrt{2}}\right\}, \quad \left\{\frac{e_1+e_4}{\sqrt{2}}, \frac{e_1-e_4}{\sqrt{2}}, \frac{e_2+e_3}{\sqrt{2}}, \frac{e_2-e_3}{\sqrt{2}}\right\}$$

are two orthonormal basises of span $\{e_1, e_2, e_3, e_4\}$. Therefore, by Lemma 2.1, we have

$$\begin{aligned}4\sum_{1\le i<j\le 4} R_{ijij} =&K(e_1+e_3, e_1-e_3)+K(e_1+e_3, e_2+e_4)+K(e_1+e_3, e_2-e_4)\\&+K(e_1-e_3, e_2+e_4)+K(e_1-e_3, e_2-e_4)+K(e_2+e_4, e_2-e_4).\end{aligned} \tag{3.1}$$

$$\begin{aligned}4\sum_{1\le i<j\le 4} R_{ijij} =&K(e_1+e_4, e_1-e_4)+K(e_1+e_4, e_2+e_3)+K(e_1+e_4, e_2-e_3)\\&+K(e_1-e_4, e_2+e_3)+K(e_1-e_4, e_2-e_3)+K(e_2+e_3, e_2-e_3).\end{aligned} \tag{3.2}$$

Set $X=e_1, Y=e_2, Z=e_3, W=e_4$ in (2.2), we have

$$\begin{aligned}&24R_{1234}\\&=K(e_1+e_3, e_2+e_4)+K(e_1-e_3, e_2-e_4)\\&\quad+K(e_2+e_3, e_1-e_4)+K(e_2-e_3, e_1+e_4)\\&\quad-K(e_1+e_3, e_2-e_4)-K(e_1-e_3, e_2+e_4)\\&\quad-K(e_2+e_3, e_1+e_4)-K(e_2-e_3, e_1-e_4)\\&=K(e_1+e_3, e_1-e_3)+K(e_1+e_3, e_2+e_4)+K(e_1+e_3, e_2-e_4)\\&\quad+K(e_1-e_3, e_2+e_4)+K(e_1-e_3, e_2-e_4)+K(e_2+e_4, e_2-e_4)\\&\quad+K(e_1+e_4, e_1-e_4)+K(e_1+e_4, e_2+e_3)+K(e_1+e_4, e_2-e_3)\\&\quad+K(e_1-e_4, e_2+e_3)+K(e_1-e_4, e_2-e_3)+K(e_2+e_3, e_2-e_3)\\&\quad-2(K(e_1+e_3, e_2-e_4)+K(e_1-e_3, e_2+e_4\\&\quad+K(e_2+e_3, e_1+e_4)+K(e_2-e_3, e_1-e_4))\\&\quad-K(e_1+e_3, e_1-e_3)-K(e_2+e_4, e_2-e_4)\\&\quad-K(e_1+e_4, e_1-e_4)-K(e_2+e_3, e_2-e_3)\\&=8\sum_{1\le i<j\le 4} R_{ijij}-4\left(R_{1313}+R_{1414}+R_{2323}+R_{2424}\right)\\&\quad-2(K(e_1+e_3, e_2-e_4)+K(e_1-e_3, e_2+e_4)\end{aligned}$$

$$+K(e_2+e_3,e_1+e_4)+K(e_2-e_3,e_1-e_4)).$$

In the last equality, we have used (3.1) and (3.2). □

The following theorem obtained by Gu-Xu-Zhao [14] independently. We list a proof here for reader's convenience.

Theorem 3.2 *Let $M^n(n \geq 4)$ be a closed and simply connected Riemannian manifold. Assume the following pinching condition holds,*

$$K_{min} > \left(1 - \frac{12}{n^2-n+12}\right) R_0,$$

then M is diffeomorphic to $\mathbb{S}^n$.

Proof of Theorem Theorem 3.2 By Theorem B, it is sufficient to prove (2.6) holds for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and $\lambda \in [0, 1]$. By Lemma 3.1, we have

$$\begin{aligned}
&12\,(R_{1313}+R_{2323}+R_{1234})\\
&\quad = 4\sum_{1\le i<j\le 4} R_{ijij} - 2\,(R_{1414}+R_{2424}) + 10\,(R_{1313}+R_{2323})\\
&\qquad - (K(e_1+e_3,e_2-e_4)+K(e_1-e_3,e_2+e_4))\\
&\qquad - (K(e_2+e_3,e_1+e_4)+K(e_2-e_3,e_1-e_4))\\
&\quad = 4\sum_{1\le i<j\le 4} R_{ijij} - 2\,(R_{1414}+R_{2424}) + 10\,(R_{1313}+R_{2323})\\
&\qquad - \left(4\sum_{1\le i<j\le 4} R_{ijij} - K(e_1+e_3,e_2+e_4) - K(e_1-e_3,e_2-e_4) - 4R_{1313} - 4R_{2424}\right)\\
&\qquad - \left(4\sum_{1\le i<j\le 4} R_{ijij} - K(e_2+e_3,e_1-e_4) - K(e_2-e_3,e_1+e_4) - 4R_{2323} - 4R_{1414}\right)\\
&\quad = -4\sum_{1\le i<j\le 4} R_{ijij} + 2\,(R_{1414}+R_{2424}) + 14\,(R_{1313}+R_{2323})\\
&\qquad + K(e_1+e_3,e_2+e_4)+K(e_1-e_3,e_2-e_4)\\
&\qquad + K(e_2+e_3,e_1-e_4)+K(e_2-e_3,e_1+e_4),
\end{aligned}$$

where in the second equality, we have used (3.1) and (3.2). Thus,

$$\begin{aligned}
&12\,(R_{1313}+R_{2323}+R_{1234})\\
&\quad \geq -2\left(n(n-1)R_0 - 2\sum_{i=1}^{4}\sum_{j=5}^{n} R_{ijij} - \sum_{5\le i,j\le n} R_{ijij}\right) + 48K_{min}\\
&\quad \geq -2\,[n(n-1)R_0 - (2\times 4(n-4) + (n-4)(n-5))\,K_{min}] + 48K_{min}\\
&\quad = 2\,(-n(n-1)R_0 - (n(n-1)+12)\,K_{min})\,.
\end{aligned}$$

Hence, if

$$K_{min} > \left(1 - \frac{12}{n^2-n+12}\right) R_0,$$

we obtain

$$R_{1313} + R_{2323} + R_{1234} > 0.$$

Replace e_4 by $-e_4$, we obtain

$$R_{1313} + R_{2323} - R_{1234} > 0.$$

Hence,

$$R_{1313} + R_{2323} - |R_{1234}| > 0,$$

Similarly,

$$R_{1414} + R_{2424} - |R_{1234}| > 0.$$

Therefore,

$$R_{1313} + R_{2323} + \lambda^2(R_{1414} + R_{2424}) > (1 + \lambda^2)\,|R_{1234}| \geq 2\lambda R_{1234}.$$

Our conclusion follows immediately from Theorem B. □

When the dimension $n = 4$, the following example indicates that our pinching constant is optimal.

Example 3.3 Consider the Fubini-Study metric on $\mathbb{C}P^m$, then we have

$$R(X, Y, X, Y) = 1 + 3\,|\langle JX, Y\rangle|^2\,,$$

for every orthonormal two-frame $\{X, Y\}$, where J is the complex structure. Let $n = 2m$, consider a local orthonormal frame $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$, we have

$$\begin{aligned}
R(e_i, e_j, e_i, e_j) &= 1, \quad \forall 1 \leq i \neq j \leq m,\\
R(e_i, Je_j, e_i, Je_j) &= 1, \quad \forall 1 \leq i \neq j \leq m,\\
R(e_i, Je_i, e_i, Je_i) &= 4, \quad \forall 1 \leq i \leq m,\\
R(Je_i, Je_j, Je_i, Je_j) &= 1, \quad \forall 1 \leq i \neq j \leq m
\end{aligned}$$

Therefore,

$$\begin{aligned}
s =& 4m(m-1) + 8m = n(n+2), \quad Ric_M = 2m + 2 = n + 2, \quad K_{min} = 1, \quad K_{max} = 4,\\
R_0 =& \frac{s}{n(n-1)} = \frac{n+2}{n-1} = \frac{Ric_M}{n-1}, \quad K_{min} = \frac{n-1}{n+2}R_0 = \frac{n-1}{n+2}\frac{Ric_M}{n-1},\\
R_0 =& \frac{Ric_M}{n-1} = \frac{n+2}{4(n-1)}K_{max}.
\end{aligned}$$

When $n = 4$, we have

$$R_0 = \frac{Ric_M}{3} = \frac{1}{2}K_{max}, \quad K_{min} = \frac{1}{2}R_0.$$

Proof of Theorem 1.1 We begin with the following identity:

$$n(n-1)R_0 = \sum_{i,j=5}^{n} R_{ijij} + 2\sum_{i=1}^{4}\sum_{j=5}^{n} R_{ijij} + 2\sum_{1\leq i<j\leq 4} R_{ijij}. \tag{3.3}$$

Notice that, for $\lambda \in [0, 1]$ and $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{1\le i<j\le 4} R_{ijij} \\
&= \sum_{1\le i<j\le 4} R_{ijij} - \frac{\varepsilon}{2(1+\lambda^2)}\left((R_{1313}+R_{2323}) + \lambda^2 (R_{1414}+R_{2424}) - 2\lambda R_{1234}\right) \\
&\quad + \frac{\varepsilon}{2(1+\lambda^2)}\left((R_{1313}+R_{2323}) + \lambda^2 (R_{1414}+R_{2424}) - 2\lambda R_{1234}\right) \\
&= R_{1212} + R_{3434} + \left(1 - \frac{\varepsilon}{2(1+\lambda^2)}\right)(R_{1313}+R_{2323}) \\
&\quad + \left(1 - \frac{\varepsilon\lambda^2}{2(1+\lambda^2)}\right)(R_{1414}+R_{2424}) + \frac{\varepsilon\lambda}{1+\lambda^2} R_{1234} \\
&\quad + \frac{\varepsilon}{2(1+\lambda^2)}\left((R_{1313}+R_{2323}) + \lambda^2 (R_{1414}+R_{2424}) - 2\lambda R_{1234}\right). \qquad (3.4)
\end{aligned}
$$

According to Lemma 3.1, replace e_4 by $-e_4$, we obtain

$$
\begin{aligned}
12R_{1234} = &-4(R_{1212}+R_{3434}) - 2(R_{1313}+R_{1414}+R_{2323}+R_{2424}) \\
&+ (K(e_1+e_3, e_2+e_4) + K(e_1-e_3, e_2-e_4) \\
&+ K(e_2+e_3, e_1-e_4) + K(e_2-e_3, e_1+e_4)). \qquad (3.5)
\end{aligned}
$$

Therefore, for fixed $\varepsilon_0 = 12(\sqrt{10}-3)$,

$$
\begin{aligned}
&R_{1212} + R_{3434} + \left(1 - \frac{\varepsilon_0}{2(1+\lambda^2)}\right)(R_{1313}+R_{2323}) \\
&\quad + \left(1 - \frac{\varepsilon_0\lambda^2}{2(1+\lambda^2)}\right)(R_{1414}+R_{2424}) + \frac{\varepsilon_0\lambda}{1+\lambda^2} R_{1234} \\
&= \left(1 - \frac{\varepsilon_0\lambda}{3(1+\lambda^2)}\right)(R_{1212}+R_{3434}) \\
&\quad + \left(1 - \frac{\varepsilon_0(3+\lambda)}{6\left(1+\lambda^2\right)}\right)(R_{1313}+R_{2323}) + \left(1 - \frac{\varepsilon_0\left(3\lambda^2+\lambda\right)}{6\left(1+\lambda^2\right)}\right)(R_{1414}+R_{2424}) \\
&\quad + \frac{\varepsilon_0\lambda\left(K(e_1+e_3, e_2+e_4) + K(e_1-e_3, e_2-e_4) + K(e_2+e_3, e_1-e_4) + K(e_2-e_3, e_1+e_4)\right)}{12(1+\lambda^2)} \\
&\le \left(1 - \frac{\varepsilon_0\lambda}{3(1+\lambda^2)}\right)\cdot 2K_{max} + \left(1 - \frac{\varepsilon_0(3+\lambda)}{6\left(1+\lambda^2\right)}\right)\cdot 2K_{max} \\
&\quad + \left(1 - \frac{\varepsilon_0\left(3\lambda^2+\lambda\right)}{6\left(1+\lambda^2\right)}\right)\cdot 2K_{max} + \frac{16\varepsilon_0\lambda K_{max}}{12(1+\lambda^2)} \\
&= (6-\varepsilon_0)K_{max}, \qquad (3.6)
\end{aligned}
$$

where we have used

$$
1 - \frac{\varepsilon_0\lambda}{3(1+\lambda^2)} \ge 0, \quad 1 - \frac{\varepsilon_0(3+\lambda)}{6\left(1+\lambda^2\right)} \ge 0, \quad 1 - \frac{\varepsilon_0\left(3\lambda^2+\lambda\right)}{6\left(1+\lambda^2\right)} \ge 0, \quad \forall \lambda \in [0, 1].
$$

Thus, (3.3), (3.4) and (3.6) yield

$$\begin{aligned}
&\frac{\varepsilon_0}{1+\lambda^2}\left((R_{1313}+R_{2323})+\lambda^2(R_{1414}+R_{2424})-2\lambda R_{1234}\right)\\
&\quad\geq n(n-1)R_0-(12-2\varepsilon_0)K_{max}-\left(\sum_{i,j=5}^{n}R_{ijij}+2\sum_{i=1}^{4}\sum_{j=5}^{n}R_{ijij}\right)\\
&\quad\geq n(n-1)R_0-(12-2\varepsilon_0)K_{max}-(n-4)(n-5)K_{max}-8(n-4)K_{max}\\
&\quad= n(n-1)R_0-(n^2-n-2\varepsilon_0)K_{max}. \qquad (3.7)
\end{aligned}$$

Consequently, the assumption $R_0 > \left(1-\frac{2\varepsilon_0}{n(n-1)}\right)K_{max}$ combined with (3.7) imply

$$(R_{1313}+R_{2323})+\lambda^2(R_{1414}+R_{2424})-2\lambda R_{1234}>0.$$

Our conclusion follows from Theorem B immediately. □

Remark 3.1 If we take $\lambda = 1$, $\varepsilon_0 = 3$ in the above proof, we actually have, when

$$R_0 > \left(1-\frac{6}{n(n-1)}\right)K_{max},$$

then the isotropic curvature

$$R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}>0,$$

which implies that M is homeomorphic to a sphere. One can see from Example 3.3, the pinching constant $\left(1-\dfrac{6}{n(n-1)}\right)$ is optimal when $n = 4$.

Proof of Theorem 1.2 Let D be a constant satisfying $Ric^{[4]} > 4(n-1)D$. Then

$$4(n-1)D < Ric_{11}+Ric_{22}+Ric_{33}+Ric_{44} = \sum_{i=1}^{4}\sum_{j=5}^{n}R_{ijij}+2\sum_{1\leq i<j\leq 4}R_{ijij}. \qquad (3.8)$$

Check the proof of Theorem 1.1, we actually have proved that for every $\lambda \in [0, 1]$,

$$\begin{aligned}
\sum_{1\leq i<j\leq 4}R_{ijij} \leq&(6-\varepsilon_0)K_{max}\\
&+\frac{\varepsilon_0}{2(1+\lambda^2)}\left((R_{1313}+R_{2323})+\lambda^2(R_{1414}+R_{2424})-2\lambda R_{1234}\right),
\end{aligned}$$

where $\varepsilon_0 = 12(\sqrt{10}-3)$. Combined with (3.8), we obtain

$$\begin{aligned}
4(n-1)D <& \left(4(n-1)-2\varepsilon_0\right)K_{max}\\
&+\frac{3}{1+\lambda^2}\left((R_{1313}+R_{2323})+\lambda^2(R_{1414}+R_{2424})-2\lambda R_{1234}\right).
\end{aligned}$$

Hence, if

$$\frac{Ric^{[4]}}{4(n-1)} > \left(1-\frac{\varepsilon_0}{2(n-1)}\right)K_{max},$$

we have

$$(R_{1313}+R_{2323})+\lambda^2(R_{1414}+R_{2424})-2\lambda R_{1234}>0.$$

We complete our proof. □

Remark 3.2 Similar as Remark 3.1, one can take $\lambda = 1$ and $\varepsilon_0 = 3$ and obtain that, if

$$\frac{Ric^{[4]}}{4(n-1)} > \left(1 - \frac{3}{2(n-1)}\right) K_{max},$$

then M has positive isotropic curvature, and is homeomorphic to a sphere.

Moreover, if M is Einstein, we obtain the following

Corollary 3.4 *Let $M^n (n \geq 4)$ be a closed and simply connected Einstein manifold. If*

$$R_0 > \left(1 - \frac{3}{2(n-1)}\right) K_{max},$$

then M is isometric (by scaling) to $\mathbb{S}^n$.

Proof If M is Einstein, then $Ric = cg$ for some positive constant c, the normalized scalar curvature $R_0 = \frac{Ric^{[4]}}{4(n-1)}$. From Remark 3.2, we know the isotropic curvature is positive. Therefore, by Brendle's Theorem ([5, Theorem 1]) we obtain the conclusion. □

4 Submanifolds with pinching curvatures

In this section, we will prove some sphere theorems for a Riemannian manifold isometrically immersed into another with some pinching curvature conditions. It is worth pointing out that our pinching constants in this section also improve Gu-Xu's corresponding pinching constants in [13] and [32].

Let R denote an algebraic curvature tensor, for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and $\lambda, \mu \in [-1, 1]$, we give the following notation,

$$\mathcal{I}_{\lambda,\mu}(R) = \frac{1}{(1+\lambda^2)(1+\mu^2)} \left(R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234}\right),$$

and we denote $\mathcal{I}_{\lambda,1}(R)$ briefly by $\mathcal{I}_\lambda(R)$.

Therefore, by Gauss equation (2.3), we have

$$\mathcal{I}_\lambda(R) = \mathcal{I}_\lambda(\bar{R}^T) + \mathcal{I}_\lambda\left(\frac{1}{2} B \owedge B\right). \tag{4.1}$$

Corresponding Theorem 1.1, we have the following result:

Theorem 4.1 *Let M^n be an $n(\geq 4)$-dimensional closed submanifold in an N-dimensional Riemannian manifold $\bar{M}^N$.*

(1) *If, pointwisely,*

$$|B|^2 < \frac{2N(N-1)}{3}\left[\bar{R}_0 - \left(1 - \frac{6}{N(N-1)}\right)\bar{K}_{max}\right] + \frac{n^2|H|^2}{n-2},$$

then M has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left[\frac{n}{2}\right]$. In particular, if M is simply connected, then M is homeomorphic to a sphere.

(2) *If, pointwisely,*

$$|B|^2 < \frac{N(N-1)}{3}\left[\bar{R}_0 - \left(1 - \frac{24(\sqrt{10}-3)}{N(N-1)}\right)\bar{K}_{max}\right] + \frac{n^2|H|^2}{n-1},$$

then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{S}^n$.

Proof Let $\bar{D}$ be a constant satisfying $N(N-1)\bar{D} < \sum_{i,j=1}^{N} \bar{R}_{ijij}$. Then a similar algebraic argument as the proof of Theorem 1.1 gives a similar inequality as (3.7):

$$6\mathcal{I}_\lambda\left(\bar{R}^T\right) > N(N-1)\bar{D} - (N^2 - N - 2\varepsilon_0)\bar{K}_{max}. \tag{4.2}$$

For $\lambda = 1$, we take $\varepsilon_0 = 3$. In the proof of [13, Lemma 9], the authors give the following estimate

$$4\mathcal{I}_1\left(\frac{1}{2}B \otimes B\right) \geq \frac{n^2H^2}{n-2} - |B|^2. \tag{4.3}$$

Thus, (4.1), (4.2) and (4.3) yield

$$\begin{aligned}\mathcal{I}_1(R) >& \frac{1}{6}\left(N(N-1)\bar{D} - (N^2 - N - 6)\bar{K}_{max}\right) + \frac{1}{4}\left(\frac{n^2|H|^2}{n-2} - |B|^2\right)\\ =& \frac{1}{4}\left\{\frac{2N(N-1)}{3}\left[\bar{D} - \left(1 - \frac{6}{N(N-1)}\right)\bar{K}_{max}\right] + \frac{n^2|H|^2}{n-2} - |B|^2\right\}.\end{aligned}$$

Combined with Theorem A, we complete the proof of Claim (1).

For arbitrary $\lambda \in [0, 1]$, we take $\varepsilon_0 = 12(\sqrt{10} - 3)$. In the proof of [13, Lemma 11], the authors obtain

$$2\mathcal{I}_\lambda\left(\frac{1}{2}B \otimes B\right) \geq \frac{n^2|H|^2}{n-1} - |B|^2, \quad \forall\lambda \in [0, 1] \tag{4.4}$$

Thus, (4.1), (4.2) and (4.4) give

$$\begin{aligned}\mathcal{I}_\lambda(R) >& \frac{1}{6}\left(N(N-1)\bar{D} - (N^2 - N - 24(\sqrt{10} - 3))\bar{K}_{max}\right) + \frac{1}{2}\left(\frac{n^2|H|^2}{n-1} - |B|^2\right)\\ =& \frac{1}{2}\left\{\frac{N(N-1)}{3}\left[\bar{D} - \left(1 - \frac{24(\sqrt{10}-3)}{N(N-1)}\right)\bar{K}_{max}\right] + \frac{n^2|H|^2}{n-1} - |B|^2\right\}.\end{aligned}$$

Then Claim (2) follows easily from Theorem B. □

After a similar argument we also have the following two extrinsic sphere theorems corresponding to Theorem 3.2 and Theorem 1.2.

Theorem 4.2 *Let M^n be an $n(\geq 4)$-dimensional closed submanifold in an N-dimensional Riemannian manifold $\bar{M}^N$.*

(1) *If, pointwisely,*

$$|B|^2 < \frac{N^2 - N + 12}{3}\left(\bar{K}_{min} - \left(1 - \frac{12}{N^2 - N + 12}\right)\bar{R}_0\right) + \frac{n^2|H|^2}{n-2},$$

then M has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left[\frac{n}{2}\right]$. In particular, if M is simply connected, then M is homeomorphic to a sphere.

(2) *If, pointwisely,*

$$|B|^2 < \frac{N^2 - N + 12}{6}\left(\bar{K}_{min} - \left(1 - \frac{12}{N^2 - N + 12}\right)\bar{R}_0\right) + \frac{n^2|H|^2}{n-1},$$

then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{S}^n$.

Theorem 4.3 *Let M^n be an $n(\geq 4)$-dimensional closed submanifold in an N-dimensional Riemannian manifold $\bar{M}^N$.*

(1) *If, pointwisely,*

$$|B|^2 < \frac{8(N-1)}{3}\left(\frac{\overline{Ric}^{[4]}_{min}}{4(N-1)} - \left(1 - \frac{3}{2(N-1)}\right)\bar{K}_{max}\right) + \frac{n^2H^2}{n-2},$$

then M has positive isotropic curvature. Therefore, $\pi_k(M) = 0$ for $2 \leq k \leq \left[\frac{n}{2}\right]$. In particular, if M is simply connected, then M is homeomorphic to a sphere.

(2) *If, pointwisely,*

$$|B|^2 < \frac{4(N-1)}{3}\left(\frac{\overline{Ric}^{[4]}_{min}}{4(N-1)} - \left(1 - \frac{6(\sqrt{10}-3)}{N-1}\right)\bar{K}_{max}\right) + \frac{n^2H^2}{n-1},$$

then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to $\mathbb{S}^n$.

Also we have the following corollary corresponding to Corollary 3.4.

Corollary 4.4 *Let M^n be an $n(\geq 4)$-dimensional closed Einstein submanifold in an N-dimensional Riemannian manifold $\bar{M}^N$. If, pointwisely,*

$$|B|^2 < \frac{8(N-1)}{3}\left(\frac{\overline{Ric}^{[4]}_{min}}{4(N-1)} - \left(1 - \frac{3}{2(N-1)}\right)\bar{K}_{max}\right) + \frac{n^2H^2}{n-2},$$

then M is isometric to a spherical space form. In particular, if M is simply connected, then M is isometric to $\mathbb{S}^n$ (by scaling).

Remark 4.1 Using a similar method, we also can get a sphere theorem under pinched curvature by K_{min}. But since the pinching constant is the same as Gu-Xu's result in [13], we omit here.

Next we will use a complex orthonormal frame to state the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.5. One can verify that in suitable complex orthonormal frame, the calculations will be considerably simplified.

Proof of Theorem 1.3 Let $e_1, \cdots, e_n$ be a local orthonormal frame of TM. For $\lambda, \mu \in [0, 1]$, define

$$\varepsilon_1 = \frac{e_1 + \sqrt{-1}\lambda e_2}{\sqrt{1+\lambda^2}}, \quad \varepsilon_2 = \frac{e_3 + \sqrt{-1}\mu e_4}{\sqrt{1+\mu^2}},$$

and extend these two vectors to be a local orthonormal frame of $T^{\mathbb{C}}M$. Then a direct computation gives

$$R_{12\bar{1}\bar{2}} = R(\varepsilon_1, \varepsilon_2, \bar{\varepsilon}_1, \bar{\varepsilon}_2) = \mathcal{I}_{\lambda,\mu}(R).$$

We first claim that

$$\sum_{i,j=1}^{n} \bar{R}_{ij\bar{i}\bar{j}} \leq (n^2-n-2)\bar{K}_{max} + 2\bar{R}_{12\bar{1}\bar{2}}. \tag{4.5}$$

If this is true, then (2.5) and (4.5) give

$$n(n-1)R_0 = \sum_{i,j=1}^{n} R_{ij\bar{i}\bar{j}}$$

$$
\begin{aligned}
&=\sum_{i,j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}+n(n-1)|H|^2-\sum_{i,j=1}^{n}\sum_{\alpha}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\\
&\leq(n^2-n-2)\bar{K}_{max}+2\bar{R}_{12\bar{1}\bar{2}}+n(n-1)|H|^2-\sum_{i,j=1}^{n}\sum_{\alpha}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\\
&=(n^2-n-2)\bar{K}_{max}+2R_{12\bar{1}\bar{2}}+n(n-1)|H|^2\\
&\quad-2\left(|H|^2+\sum_{\alpha}\left(H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)+\mathring{h}^{\alpha}_{1\bar{1}}\mathring{h}^{\alpha}_{2\bar{2}}-\left|\mathring{h}^{\alpha}_{1\bar{2}}\right|^2\right)\right)-\sum_{i,j=1}^{n}\sum_{\alpha}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&-2\left(|H|^2+\sum_{\alpha}\left(H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)+\mathring{h}^{\alpha}_{1\bar{1}}\mathring{h}^{\alpha}_{2\bar{2}}-\left|\mathring{h}^{\alpha}_{1\bar{2}}\right|^2\right)\right)-\sum_{i,j=1}^{n}\sum_{\alpha}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\\
&\leq-2\left(|H|^2+\sum_{\alpha}\left(H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)+\mathring{h}^{\alpha}_{1\bar{1}}\mathring{h}^{\alpha}_{2\bar{2}}-\left|\mathring{h}^{\alpha}_{1\bar{2}}\right|^2\right)\right)\\
&\quad-\sum_{\alpha}\left(\left|\mathring{h}^{\alpha}_{1\bar{1}}\right|^2+\left|\mathring{h}^{\alpha}_{2\bar{2}}\right|^2+2\left|\mathring{h}^{\alpha}_{1\bar{2}}\right|^2+\sum_{i,j=3}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\right)\\
&\leq-2|H|^2-2\sum_{\alpha}H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)\\
&\quad-\sum_{\alpha}\left(\left|\mathring{h}^{\alpha}_{1\bar{1}}\right|^2+\left|\mathring{h}^{\alpha}_{2\bar{2}}\right|^2+2\mathring{h}^{\alpha}_{1\bar{1}}\mathring{h}^{\alpha}_{2\bar{2}}\right)-\frac{1}{n-2}\sum_{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)^2\\
&=-2|H|^2-2\sum_{\alpha}H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)-\left(1+\frac{1}{n-2}\right)\sum_{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)^2\\
&\leq-\frac{n}{n-1}|H|^2,
\end{aligned}
$$

where in the second inequality, we have used

$$
\sum_{i,j=3}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\geq\sum_{i=3}^{n}\left|\mathring{h}^{\alpha}_{i\bar{i}}\right|^2\geq\frac{\left(\sum_{i=3}^{n}\mathring{h}^{\alpha}_{i\bar{i}}\right)^2}{n-2}=\frac{\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)^2}{n-2}.
$$

Therefore, we have

$$
n(n-1)R_0\leq(n^2-n-2)\bar{K}_{max}+2R_{12\bar{1}\bar{2}}+\left(n(n-1)-\frac{n}{n-1}\right)|H|^2,
$$

which implies

$$
2R_{12\bar{1}\bar{2}}\geq n(n-1)\left[R_0-\left(\left(1-\frac{2}{n(n-1)}\right)\bar{K}_{max}+\frac{n(n-2)}{(n-1)^2}|H|^2\right)\right]. \tag{4.6}
$$

Thus, by the assumption of this theorem and (4.6), we have $R_{12\bar{1}\bar{2}}\geq 0$. Therefore, $M\times\mathbb{R}^2$ has nonnegative isotropic curvature (see for example [6, Proposition 7.18]). Also by the assumption, the isotropic curvature of $M\times\mathbb{R}^2$ is positive at some point. Consequently, M has nonnegative isotropic curvature and positive isotropic curvature at some point. Then M admits a metric with positive isotropic curvature (see [28]). Therefore, M is a topological sphere by Theorem A. But by the classification theorem of Brendle-Schoen (Theorem C), M must be diffeomorphic to $\mathbb{S}^n$.

It remains to prove the inequality (4.5). Under the orthonormal frames $\{e_i\}$, this inequality is equivalent to

$$\sum_{i,j=1}^{n} \bar{R}_{ijij} \leq (n^2-n-2)\bar{K}_{max} + 2\mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right).$$

Notice that

$$\begin{aligned}\sum_{i,j=1}^{n} \bar{R}_{ijij} =& 2\mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right) + 2\left(\sum_{1\leq i<j\leq 4} \bar{R}_{ijij} - \mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right)\right) + 2\sum_{i=1}^{4}\sum_{j=5}^{n} \bar{R}_{ijij} + \sum_{i,j=5}^{n} \bar{R}_{ijij} \\ \leq& 2\mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right) + 2\left(\sum_{1\leq i<j\leq 4} \bar{R}_{ijij} - \mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right)\right) + (n^2-n-12)\bar{K}_{max}.\end{aligned}$$

Therefore, it is sufficient to prove

$$\sum_{1\leq i<j\leq 4} \bar{R}_{ijij} - \mathcal{I}_{\lambda,\mu}\left(\bar{R}^T\right) \leq 5\bar{K}_{max}.$$

A direct computation using (3.5) yields

$$\begin{aligned}&\sum_{1\leq i<j\leq 4} \bar{R}_{ijij} - \mathcal{I}_{\lambda,\mu}(\bar{R}^T) \\ &= \sum_{1\leq i<j\leq 4} \bar{R}_{ijij} - \frac{1}{(1+\lambda^2)(1+\mu^2)} \\ &\quad \left(\bar{R}_{1313} + \lambda^2\bar{R}_{1414} + \mu^2\bar{R}_{2323} + \lambda^2\mu^2\bar{R}_{2424} - 2\lambda\mu\bar{R}_{1234}\right) \\ &= \left(1 - \frac{2\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)(\bar{R}_{1212} + \bar{R}_{3434}) + \left(1 - \frac{3+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{1313} \\ &\quad + \left(1 - \frac{3\lambda^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{1414} + \left(1 - \frac{3\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{2323} \\ &\quad + \left(1 - \frac{3\lambda^2\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{2424} \\ &\quad + \frac{\lambda\mu\left(\bar{K}(e_1+e_3, e_2+e_4) + \bar{K}(e_1-e_3, e_2-e_4) + \bar{K}(e_2+e_3, e_1-e_4) + \bar{K}(e_2-e_3, e_1+e_4)\right)}{6(1+\lambda^2)(1+\mu^2)} \\ &\leq \left(1 - \frac{2\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\cdot 2\bar{K}_{max} + \left(1 - \frac{3+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{K}_{max} \\ &\quad + \left(1 - \frac{3\lambda^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{K}_{max} \\ &\quad + \left(1 - \frac{3\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{K}_{max} + \left(1 - \frac{3\lambda^2\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{K}_{max} \\ &\quad + \frac{16\lambda\mu}{6(1+\lambda^2)(1+\mu^2)}\bar{K}_{max} \\ &= 5\bar{K}_{max}.\end{aligned}$$

□

Theorem 1.4 and Theorem 1.5 are easy consequences of the following theorem:

Theorem 4.5 *For fixed* $0 < \varepsilon \leq 1$, *set* $\delta(\varepsilon, n) = \frac{((n-4)\varepsilon+2)^2 n^2}{8(2+(n^2-4n+2)\varepsilon)}$. *Suppose* $M^n (n \geq 4)$ *is a closed simply connected submanifold of* $\bar{M}^N$ *satisfying*

$$\frac{Ric^{[2]}}{2} \geq (n-1-\varepsilon)\bar{K}_{max} + \delta(\varepsilon, n)\,|H|^2\,,$$

with strict inequality at some point, then M *is diffeomorphic to* $\mathbb{S}^n$.

Proof Let $\{e_i\}$ be a local orthonormal frame of TM. For $\lambda, \mu \in [0, 1]$, define

$$\varepsilon_1 = \frac{e_1 + \sqrt{-1}\mu e_2}{\sqrt{1+\mu^2}}, \quad \varepsilon_2 = \frac{e_3 + \sqrt{-1}\lambda e_4}{\sqrt{1+\lambda^2}}, \quad \varepsilon_3 = \frac{\mu e_1 - \sqrt{-1}e_2}{\sqrt{1+\mu^2}}, \quad \varepsilon_4 = \frac{\lambda e_3 - \sqrt{-1}e_4}{\sqrt{1+\lambda^2}},$$
$$\varepsilon_i = e_i, \quad 5 \leq i \leq n.$$

Then $\{\varepsilon_i\}$ is a local orthonormal frame of $T^{\mathbb{C}}M$. Similar as the proof of Theorem 1.3, it is sufficient to prove $R_{1\bar{2}\bar{1}2} \geq 0$ and the strict inequality holds for all frame $\{e_i\}$ and all numbers $\lambda, \mu \in [0, 1]$ at some point. Ricci curvature formula (2.5) gives

$$\begin{aligned} &\frac{1}{2}\left(Ric_{1\bar{1}} + Ric_{2\bar{2}}\right) \\ &\quad = \frac{1}{2}\left(\sum_{i\neq 1}\bar{R}_{1i\bar{1}\bar{i}} + \sum_{i\neq 2}\bar{R}_{2i\bar{2}\bar{i}}\right) + (n-1)\,|H|^2 \\ &\quad + \frac{1}{2}\sum_{\alpha}\left((n-2)H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}} + \mathring{h}^{\alpha}_{2\bar{2}}\right) - \sum_{i=1}^{n}\left(\left|\mathring{h}^{\alpha}_{1\bar{i}}\right|^2 + \left|\mathring{h}^{\alpha}_{2\bar{i}}\right|^2\right)\right), \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\frac{1}{n-2}\sum_{i=3}^{n} Ric_{i\bar{i}} \\ &\quad = \frac{1}{n-2}\left[\sum_{i=3}^{n}\sum_{j\neq i}\bar{R}_{ij\bar{i}\bar{j}} - \sum_{\alpha}\left((n-2)H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}} + \mathring{h}^{\alpha}_{2\bar{2}}\right) + \sum_{i=3}^{n}\sum_{j=1}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\right)\right] \\ &\quad + (n-1)\,|H|^2\,. \end{aligned} \tag{4.8}$$

Assume

$$Ric_{i\bar{i}} + Ric_{j\bar{j}} \geq 2D, \quad \forall 1 \leq i < j \leq n.$$

Then

$$D \leq \frac{Ric_{1\bar{1}} + Ric_{2\bar{2}}}{2}, \quad D \leq \frac{\sum_{3\leq j<k\leq n}\left(Ric_{j\bar{j}} + Ric_{k\bar{k}}\right)}{(n-2)(n-3)} = \frac{\sum_{i=3}^{n} Ric_{i\bar{i}}}{n-2}.$$

Hence for every $0 < \varepsilon \leq 1$, by using (4.7), (4.8) and (2.4), we get

$$\begin{aligned} D \leq\ & \varepsilon \cdot \frac{Ric_{1\bar{1}} + Ric_{2\bar{2}}}{2} + (1-\varepsilon)\cdot\frac{\sum_{i=3}^{n} Ric_{i\bar{i}}}{n-2} \\ =\ & \frac{\varepsilon}{2}\sum_{i=1}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}} + \bar{R}_{2i\bar{2}\bar{i}}\right) + \frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}} + (n-1)\,|H|^2 - \frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\alpha}\left[\frac{n\varepsilon-2}{2}H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)-\frac{\varepsilon}{2}\sum_{i=1}^{n}\left(\left|\mathring{h}^{\alpha}_{1\bar{i}}\right|^2+\left|\mathring{h}^{\alpha}_{2\bar{i}}\right|^2\right)\right]\\
&=\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}}+\bar{R}_{2i\bar{2}\bar{i}}\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}+\varepsilon R_{12\bar{1}\bar{2}}+(n-1-\varepsilon)\left|H\right|^2\\
&-\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\\
&+\sum_{\alpha}\left[\frac{(n-2)\varepsilon-2}{2}H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)-\frac{\varepsilon}{2}\left(\left|\mathring{h}^{\alpha}_{1\bar{1}}\right|^2+2\mathring{h}^{\alpha}_{1\bar{1}}\mathring{h}^{\alpha}_{2\bar{2}}+\left|\mathring{h}^{\alpha}_{2\bar{2}}\right|^2\right)\right.\\
&\left.-\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\left|\mathring{h}^{\alpha}_{1\bar{i}}\right|^2+\left|\mathring{h}^{\alpha}_{2\bar{i}}\right|^2\right)\right]\\
&\leq\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}}+\bar{R}_{2i\bar{2}\bar{i}}\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}+\varepsilon R_{12\bar{1}\bar{2}}+(n-1-\varepsilon)\left|H\right|^2\\
&+\sum_{\alpha}\left[\frac{(n-2)\varepsilon-2}{2}H^{\alpha}\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)-\left(\frac{\varepsilon}{2}+\frac{1-\varepsilon}{(n-2)^2}\right)\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)^2\right]\\
&\leq\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}}+\bar{R}_{2i\bar{2}\bar{i}}\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}+\varepsilon R_{12\bar{1}\bar{2}}+\delta(\varepsilon,n)\left|H\right|^2
\end{aligned}
$$

where in the second inequality, we have used

$$\sum_{i=3}^{n}\sum_{j=1}^{n}\left|\mathring{h}^{\alpha}_{i\bar{j}}\right|^2\geq\sum_{i=3}^{n}\left|\mathring{h}^{\alpha}_{i\bar{i}}\right|^2\geq\frac{\left(\sum_{i=3}^{n}\mathring{h}^{\alpha}_{i\bar{i}}\right)^2}{n-2}=\frac{\left(\mathring{h}^{\alpha}_{1\bar{1}}+\mathring{h}^{\alpha}_{2\bar{2}}\right)^2}{n-2}.$$

Therefore,

$$\varepsilon R_{12\bar{1}\bar{2}}\geq D-\left(\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}}+\bar{R}_{2i\bar{2}\bar{i}}\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}+\delta(\varepsilon,n)\left|H\right|^2\right). \tag{4.9}$$

We claim that

$$\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}_{1i\bar{1}\bar{i}}+\bar{R}_{2i\bar{2}\bar{i}}\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}_{ij\bar{i}\bar{j}}\leq(n-1-\varepsilon)\bar{K}_{max}. \tag{4.10}$$

If this is true, then combined with (4.9), we have

$$\varepsilon R_{12\bar{1}\bar{2}}\geq D-(n-1-\varepsilon)\bar{K}_{max}+\delta(\varepsilon,n)\left|H\right|^2. \tag{4.11}$$

By the assumption of the theorem, we get

$$
\begin{aligned}
&Ric(\varepsilon_1,\bar{\varepsilon}_1)+Ric(\varepsilon_2,\bar{\varepsilon}_2)\\
&\quad=\frac{Ric(e_1,e_1)+\mu^2Ric(e_2,e_2)}{1+\mu^2}+\frac{Ric(e_3,e_3)+\lambda^2Ric(e_4,e_4)}{1+\lambda^2}\\
&\quad=\frac{\left(Ric(e_1,e_1)+Ric(e_3,e_3)\right)+\lambda^2\left(Ric(e_1,e_1)+Ric(e_4,e_4)\right)}{\left(1+\lambda^2\right)\left(1+\mu^2\right)}
\end{aligned}
$$

$$+\frac{\mu^2\left(Ric(e_2,e_2)+Ric(e_3,e_3)\right)+\lambda^2\mu^2\left(Ric(e_2,e_2)+Ric(e_4,e_4)\right)}{\left(1+\lambda^2\right)\left(1+\mu^2\right)}$$
$$\geq 2\left((n-1-\varepsilon)\bar{K}_{max}+\delta(\varepsilon,n)\left|H\right|^2\right).$$

Therefore, by the arbitrariness of e_1, e_2, e_3, e_4, we can take

$$D=(n-1-\varepsilon)\bar{K}_{max}+\delta(\varepsilon,n)\left|H\right|^2.$$

Combining the above inequality with (4.11), we have

$$\varepsilon R(\varepsilon_1,\varepsilon_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2)\geq D-\left((n-1-\varepsilon)\bar{K}_{max}+\delta(\varepsilon,n)\left|H\right|^2\right)=0.$$

Therefore we have $R_{12\bar{1}\bar{2}}\geq 0$, and strict inequality holds for all frame $\{e_i\}$ and all numbers $\lambda, \mu \in [0,1]$ at some point.

What is left is to prove the inequality (4.10). Under the given basis of $T^{\mathbb{C}}M$, a direct computation gives

$$\bar{R}(\varepsilon_1,\varepsilon_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2)$$
$$=\frac{\bar{R}_{1313}+\mu^2\bar{R}_{2323}+\lambda^2\bar{R}_{1414}+\lambda^2\mu^2\bar{R}_{2424}-2\lambda\mu\bar{R}_{1234}}{\left(1+\lambda^2\right)\left(1+\mu^2\right)} \tag{4.12}$$

$$\sum_{i=1}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)=\sum_{i=1}^{n}\left[\frac{\bar{R}_{1i1i}+\mu^2\bar{R}_{2i2i}}{1+\mu^2}+\frac{\bar{R}_{3i3i}+\lambda^2\bar{R}_{4i4i}}{1+\lambda^2}\right],$$
$$\sum_{i=1}^{n}\left(\bar{R}(\varepsilon_3,\varepsilon_i,\bar{\varepsilon}_3,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_4,\varepsilon_i,\bar{\varepsilon}_4,\bar{\varepsilon}_i)\right)=\sum_{i=1}^{n}\sum_{i=1}^{n}\left[\frac{\mu^2\bar{R}_{1i1i}+\bar{R}_{2i2i}}{1+\mu^2}+\frac{\lambda^2\bar{R}_{3i3i}+\bar{R}_{4i4i}}{1+\lambda^2}\right],$$
$$\sum_{j=1}^{n}\bar{R}(\varepsilon_i,\varepsilon_j,\bar{\varepsilon}_i,\bar{\varepsilon}_j)=\sum_{j=1}^{n}\bar{R}_{ijij},\quad 5\leq i\leq n. \tag{4.13}$$

Note that,

$$\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}(\varepsilon_i,\varepsilon_j,\bar{\varepsilon}_i,\bar{\varepsilon}_j)$$
$$=\frac{\varepsilon}{2}\sum_{i=1}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)-\varepsilon\bar{R}(\varepsilon_1,\varepsilon_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2)$$
$$+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}(\varepsilon_i,\varepsilon_j,\bar{\varepsilon}_i,\bar{\varepsilon}_j)$$
$$\leq\frac{\varepsilon}{2}\sum_{i=1}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)-\varepsilon\bar{R}(\varepsilon_1,\varepsilon_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2)$$
$$+\frac{1-\varepsilon}{n-2}\cdot(n-2)(n-1)\bar{K}_{max}.$$

By using (4.12) and (4.13), we have

$$\frac{1}{2}\sum_{i=1}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)-\bar{R}(\varepsilon_1,\varepsilon_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2)$$
$$=\frac{1}{2}\sum_{i=1}^{4}\left[\frac{\bar{R}_{1i1i}+\mu^2\bar{R}_{2i2i}}{1+\mu^2}+\frac{\bar{R}_{3i3i}+\lambda^2\bar{R}_{4i4i}}{1+\lambda^2}\right]$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=5}^{n}\left[\frac{\bar{R}_{1i1i}+\mu^2\bar{R}_{2i2i}}{1+\mu^2}+\frac{\bar{R}_{3i3i}+\lambda^2\bar{R}_{4i4i}}{1+\lambda^2}\right]\\
&-\frac{\bar{R}_{1313}+\mu^2\bar{R}_{2323}+\lambda^2\bar{R}_{1414}+\lambda^2\mu^2\bar{R}_{2424}-2\lambda\mu\bar{R}_{1234}}{\left(1+\lambda^2\right)\left(1+\mu^2\right)}\\
\le&\frac{1}{2}\sum_{i=1}^{4}\left[\frac{\bar{R}_{1i1i}+\mu^2\bar{R}_{2i2i}}{1+\mu^2}+\frac{\bar{R}_{3i3i}+\lambda^2\bar{R}_{4i4i}}{1+\lambda^2}\right]+(n-4)\bar{K}_{max}\\
&-\frac{\bar{R}_{1313}+\mu^2\bar{R}_{2323}+\lambda^2\bar{R}_{1414}+\lambda^2\mu^2\bar{R}_{2424}-2\lambda\mu\bar{R}_{1234}}{\left(1+\lambda^2\right)\left(1+\mu^2\right)}\\
=&\left(\frac{1}{2}-\frac{2\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\left(\bar{R}_{1212}+\bar{R}_{3434}\right)\\
&+\left(\frac{2+\mu^2+\lambda^2}{2(1+\lambda^2)(1+\mu^2)}-\frac{3+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{1313}\\
&+\left(\frac{1+2\lambda^2+\lambda^2\mu^2}{2(1+\lambda^2)(1+\mu^2)}-\frac{3\lambda^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{1414}\\
&+\left(\frac{1+2\mu^2+\lambda^2\mu^2}{2(1+\lambda^2)(1+\mu^2)}-\frac{3\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{2323}\\
&+\left(\frac{\lambda^2+\mu^2+2\lambda^2\mu^2}{2(1+\lambda^2)(1+\mu^2)}-\frac{3\lambda^2\mu^2+\lambda\mu}{3(1+\lambda^2)(1+\mu^2)}\right)\bar{R}_{2424}+(n-4)\bar{K}_{max}\\
&+\frac{\lambda\mu\left(\bar{K}(e_1+e_3,e_2+e_4)+\bar{K}(e_1-e_3,e_2-e_4)+\bar{K}(e_2+e_3,e_1-e_4)+\bar{K}(e_2-e_3,e_1+e_4)\right)}{6(1+\lambda^2)(1+\mu^2)}\\
\le&\frac{1}{6(1+\lambda^2)(1+\mu^2)}\left[\left(3(1+\lambda^2)(1+\mu^2)-4\lambda\mu\right)\cdot 2\right.\\
&+\left(3(\lambda^2+\mu^2)-2\lambda\mu\right)+\left(3(1+\lambda^2\mu^2)-2\lambda\mu\right)\\
&\left.+\left(3(1+\lambda^2\mu^2)-2\lambda\mu\right)+\left(3(\lambda^2+\mu^2)-2\lambda\mu\right)+16\lambda\mu\right]\bar{K}_{max}+(n-4)\bar{K}_{max}\\
=&(n-2)\bar{K}_{max},
\end{aligned}
$$

where in the last inequality, we have used the fact that all the coefficients are non-negative for $\lambda, \mu \in [0, 1]$, thus we can replace $\bar{R}_{ijij}$ with $\bar{K}_{max}$. Therefore,

$$
\begin{aligned}
&\frac{\varepsilon}{2}\sum_{i=3}^{n}\left(\bar{R}(\varepsilon_1,\varepsilon_i,\bar{\varepsilon}_1,\bar{\varepsilon}_i)+\bar{R}(\varepsilon_2,\varepsilon_i,\bar{\varepsilon}_2,\bar{\varepsilon}_i)\right)+\frac{1-\varepsilon}{n-2}\sum_{i=3}^{n}\sum_{j=1}^{n}\bar{R}(\varepsilon_i,\varepsilon_j,\bar{\varepsilon}_i,\bar{\varepsilon}_j)\\
&\quad\le\varepsilon(n-2)\bar{K}_{max}+\frac{1-\varepsilon}{n-2}\cdot(n-2)(n-1)\bar{K}_{max}\\
&\quad=(n-1-\varepsilon)\bar{K}_{max}.
\end{aligned}
$$

We complete the proof. □

If we take $\varepsilon = 1$ in Theorem 4.5, we have Theorem 1.4. If we take $\varepsilon = \frac{2}{n-2}$ in Theorem 4.5, we have Theorem 1.5.

We also can take $\varepsilon = \frac{2(n^2-6n+10)}{(n-4)(n^2-4n+2)}$ in Theorem 4.5 to make the coefficient $\delta(\varepsilon, n)$ to be minimal when $n \ge 6$.

Corollary 4.6 *Suppose $M^n (n \ge 6)$ is a closed simply connected submanifold of $\bar{M}^N$ satisfying*

$$\frac{Ric^{[2]}}{2} \geq \left(n - 1 - \frac{2(n^2 - 6n + 10)}{(n - 4)(n^2 - 4n + 2)}\right) \bar{K}_{max} + \frac{(n - 2)(n - 3)(n - 4)n^2}{(n^2 - 4n + 2)^2} |H|^2,$$

with strict inequality at some point, then M is diffeomorphic to $\mathbb{S}^n$.

Remark 4.2 It is easy to check, for $n \geq 6$,

$$n - 1 - \frac{2(n^2 - 6n + 10)}{(n - 4)(n^2 - 4n + 2)} < \frac{(n - 2)(n - 3)(n - 4)n^2}{(n^2 - 4n + 2)^2} < \frac{n(n - 3)}{n - 2}.$$

Therefore, when $n \geq 6$, Corollary 4.6 implies Theorem 1.5.

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