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Optimal lower eigenvalue estimates for Hodge-Laplacian and applications ☆

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Abstract

We consider the eigenvalue problem for Hodge-Laplacian on a Riemannian manifold M isometrically immersed into another Riemannian manifold $\bar{M}$. We first assume the pull back Weitzenböck operator of $\bar{M}$ bounded from below, and obtain an extrinsic lower bound for the first eigenvalue of Hodge-Laplacian. As applications, we obtain some rigidity results. Second, when the pull back Weitzenböck operator of $\bar{M}$ bounded from both sides, we give a lower bound of the first eigenvalue by the Ricci curvature of M and some extrinsic geometry. As a consequence, we prove a weak Ejiri type theorem, that is, if the Ricci curvature bounded from below pointwisely by a function of the norm square of the mean curvature vector, then M is a homology sphere. In the end, we give an example to show that all the eigenvalue estimates are optimal when $\bar{M}$ is the space form.

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Keywords: Hodge Laplacian; Eigenvalue estimate; Rigidity theorem; Homology sphere theorem

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1. Introduction

Let M^n be an n-dimensional Riemannian manifold. For each integer $0 \le p \le n$, the Hodge-Laplacian (or Laplace de Rham operator) acting on p-forms of M is defined by

$$\Delta = d\delta + \delta d : \Omega^p(M) \to \Omega^p(M),$$

where d and δ are the differential and co-differential operator. Hodge-Laplacian is an natural generalization (up to a sign) of Laplace–Beltrami operator acting on scalar functions (i.e., 0-forms). For each $0 \le p \le n$, denote by $\lambda_{1,p}$ the first eigenvalue of Hodge-Laplacian, i.e.,

$$\lambda_{1,p} = \inf_{0 \neq \omega \in \Omega^p(M)} \left\{ \frac{\int_M |d\omega|^2 + |\delta\omega|^2}{\int_M |\omega|^2} \right\}.$$

Eigenvalue estimates for Laplace–Beltrami operator acting on scalar functions are intensively studied in a huge literature. Compare to this, eigenvalue problems for Hodge-Laplacian attracted less attention, although they also play an important role in revealing relations between geometry (curvature, etc.) and topology (cohomology, etc.) of manifolds. One of the difficulties of the eigenvalue estimates for Hodge-Laplacian is the algebraic complexity of $\Omega^p(M)$ (compare to $\Omega^0(M)$).

In recent years, a number of authors devoted to this problem (e.g. [4,6,9–13,15]). Among them, Guerini–Savo [4], Kwong [6], Raulot–Savo [9,10] and Savo [12] investigated eigenvalues for Hodge-Laplacian on a manifold with boundary; Savo [11] and Smoczyk [15] studied eigenvalues for Hodge-Laplacian on submanifolds in Euclidean space or a sphere; Raulot–Savo [9] and Savo [13] also studied eigenvalues for Hodge-Laplacian on a hypersurface immersed into another Riemannian manifold.

As we note from above, when the target manifold is not a space form, all the extrinsic results are of codimension one (be a hypersurface or boundary of a Riemannian manifold). It is natural to study eigenvalue problems of Hodge-Laplacian on a Riemannian manifold immersed into another with arbitrary codimension. To this end, in the present paper, we first give some optimal extrinsic lower eigenvalue estimates of Hodge-Laplacian on a Riemannian manifold immersed into another with arbitrary codimension. After that, as applications, we will prove some rigidity results, such as the homology sphere theorems.

Let $i : M^n \to \bar{M}^{n+m}$ be an isometric immersion from n-dimensional Riemannian manifold M into $(n+m)$-dimensional Riemannian manifold $\bar{M}$. Let $\nu \in T^\perp M$ be a unit normal vector, and denote by S_ν the shape operator associated with ν. Assume $\{k_i\}_{i=1}^n$ are the principle curvatures of S_ν. Denote by I_p the set of all p-multi-indices

$$I_p = \left\{ \left\{ j_1, \cdots, j_p \right\} \middle| \, 1 \le j_1 \le \cdots \le j_p \le n \right\}.$$

For a given $\alpha = \left\{ j_1, \cdots, j_p \right\} \in I_p$, set $\alpha_\star = \{1, \cdots, n\} \backslash \alpha$, and call

$$K_\alpha = k_{j_1} + \cdots + k_{j_p}$$

a *p-curvature* of S_ν. Set

$$\begin{cases} \beta_p(x) = \dfrac{1}{p(n-p)} \inf\limits_{\alpha \in I_p} K_\alpha(x) K_{\alpha_\star}(x), \\ \beta_p(M) = \inf\limits_{x \in M} \beta_p(x). \end{cases}$$

When codimension $m = 1$ and the curvature operator of $\bar{M}$ is bounded from below, Savo [13, Theorem 7] obtained an optimal extrinsic lower bound of $\lambda_{1,p}$:

Theorem A *(Savo [13]). Let M^n be a closed hypersurface of $\bar{M}^{n+1}$, a manifold with curvature operator $\bar{\mathcal{R}}$ bounded from below by $c \in \mathbb{R}$. Let $1 \leq p \leq \frac{n}{2}$. Then*

$$\lambda_{1,p}(M) \geq p(n-p+1)(c + \beta_p(M)),$$

where $\beta_p(M)$ is a constant defined above. If M is a geodesic sphere in a simply connected manifold of constant curvature c, then equality holds.

Remark 1.1. By Poincaré duality, $\lambda_{1,p}(M) = \lambda_{1,n-p}(M)$, for all $0 \leq p \leq n$. Moreover, $\lambda_{1,0} = \lambda_{1,n} = 0$. Therefore, we always assume $1 \leq p \leq \frac{n}{2}$ if there is no other explanation.

The main tool that Savo used to prove the above theorem is the Bochner formula, that is, for all $\omega \in \Omega^p(M)$,

$$\frac{1}{2}\Delta |\omega|^2 = |\nabla \omega|^2 - \langle \Delta\omega, \omega \rangle + \left\langle W^{[p]}(\omega), \omega \right\rangle,$$

where $W^{[p]} : \Omega^p(M) \to \Omega^p(M)$ is usually called *(p-th) Weitzenböck operator*. When $p = 1$, $W^{[1]}$ is nothing but the Ricci tensor. But when $2 \leq p \leq n-2$, $W^{[p]}$ is complicated and is hard to be controlled in a general case. However, it is crucial and necessary to control the term $\left\langle W^{[p]}(\omega), \omega \right\rangle$ in eigenvalue estimates or in other problems. One can also define the Weitzenböck operator $\bar{W}^{[p]}$ of $\bar{M}$. Denote by $i^*\bar{W}^{[p]}$ the *pull-back Weitzenböck operator*, which is the restriction of $\bar{W}^{[p]}$ on $\Omega^p(M)$. One can check (cf. [2]) that, $\bar{\mathcal{R}} \geq c$ implies $i^*\bar{W}^{[p]} \geq p(n-p)c$.

For higher codimension, we follow the ideas in [2] and [13] and obtain an optimal lower bound for the first eigenvalue of Hodge-Laplacian acting on p-forms. In higher codimension case, there are more than one normal directions, and it is more difficult for dealing with the algebraic complexity of $\Omega^p(M)$.

Theorem 1.1. *Suppose M^n is a closed submanifold in $\bar{M}^{n+m}$ with the pull back Weitzenböck operator $i^*\bar{W}^{[p]} \geq p(n-p)c$ for some constant c and $1 \leq p \leq \frac{n}{2}$. Then*

$$\lambda_{1,p}(M) \geq p(n-p+1)\left(c + \gamma_p\right),$$

where

$$\gamma_p = \min_{x \in M} \left\{ \left(-\frac{1}{n} \left|\mathring{B}\right|^2 - \frac{(n-2p)\,|H|}{\sqrt{np(n-p)}} \left|\mathring{B}\right| + |H|^2 \right)(x) \right\},$$

H is the mean curvature vector and $\mathring{B}$ is the traceless part of the second fundamental form B. Moreover, if M is totally umbilical, then

$$\lambda_{1,p}(M) \geq p(n-p+1)\min_M \left(c+|H|^2\right).$$

The following corollary is a direct consequence of Theorem 1.1.

Corollary 1.2. *Assume the assumptions of Theorem 1.1 hold, we also have the following eigenvalue estimates:*

(i) $\lambda_{1,p}(M) \geq p(n-p+1)\min_M \left(c-\frac{n}{4p(n-p)}\left|\mathring{B}\right|^2\right)$.
(ii) $\lambda_{1,p}(M) \geq p(n-p+1)\min_M \left(c-\frac{1}{2\sqrt{p(n-p)}}|B|^2\right)$.
(iii) *If n is even, then*

$$\lambda_{1,n/2}(M) \geq \frac{n(n+2)}{4}\min_M \left(c-\frac{1}{n}\left|\mathring{B}\right|^2+|H|^2\right).$$

Remark 1.2. (i) It is worth pointing out that all the eigenvalue estimates in Theorem 1.1 and Corollary 1.2 are optimal. To see this, let $\bar{M}^{n+m}$ be a Riemannian manifold with constant sectional curvature $c>0$, and M^n be a geodesic sphere in $\bar{M}^{n+m}$. In this case, $i^*W^{[p]} \equiv p(n-p)c$ and M is totally umbilical. On the other hand, it is shown (cf. [2]) that $\lambda_{1,p}(M) = p(n-p+1)\left(c+|H|^2\right)$. Thus, the eigenvalue estimates is optimal when M is umbilical. When M is not umbilical, the eigenvalue estimates are also optimal by computing the first eigenvalue of Clifford torus (see the example in Appendix A).

(ii) When codimension $m=1$, Savo [13] obtained Theorem 1.1 and Corollary 1.2. Moreover, Theorem 1.1 gives Theorem A.

As an application, we have the following rigidity result.

Theorem 1.3. *Suppose M^n is a closed submanifold in $\bar{M}^{n+m}$, $p \in \{1,\dots,n-1\}$, $i^*\bar{W}^{[p]} \geq c \geq 0$ and $|B|^2 \leq \alpha(c,p,n,H)$, then the p-th Betti number $b_p \leq \binom{n}{p}$, where*

$$\alpha(c,p,n,H) := nc + \frac{n^3|H|^2}{2p(n-p)} - \frac{n|n-2p||H|\sqrt{n^2|H|^2+4cp(n-p)}}{2p(n-p)}.$$

Moreover, if $b_p>0$, then $|B|^2 \equiv \alpha(c,p,n,H)$. In particular, if $\chi(M) \neq 1+(-1)^n$, then $|B|^2 \equiv \alpha(c,p,n,H)$ for some $1 \leq p \leq n-1$.

The following two corollaries are direct consequences of the above rigidity theorem.

Corollary 1.4. *If the assumptions of Theorem 1.3 hold, and moreover if $|B|^2 \leq \alpha(c,1,n,H)$ and $b_p>0$ for some $1<p<n-1$, then either $\mathring{B} \equiv 0$ or M is minimal satisfying $|B|^2 \equiv nc$.*

Corollary 1.5. *If the assumptions of Theorem 1.3 hold, and moreover if* $n = 0$ (mod 4), $|B|^2 \le \alpha(c, 1, n, H)$ *and the signature* $sig(M)$ *of* M *is nonzero, then either* $\mathring{B} \equiv 0$ *or* M *is minimal satisfying* $|B|^2 \equiv nc$.

Moreover, if the strict inequality holds, we have a homology sphere theorem as follows.

Theorem 1.6. *If the assumptions of Theorem 1.3 hold, and moreover if* $|B|^2 < \alpha(c, 1, n, H)$ *and* M *is simply connected, then* M *is a homology sphere, i.e.,* $H^i(M, \mathbb{R}) = 0$ *for all* $1 \le i \le n-1$.

Remark 1.3. About Theorem 1.6, we should remark that,

- If $\bar{M}$ is a space form, then Shiohama–Xu [14] obtained a topological sphere theorem under the same condition, i.e., the condition $|B|^2 < \alpha(c, 1, n, H)$ implies that M is homeomorphic to a sphere.
- Applying the rational Hurewicz theorem, in the conclusion, one actually have $\pi_i(M, \mathbb{Q}) = 0$ for all $1 \le i \le n-1$ (which can be called a *rational homotopy sphere*).

It is worth noting that, constant γ_p in Theorem 1.1 depends on $|H|^2$ and $\left|\mathring{B}\right|^2$. But by the Gauss equation (2.1), we have

$$Scal_M = \sum_{i,j} \bar{R}_{ijij} + n(n-1)\,|H|^2 - \left|\mathring{B}\right|^2.$$

Moreover, by the definition of $i^*\bar{W}^{[p]}$,

$$\left\langle i^*\bar{W}^{[1]}(\eta^i), \eta^i \right\rangle = \sum_{j=1}^{n} \bar{R}_{ijij}.$$

We see that γ_p actually depends on the scalar curvature of M, $|H|^2$ and $i^*\bar{W}^{[1]}$. In this direction, assume $i^*\bar{W}^{[p]}$ bounded from below and $i^*\bar{W}^{[1]}$ bounded from above, we also obtain a lower eigenvalue estimate for Hodge-Laplacian by the Ricci curvature of M and $|H|^2$.

Theorem 1.7. *Suppose* M^n *is a closed submanifold of* $\bar{M}^{n+m}$ *with* $i^*\bar{W}^{[p]} \ge p(n-p)c_*$ *and* $i^*\bar{W}^{[1]} \le (n-1)c^*$, *where* $c^* \ge c_*$ *are two constants, then for* $1 \le p \le n/2$,

$$\begin{aligned}
&\frac{n-p}{n-p+1}\frac{n-2}{(n+2)p(n-p)-n^2}\lambda_{1,p}(M)\\
&\ge \left(Ric_{min} - (n-1)\left(c^* + |H|^2\right) + \frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\left(c_* + |H|^2\right)\right).
\end{aligned}$$

As an application, we will give a new *Ejiri type homology sphere theorem*. There are several types of sphere theorems depending on the curvature assumptions added on the submanifold or the target manifold. These curvature assumptions include pinched sectional curvature, bounded Ricci curvature, etc. In Section 4, we will restrict our attention to the case that the Ricci curvature of the submanifold is bounded from below. As far as we know, the first such type result was given by Ejiri in 1979 for minimal submanifolds of a sphere. Recently, Gu–Xu generalized Ejiri's result to submanifolds of space forms with parallel mean curvature vector.

Theorem B *(Ejiri [1], Gu–Xu [17]). Let M be an $n(\geq 3)$-dimensional complete submanifold with parallel mean curvature vector H in $F^{n+m}(c)$ with $c+|H|^2>0$. If the Ricci curvature of M satisfies*

$$Ric_M \geq (n-2)\left(c+|H|^2\right),$$

then M is either the totally geodesic submanifold $\mathbb{S}^n\left(\frac{1}{\sqrt{c+|H|^2}}\right)$, the Clifford torus $\mathbb{S}^l\left(\frac{1}{\sqrt{2(c+|H|^2)}}\right)\times \mathbb{S}^l\left(\frac{1}{\sqrt{2(c+|H|^2)}}\right)$ in $\mathbb{S}^{n+1}\left(\frac{1}{\sqrt{c+|H|^2}}\right)$ with $n=2l$, or $\mathbb{C}P^2\left(\frac{4}{3}(c+|H|^2)\right)$ in $\mathbb{S}^7\left(\frac{1}{c+|H|^2}\right)$. Here $\mathbb{C}P^2\left(\frac{4}{3}(c+|H|^2)\right)$ denotes the 2-dimensional complex projective space minimally immersed into $\mathbb{S}^7\left(\frac{1}{c+|H|^2}\right)$ with constant holomorphic sectional curvature $\frac{4}{3}(1+|H|^2)$.

Gu–Xu [17] also obtain the following topological sphere theorem without the assumption of parallel mean curvature vector.

Theorem C *(Gu–Xu [17]). Let M be an n-dimensional closed submanifold with mean curvature vector H in $F^{n+m}(c)$ with $c\geq 0$. If the Ricci curvature of M satisfies*

$$Ric_M > (n-2)\left(c+|H|^2\right), \tag{1.1}$$

then M is homeomorphic to a sphere.

The original version of Gu–Xu's theorem assume that $n\geq 4$. The case $n=2$ is a consequence of Gauss–Bonnet formula. The case $n=3$ is a consequence of Lawson–Simons theorem and Perelman's solution of Poincaré conjecture.

The key idea to prove Theorem C is to claim that there is no stable integral p-currents for $0<p<n$ under the assumption (1.1). The *p-th weak Ricci curvature* of the p-plane $e_1\wedge e_2\wedge\cdots\wedge e_p$ introduced by Gu–Xu [3] is defined by

$$Ric(e_1\wedge e_2\wedge\cdots\wedge e_p):=\sum_{i=1}^{p} Ric_{ii}.$$

One can verify that $Ric(e_1\wedge e_2\wedge\cdots\wedge e_p)$ is well defined, i.e., it is depending only on the p-plane $e_1\wedge e_2\wedge\cdots\wedge e_p$. With an obvious modification of original results of Gu–Xu [17] and Gu–Leng–Xu [18], one can obtain the following Theorem (for readers' convenience, we list a proof in Section 4).

Theorem D. *[17,18] Let M be an $n(\geq 4)$-dimensional closed submanifold with mean curvature vector H in $F^{n+m}(c)$ with $c\geq 0$. If*

$$\frac{Ric_{(p)}}{p} > \left(n-1-\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\right)\left(c+|H|^2\right), \quad 1<p<n-1,$$

where $Ric_{(p)}$ is the lower bound of the p-th Ricci curvature, then there is no stable integral p-currents.

Note that the target manifold in the above three results are all of constant curvature. In Section 4, as an application of Theorem 1.7, we generalize Theorem C and Theorem D to a more general case, that is, $\bar{M}$ is not necessarily of constant curvature.

Theorem 1.8. *Suppose M^n is a closed submanifold of $\bar{M}^{n+m}$ with $i^*\bar{W}^{[p]} \geq p(n-p)c_*$ and $i^*\bar{W}^{[1]} \leq (n-1)c^*$, where $c^* \geq c_*$ are two constants. If*

$$Ric > (n-1)\left(c^* + |H|^2\right) - \frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\left(c_* + |H|^2\right),$$

holds for some $0 < p < n$, then the p-th Betti number is zero. In particular, if M is simply connected, and

$$Ric > \begin{cases} (n-1)\left(c^* + |H|^2\right) - \left(c_* + |H|^2\right), & n \text{ is even}, \\ (n-1)\left(c^* + |H|^2\right) - \dfrac{(n-2)(n^2-1)}{n^3-2n^2-n-2}\left(c_* + |H|^2\right), & n \text{ is odd}, \end{cases} \tag{1.2}$$

then M is a homology sphere.

Remark 1.4.

1. Suppose the sectional curvature of $\bar{M}$ is bounded below by $\bar{K}_{\min}$ and above by $\bar{K}_{\max}$, then we can take
$$c^* = (n-1)\bar{K}_{\max},$$
$$c_* = \frac{2[n/2]+1}{3}\left(\bar{K}_{\min} - \frac{2[n/2]-2}{2[n/2]+1}\bar{K}_{\max}\right).$$
Therefore, our assumption is indeed weaker than constant curvature assumption.
2. The condition (1.2) is sharp for all p and all n (no matter n is even or odd) when $\bar{M} = F^{n+m}(c)$ (see the example in Appendix A).

We call the above result a *weak Ejiri type theorem*. It is weak in the sense that M is just a homology sphere in our conclusion. Therefore, it can be seen as a generalization of Theorem C in the rational homotopy sense. We emphasize that the proof of Theorem 1.8 bases on the Bochner's method which is quite different from Ejiri's and Gu–Xu's.

The paper is organized as follows. In Section 2, we set up notation and terminology, and review some of the standard facts on submanifolds geometry and Hodge-Laplacian. In Section 3, we give the proofs of Theorem 1.1 and Theorem 1.3. We also give another two applications of Theorem 1.1. In Section 4, we give the proof of Theorem 1.7 and Theorem 1.8. In Appendix A, we calculate an example of Clifford torus to show that the eigenvalue estimates and sphere theorems are all optimal when the target manifold is the standard sphere.

2. Preliminaries

In this section, we first recall some of the standard facts on submanifold geometry.

Let $i : M^n \to \bar{M}^{n+m}$ be an isometric immersion from a closed n-dimensional Riemannian manifold M to an $(n+m)$-dimensional Riemannian manifold $\bar{M}$. Let $e_1, \cdots, e_n, \nu_1, \cdots, \nu_m$ be an orthonormal frame on $\bar{M}$ such that $e_1, \cdots, e_n$ are tangent to M and $\nu_1, \cdots, \nu_m$ are perpendicular to M, and $\eta^1, \cdots, \eta^n$ be the dual of $e_1, \cdots, e_n$. Let R (resp. $\bar{R}$) be the $(0,4)$-type curvature tensor of M (resp. $\bar{M}$), and $\mathcal{R} : \Lambda^2 TM \to \Lambda^2 TM$ be the curvature operator defined by

$$\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l \rangle = R(e_i, e_j, e_k, e_l) =: R_{ijkl}.$$

From now on, we assume the Latin subscripts (or superscripts) $i, j, k, l, \cdots$ range from 1 to n, and the Greek subscripts (or superscripts) $\alpha, \beta, \gamma, \cdots$ range from 1 to m, and we will adopt the Einstein summation rule. The second fundamental form and the mean curvature vector are given by

$$B = h^\alpha_{ij} \eta^i \otimes \eta^j \otimes \nu_\alpha, \quad H = \frac{1}{n} \sum_i h^\alpha_{ii} \nu_\alpha =: H^\alpha \nu_\alpha,$$

and write $\mathring{B} = B - H \otimes g$ which is the traceless part of B, where g is the metric on M. Let A be the shape operator defined by

$$\langle B(X, Y), \nu \rangle = \left\langle A^\nu(X), Y \right\rangle, \quad \text{for all} \quad X, Y \in TM \quad \text{and} \quad \nu \in T^\perp M,$$

Write

$$A^\alpha := A^{\nu^\alpha} \quad \text{and} \quad \mathring{A}^\alpha := A^\alpha - H^\alpha g.$$

Recall the Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_{\alpha=1}^{m} \left(h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk} \right). \tag{2.1}$$

Second, we summarize the relevant material on Hodge-Laplacian and some facts of its first eigenvalue.

Let Δ be the Hodge-Laplacian, i.e., $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$. Since $[\mathrm{d}, \Delta] = [\delta, \Delta] = 0$, we have

$$\Delta : \mathrm{d}\Omega^p(M) \longrightarrow \mathrm{d}\Omega^p(M),$$
$$\Delta : \delta\Omega^p(M) \longrightarrow \delta\Omega^p(M).$$

Let $\lambda^e_{1,p}$ and $\lambda^{ce}_{1,p}$ be the first eigenvalue of Δ acting on the exact and co-exact p-form on M respectively. It is easy to check that the first eigenvalue $\lambda_{1,p}$ of Δ satisfies

$$\lambda_{1,p} \le \min \left\{ \lambda^e_{1,p}, \lambda^{ce}_{1,p} \right\}.$$

By Hodge decomposition, we know that

$$\lambda_{1,p} = \min\left\{\lambda^{e}_{1,p}, \lambda^{ce}_{1,p}\right\},$$

provided $H^p(M,\mathbb{R}) = 0$. By Hodge duality,

$$\lambda^{e}_{1,p} = \lambda^{ce}_{1,n-p}, \quad \lambda_{1,p} = \lambda_{1,n-p}.$$

Thus, by differentiating eigenforms, we obtain that if $H^p(M,\mathbb{R}) = 0$, then

$$\lambda^{ce}_{1,p-1} \leq \lambda^{e}_{1,p}, \quad \lambda^{e}_{1,p+1} \leq \lambda^{ce}_{1,p}.$$

In particular, if $H^{p-1}(M,\mathbb{R}) = H^p(M,\mathbb{R}) = 0$, we have

$$\lambda^{ce}_{1,p-1} = \lambda^{e}_{1,p}.$$

Moreover, if $H^p(M,\mathbb{R}) = 0$, then

$$\min\left\{\lambda_{1,p-1}, \lambda_{1,p+1}\right\} \leq \lambda_{1,p}.$$

For example,

$$\lambda^{e}_{1,p}\left(\mathrm{S}^n(1)\right) = p(n-p+1), \quad \lambda^{ce}_{1,p}\left(\mathrm{S}^n(1)\right) = (p+1)(n-p).$$

Third, we briefly sketch the Weitzenböck formula and Bochner formula for differential forms.

For every p-form ω on M, using the local orthonormal frame, the Weitzenböck operator $W^{[p]} : \Omega^p(M) \to \Omega^p(M)$ is given by (cf. [5]),

$$W^{[p]}(\omega) = \eta^i \wedge \iota_{e_j} R(e_i, e_j)\omega.$$

Similarly, the pull back Weitzenböck operator $i^*\bar{W}^{[p]} : \Omega^p(M) \to \Omega^p(M)$ is given by

$$i^*\bar{W}^{[p]}(\omega) = \eta^i \wedge \iota_{e_j} \bar{R}(e_i, e_j)\omega.$$

The following two formulas for p-forms are well known,

$$\Delta\omega = \nabla^\star\nabla\omega + W^{[p]}(\omega), \tag{2.2}$$

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 - \langle\Delta\omega, \omega\rangle + \left\langle W^{[p]}(\omega), \omega\right\rangle, \tag{2.3}$$

where $\nabla^\star\nabla$ is the connection Laplacian. Equalities (2.2) and (2.3) are usually called Weitzenböck formula and Bochner formula. Introduce

$$\mathfrak{ad}_{e_i\wedge e_j}\omega := e_i \cdot e_j \cdot \omega - e_j \cdot e_i \cdot \omega = 2\eta^j \wedge \iota_{e_i}\omega - 2\eta^i \wedge \iota_{e_j}\omega,$$

where $\cdot$ stands for the Clifford multiplication. A direct computation gives (cf. [7])

$$\left\langle W^{[p]}(\omega), \omega\right\rangle = \frac{1}{4}\langle\mathcal{R}(\theta_I), \theta_J\rangle\left\langle\mathfrak{ad}_{\theta_I}\omega, \mathfrak{ad}_{\theta_J}\omega\right\rangle,$$

where $\{\theta_I\}$ is a local orthonormal frame of $\Lambda^2 TM$.

In the end of this section, let us recall the following theorem due to Lawson–Simons [8] ($c > 0$) and Xin [16] ($c = 0$).

Theorem E *(Lawson–Simons, Xin). Suppose $M^n \subset F^{n+m}(c)$, $c \geq 0$ and for every orthonormal frame $\{e_i\}$ of TM,*

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right) < p(n-p)c, \tag{2.4}$$

then there is no stable integral p-currents, where $F^{n+m}(c)$ is the $(n+m)$-dimensional space form with sectional curvature c.

3. Eigenvalue estimate and its applications

In this section, we give the proofs of Theorem 1.1 and Theorem 1.3 and their applications.

To prove Theorem 1.1, we need the following lemma which is due to Gallot and Meyer [2], and we give a proof here for completeness.

Lemma 3.1. *Let M^n be an n-dimensional closed Riemannian manifold. For $1 \leq p \leq \frac{n}{2}$, assume the Weitzenböck operator $W^{[p]} \geq p(n-p)\Lambda$ for some $\Lambda > 0$, then*

$$\lambda_{1,p}(M) \geq p(n-p+1)\Lambda.$$

Proof. Introduce the twistor operator P on M acting on p-form ω by

$$P_X\omega := \nabla_X\omega - \frac{1}{p+1}\iota_X \mathrm{d}\omega + \frac{1}{n+1-p}X^{\flat} \wedge \delta\omega,$$

where $X^{\flat}$ is the dual 1-form defined by $X^{\flat}(e_i) = \langle X, e_i\rangle$. Then the following identity holds,

$$|\nabla\omega|^2 = |P\omega|^2 + \frac{1}{p+1}|\mathrm{d}\omega|^2 + \frac{1}{n+1-p}|\delta\omega|^2.$$

Now applying Bochner formula (2.3), and by the assumption $W^{[p]} \geq p(n-p)\Lambda$, we obtain

$$\begin{aligned}
&\frac{p}{p+1}\int_M |\mathrm{d}\omega|^2 + \frac{n-p}{n+1-p}\int_M |\delta\omega|^2\\
&= \int_M |P\omega|^2 + \left\langle W^{[p]}(\omega), \omega\right\rangle\\
&\geq p(n-p)\Lambda \int |\omega|^2.
\end{aligned}$$

By hypothesis $1 \leq p \leq \frac{n}{2}$, we have $\frac{p}{p+1} \leq \frac{n-p}{n+1-p}$. Therefore,

$$\frac{n-p}{n+1-p}\left(\int_M |\mathrm{d}\omega|^2 + \int_M |\delta\omega|^2\right) \geq p(n-p)\Lambda \int |\omega|^2 .$$

The conclusion follows from the variational characteristic of the first eigenvalue. □

Proof of Theorem 1.1. We will adopt the notations in Section 2, and for simplification, we introduce two more notations,

$$S(\omega) := \eta^i \wedge \iota_{A^\alpha(e_i)}\omega \otimes \nu_\alpha, \quad \mathring{S}(\omega) := \eta^i \wedge \iota_{\mathring{A}^\alpha(e_i)}\omega \otimes \nu_\alpha .$$

Direct calculations yield

$$W^{[p]} = \eta^i \wedge \iota_{e_j} R(e_j, e_i) = R_{ijkl}\eta^j \wedge \iota_{e_i}\left(\eta^k \wedge \iota_{e_l}\right),$$
$$i^*\bar{W}^{[p]} = \eta^i \wedge \iota_{e_j} \bar{R}(e_j, e_i) = \bar{R}_{ijkl}\eta^j \wedge \iota_{e_i}\left(\eta^k \wedge \iota_{e_l}\right).$$

Hence, by using Gauss equation (2.1),

$$\begin{aligned}
&\left\langle W^{[p]}(\omega), \omega\right\rangle - \left\langle i^*\bar{W}^{[p]}(\omega), \omega\right\rangle \\
&= \left(R_{ijkl} - \bar{R}_{ijkl}\right)\left\langle \eta^i \wedge \iota_{e_j}\omega, \eta^k \wedge \iota_{e_l}\omega\right\rangle \\
&= \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right)\left\langle \eta^i \wedge \iota_{e_j}\omega, \eta^k \wedge \iota_{e_l}\omega\right\rangle \\
&= \sum_\alpha \left\langle \eta^i \wedge \iota_{A^\alpha(e_l)}\omega, A^\alpha(e_i) \wedge \iota_{e_l}\omega\right\rangle - \sum_\alpha \left\langle \eta^i \wedge \iota_{A^\alpha(e_k)}\omega, \eta^k \wedge \iota_{A^\alpha(e_i)}\omega\right\rangle \\
&= \sum_\alpha \left\langle \iota_{A^\alpha(e_l)}\omega, nH^\alpha \iota_{e_l}\omega + A^\alpha(e_i) \wedge \iota_{e_l}\iota_{e_i}\omega\right\rangle - \sum_\alpha \left\langle \iota_{A^\alpha(e_k)}\omega, \iota_{A^\alpha(e_k)}\omega - \eta^k \wedge \iota_{e_i}\iota_{A^\alpha(e_i)}\omega\right\rangle \\
&= \sum_\alpha \left\langle \eta^l \wedge \iota_{nH^\alpha A^\alpha(e_l)}\omega, \omega\right\rangle - \sum_\alpha \left|\sum_l \eta^l \wedge \iota_{A^\alpha(e_l)}\omega\right|^2 + \sum_\alpha \left|\sum_k \iota_{e_k}\iota_{A^\alpha(e_k)}\omega\right|^2 \\
&= \sum_\alpha \left\langle \eta^l \wedge \iota_{A^\alpha(e_l)}\omega, nH^\alpha\omega\right\rangle - \sum_\alpha \left|\sum_l \eta^l \wedge \iota_{A^\alpha(e_l)}\omega\right|^2 \\
&= -\sum_\alpha \left|\sum_l \eta^l \wedge \iota_{A^\alpha(e_l)}\omega - \frac{n}{2}H^\alpha\omega\right|^2 + \frac{n^2}{4}|H|^2|\omega|^2 \\
&= -\left|S(\omega) - \frac{n}{2}H\omega\right|^2 + \frac{n^2}{4}|H|^2|\omega|^2 .
\end{aligned}$$

Therefore, by the assumption of the theorem, we have

$$\begin{aligned}\left\langle W^{[p]}(\omega),\omega\right\rangle &= \left\langle i^*\bar{W}^{[p]}(\omega),\omega\right\rangle - \left|S(\omega)-\frac{n}{2}H\omega\right|^2 + \frac{n^2}{4}|H|^2|\omega|^2 \\ &= \left\langle i^*\bar{W}^{[p]}(\omega),\omega\right\rangle - \left|\mathring{S}(\omega)-\frac{n-2p}{2}H\omega\right|^2 + \frac{n^2}{4}|H|^2|\omega|^2 \\ &= \left\langle i^*\bar{W}^{[p]}(\omega),\omega\right\rangle - \left|\mathring{S}(\omega)\right|^2 + (n-2p)\left\langle \mathring{S}(\omega), H\omega\right\rangle + p(n-p)|H|^2|\omega|^2 \\ &\geq p(n-p)c|\omega|^2 - \left|\mathring{S}(\omega)\right|^2 + (n-2p)\left\langle \mathring{S}(\omega), H\omega\right\rangle + p(n-p)|H|^2|\omega|^2 .\end{aligned}$$

Hence, according to Lemma 3.1 and the above inequality, to prove the theorem, it is sufficient to prove that,

$$\left|\mathring{S}|_{\Omega^p(M)}\right|^2_{op} \leq \frac{p(n-p)}{n}\left|\mathring{B}\right|^2 ,$$

where $|\cdot|_{op}$ stands for the operator norm when acting on p-forms.

By definition, acting on p-forms,

$$\mathring{S}(\omega) = \eta^i \wedge \iota_{\mathring{A}^\alpha(e_i)}\omega \otimes \nu_\alpha =: \mathring{S}^\alpha(\omega)\otimes\nu_\alpha ,$$

we have

$$\left|\mathring{S}\right|^2_{op} = \left(\sup_{0\neq\omega\in\Omega^p(M)} \frac{\left|\mathring{S}(\omega)\right|}{|\omega|}\right)^2 \leq \sum_{\alpha=1}^{m}\left|\mathring{S}^\alpha\right|^2_{op} .$$

On the other hand, $\left|\mathring{B}\right|^2 = \sum_{\alpha=1}^m \left|\mathring{A}^\alpha\right|^2$. Hence, the rest of the proof can be reduced to codimension $m=1$ case, which has already done (cf. [9, Formula (18)]). □

Conclusion (iii) of Corollary 1.2 is a direct consequence of Theorem 1.1 when $p=\frac{n}{2}$. Conclusions (i) and (ii) of Corollary 1.2 follow from the corollary below.

Corollary 3.2. *Assume the assumptions of Theorem 1.1 hold, then for every $\varepsilon > -1$, we have*

$$\lambda_{1,p}(M) \geq p(n+1-p)\min_M\left(c - \frac{1}{n(1+\varepsilon)}\left(\frac{n^2}{4p(n-p)}+\varepsilon\right)\left|\mathring{B}\right|^2 - \varepsilon|H|^2\right).$$

Proof. Direct calculations by Theorem 1.1 and Young inequality. □

Proof of Theorem 1.3. Notice that

$$c \geq \max\left\{\frac{1}{n}\left|\mathring{B}\right|^2 + \frac{|n-2p|\,|H|}{\sqrt{np(n-p)}}\left|\mathring{B}\right| - |H|^2\right\}$$

is equivalent to

$$|B|^2 \le nc + \frac{n^3 |H|^2}{2p(n-p)} - \frac{n\,|n-2p|\,|H| \sqrt{n^2 |H|^2 + 4cp(n-p)}}{2p(n-p)}.$$

Hence, by assumption, $\lambda_{1,p}(M) \ge 0$. Thus, every harmonic p-form is a conformal killing form, i.e., $P\omega = 0$, and is parallel. Moreover, if for some point, the strictly inequality holds, then there is no nontrivial harmonic p-form. In other words, $H^p(M, \mathbb{R}) = 0$.

Notice that

$$\min_{p\in\{1,\dots,n-1\}} \left\{ nc + \frac{n^3 |H|^2}{2p(n-p)} - \frac{n\,|n-2p|\,|H| \sqrt{n^2 |H|^2 + 4cp(n-p)}}{2p(n-p)} \right\}$$
$$= nc + \frac{n^3 |H|^2}{2(n-1)} - \frac{n(n-2)\,|H| \sqrt{n^2 |H|^2 + 4c(n-1)}}{2(n-1)}.$$

Consequently, if

$$|B|^2 \le nc + \frac{n^3 |H|^2}{2(n-1)} - \frac{n(n-2)\,|H| \sqrt{n^2 |H|^2 + 4c(n-1)}}{2(n-1)},$$

then

$$b_p := \dim_{\mathbb{R}} H^p(M, \mathbb{R}) \le \binom{n}{p}, \quad p \in \{0, 1, \dots, n\}.$$

Moreover, if the inequality holds strictly at some point, then

$$b_p := \dim_{\mathbb{R}} H^p(M, \mathbb{R}) = 0, \quad p \in \{1, \dots, n-1\}.$$

Finally, if $\chi(M) \neq 1 + (-1)^n$, there must be some $p \in \{1, \dots, n-1\}$ such that the Betti number $b_p > 0$. We finish the proof. $\square$

Proof of Theorem 1.6. Since $|B|^2 < \alpha(c, 1, n, H)$, applying the estimate of the lower bound of the first p-eigenvalue, we know that the p-th Betti number is zero for $0 < p < n$, i.e., M is a homology sphere. $\square$

Besides the corollaries and theorems mentioned in the introduction, we have two more applications of Theorem 1.1.

Theorem 3.3. *If the assumptions of Theorem 1.3 hold, and moreover if* $\left|\mathring{B}\right|^2 \le 4cp(n-p)/n$ *and the strictly inequality holds at some point, then* $b_p = 0$ *for* $p = 1, \dots, n-1$. *Therefore, if* $\left|\mathring{B}\right|^2 < 4c(n-1)/n$ *and* M *is simply connected, then* M *is a rational homotopy sphere.*

Proof. A direct computation gives

$$\min_{H}\left\{\alpha(c,p,n,H)-n|H|^2\right\}=\frac{4cp(n-p)}{n},$$

and the equality holds if and only if

$$n^2|H|^2+4cp(n-p)=n^2c.$$

Therefore, if $\left|\mathring{B}\right|^2\leq\frac{4cp(n-p)}{n}$ and strictly inequality holds at some point, then $b_p=0$. □

Similarly,

Theorem 3.4. *If the assumptions of Theorem 1.3 hold, and moreover if $|B|^2\leq 2c\sqrt{p(n-p)}$ and the strictly inequality holds at some point, then $b_p=0$ for $p=1,\ldots,n-1$. Therefore, if $|B|^2<2c\sqrt{n-1}$ and M is simply connected, then M is a rational homotopy sphere.*

Proof. Since

$$\min_{H}\alpha(c,p,n,H)=2c\sqrt{p(n-p)},$$

we obtain the theorem. □

4. Eigenvalue estimate and Ejiri's type Theorem

In this section, we will prove Theorem 1.8.

First, for readers' convenience, we provide here a different but simple proof of Theorem D.

Proof of Theorem D. We need to verify Lawson–Simons condition (2.4).

We first study the case of $p=1$.

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2-h_{ii}^{\alpha}h_{jj}^{\alpha}\right)\\
&=\sum_{j=2}^{n}\sum_{\alpha=1}^{m}\left(h_{1j}^{\alpha}\right)^2-Ric_{11}+(n-1)c\\
&\leq\frac{1}{2}\left(|B|^2-n|H|^2\right)-Ric_{11}+(n-1)c\\
&=\frac{1}{2}\left(n(n-1)\left(c+|H|^2\right)-\sum_{i=1}^{n}R_{ii}\right)-Ric_{11}+(n-1)c\\
&=\frac{1}{2}n(n-1)\left(c+|H|^2\right)-\frac{1}{2}\sum_{i=1}^{n}R_{ii}-Ric_{11}+(n-1)c.
\end{aligned}$$

Hence, if

$$Ric > \frac{n(n-1)}{n+2}\left(c+|H|^2\right),$$

then

$$\sum_{j=2}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{1j}^{\alpha}\right)^2 - h_{11}^{\alpha}h_{jj}^{\alpha}\right) < (n-1)c,$$

which means that there is no stable integral 1-currents.

Now we consider the case of $2 \le p \le n/2$.

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right)\\
=&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}2\left(\mathring{h}_{ij}^{\alpha}\right)^2 + \sum_{\alpha=1}^{m}\left(\left(\sum_{i=1}^{p}\mathring{h}_{ii}^{\alpha}\right)^2 - (n-2p)H^{\alpha}\sum_{i=1}^{p}\mathring{h}_{ii}^{\alpha}\right) - p(n-p)\,|H|^2\\
=&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}2\left(\mathring{h}_{ij}^{\alpha}\right)^2 + \sum_{\alpha=1}^{m}\left(\left(\sum_{i=1}^{p}\mathring{h}_{ii}^{\alpha} - \frac{n-2}{2}pH^{\alpha}\right)^2 + (p-1)nH^{\alpha}\sum_{i=1}^{p}\mathring{h}_{ii}^{\alpha}\right)\\
&-\frac{(n-2)^2}{4}p^2\,|H|^2 - p(n-p)\,|H|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right)\\
=&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}2\left(\mathring{h}_{ij}^{\alpha}\right)^2 + \sum_{\alpha=1}^{m}\Bigg(\left(\sum_{j=p+1}^{n}\mathring{h}_{jj}^{\alpha} - \frac{n-2}{2}(n-p)H^{\alpha}\right)^2\\
&+(n-p-1)nH^{\alpha}\sum_{j=p+1}^{n}\mathring{h}_{jj}^{\alpha}\Bigg)\\
&-\frac{(n-2)^2}{4}(n-p)^2\,|H|^2 - p(n-p)\,|H|^2.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right)$$

$$
\begin{aligned}
&= \frac{n-p-1}{n-2}\left(\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}2\left(\mathring{h}_{ij}^{\alpha}\right)^2+\sum_{\alpha=1}^{m}\left(\sum_{i=1}^{p}\mathring{h}_{ii}^{\alpha}-\frac{n-2}{2}pH^{\alpha}\right)^2-\frac{(n-2)^2}{4}p^2\,|H|^2\right)\\
&\quad+\frac{p-1}{n-2}\left(\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}2\left(\mathring{h}_{ij}^{\alpha}\right)^2+\sum_{\alpha=1}^{m}\left(\sum_{j=p+1}^{n}\mathring{h}_{jj}^{\alpha}-\frac{n-2}{2}(n-p)H^{\alpha}\right)^2\right.\\
&\quad\left.-\frac{(n-2)^2}{4}(n-p)^2\,|H|^2\right)-p(n-p)\,|H|^2 .
\end{aligned}
$$

Thus, for $2 \le p \le n/2$,

$$
\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2-h_{ii}^{\alpha}h_{jj}^{\alpha}\right)\\
&\le \frac{n-p-1}{n-2}\left(\sum_{i=1}^{p}\sum_{j\neq i}\sum_{\alpha=1}^{m}p\left(\mathring{h}_{ij}^{\alpha}\right)^2+\sum_{\alpha=1}^{m}p\sum_{i=1}^{p}\left(\mathring{h}_{ii}^{\alpha}-\frac{n-2}{2}H^{\alpha}\right)^2-\frac{(n-2)^2}{4}p^2\,|H|^2\right)\\
&\quad+\frac{p-1}{n-2}\left(\sum_{j=p+1}^{n}\sum_{i\neq j}\sum_{\alpha=1}^{m}(n-p)\left(\mathring{h}_{ij}^{\alpha}\right)^2+\sum_{\alpha=1}^{m}(n-p)\sum_{j=p+1}^{n}\left(\mathring{h}_{jj}^{\alpha}-\frac{n-2}{2}H^{\alpha}\right)^2\right.\\
&\quad\left.-\frac{(n-2)^2}{4}(n-p)^2\,|H|^2\right)-p(n-p)\,|H|^2\\
&=\frac{(n-p-1)p}{n-2}\sum_{i=1}^{p}\sum_{\alpha=1}^{m}\left(\sum_{j\neq i}\left(\mathring{h}_{ij}^{\alpha}\right)^2+\left(\mathring{h}_{ii}^{\alpha}-\frac{n-2}{2}H^{\alpha}\right)^2-\frac{n^2}{4}\left|H^{\alpha}\right|^2+(n-1)\left|H^{\alpha}\right|^2\right)\\
&\quad+\frac{(p-1)(n-p)}{n-2}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(\sum_{i\neq j}\left(\mathring{h}_{ij}^{\alpha}\right)^2+\left(\mathring{h}_{jj}^{\alpha}-\frac{n-2}{2}H^{\alpha}\right)^2\right.\\
&\quad\left.-\frac{n^2}{4}\left|H^{\alpha}\right|^2+(n-1)\left|H^{\alpha}\right|^2\right)-p(n-p)\,|H|^2\\
&\le\frac{(n-p-1)p}{n-2}(-K_p+(n-1)p\,|H|^2)+\frac{(p-1)(n-p)}{n-2}\frac{n-p}{p}(-K_p+(n-1)p\,|H|^2)\\
&\quad-p(n-p)\,|H|^2\\
&=-\frac{(n+2)p(n-p)-n^2}{(n-2)p}(K_p-(n-1)p\,|H|^2)-p(n-p)\,|H|^2 ,
\end{aligned}
$$

where

$$
K_p=\min_{\{e_i\}}\left\{\sum_{i=1}^{p}Ric_{ii}-(n-1)pc\right\}=:Ric_{(p)}-(n-1)pc.
$$

Hence,

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right)$$
$$\leq \frac{(n+2)p(n-p)-n^2}{(n-2)}\left((n-1)(c+|H|^2) - \frac{Ric_{(p)}}{p}\right) - p(n-p)\,|H|^2 .$$

Consequently, if

$$\frac{(n+2)p(n-p)-n^2}{(n-2)}\left((n-1)(c+|H|^2) - \frac{Ric_{(p)}}{p}\right) - p(n-p)\,|H|^2 < p(n-p)c,$$

or equivalently,

$$\frac{Ric_{(p)}}{p} > \left(n-1-\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\right)\left(c+|H|^2\right),$$

we have

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha=1}^{m}\left(2\left(h_{ij}^{\alpha}\right)^2 - h_{ii}^{\alpha}h_{jj}^{\alpha}\right) < p(n-p)c.$$

Finally, since $n \geq 4$ and $c+|H|^2 \geq 0$, we have

$$\frac{Ric_{(p)}}{p} \geq Ric_{\min},$$
$$(n-2)\left(c+|H|^2\right) \geq \max_{1<p<n-1}\left(n-1-\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\right)\left(c+|H|^2\right),$$
$$(n-2)\left(c+|H|^2\right) \geq \frac{n(n-1)}{n+2}\left(c+|H|^2\right).$$

Thus, if $Ric > (n-2)\left(c+|H|^2\right)$ and $n \geq 4$, then there is no stable integral p-currents for $0 < p < n$. $\square$

The idea of proving Theorem 1.7 is similar as the proof of Theorem 1.1. But since the Ricci curvature is the sum of sectional curvatures, we must confront a more complicated algebra than the proof of Theorem 1.1. Hence, before proving Theorem 1.7, we need an algebraic lemma.

Given a matrix $A \in M_{n\times n}(\mathbb{R})$, we extend A linearly into an operator $A : \Lambda^*\mathbb{R}^n \longrightarrow \Lambda^*\mathbb{R}^n$ satisfying

$$A(\omega \wedge \eta) = A(\omega)\wedge \eta + \omega \wedge A(\eta), \quad \forall \omega, \eta \in \Lambda^*\mathbb{R}^n.$$

For the matrix A, as an operator of $\mathbb{R}^n$ to itself, we denote by $|A|_2$ its operator norm, i.e.,

$$|A|_2 := \max_{0 \neq x \in \mathbb{R}^n} \frac{|Ax|}{|x|}.$$

It is obvious that $|A|_2^2$ is the largest eigenvalue of A^*A.

Now we state the following algebraic Lemma.

Lemma 4.1. *For every symmetric matrices $A^\alpha \in M_{n\times n}(\mathbb{R})$, $1 \leq \alpha \leq m$, we have*

$$\sum_{\alpha=1}^{m} \left|A^\alpha \omega\right|^2 \leq p^2 \left|\sum_{\alpha=1}^{m} (A_\alpha)^2\right|_2 |\omega|^2, \quad \forall \omega \in \Lambda^p \mathbb{R}^n, \quad 0 \leq p \leq n.$$

Proof. If $p = 1$, a direct verification claims that this conjecture is true, i.e., for every real numbers $x_1, \dots, x_n$, we have

$$\sum_{\alpha=1}^{m}\sum_{i=1}^{n}\left|\sum_{j=1}^{n} A_{ij}^\alpha x_j\right|^2 \leq \left|\sum_{\alpha=1}^{m}(A^\alpha)^2\right|_2 \sum_{j=1}^{n}\left|x_j\right|^2.$$

Set

$$\omega = \frac{1}{p!}\omega_{i_1\dots i_p} e_{i_1}\wedge\cdots\wedge e_{i_p} = \sum_{1\leq i_1<\cdots<i_p\leq n} \omega_{i_1\dots i_p} e_{i_1}\wedge\cdots\wedge e_{i_p},$$

then

$$\begin{aligned} A^\alpha\omega &= \frac{1}{p!}\omega_{i_1\dots i_p} A^\alpha(e_{i_1})\wedge\cdots\wedge e_{i_p} + \cdots + \frac{1}{p!}\omega_{i_1\dots i_p} e_{i_1}\wedge\cdots\wedge A^\alpha(e_{i_p}) \\ &= \frac{1}{(p-1)!}\omega_{i_1 i_2\dots i_p} A_{i_1 k}^\alpha e_k \wedge e_{i_2}\cdots\wedge e_{i_p} \\ &= p \sum_{1\leq k<i_2<\cdots<i_p\leq n} \tilde{\omega}_{k i_2\dots i_p} e_k\wedge e_{i_2}\cdots\wedge e_{i_p}, \end{aligned}$$

where $\tilde{\omega}_{k i_2\dots i_p}$ is the antisymmetrizer of $\sum_{i_1=1}^{n}\omega_{i_1 i_2\dots i_p} A_{i_1 k}^\alpha$, i.e.,

$$\tilde{\omega}_{k i_2\dots i_p} = \frac{1}{p}\left(\sum_{i_1=1}^{n}\omega_{i_1 i_2\dots i_p}A_{i_1 k}^\alpha - \sum_{i_1=1}^{n}\omega_{i_1 i_2\dots i_{p-1} k}A_{i_1 i_p}^\alpha - \cdots - \sum_{i_1=1}^{n}\omega_{i_1 k i_3\dots i_p}A_{i_1 i_2}^\alpha\right).$$

Therefore, we obtain

$$\begin{aligned} \sum_{\alpha=1}^{m}\left|A^\alpha\omega\right|^2 &\leq \frac{p^2}{p!}\sum_{\alpha=1}^{m}\sum_{i_2,\dots,i_p,k}\left|\tilde{\omega}_{k i_2\dots i_p}\right|^2 \\ &\leq \frac{p}{p!}\sum_{\alpha=1}^{m}\sum_{\#\{i_2,\dots,i_p,k\}=p}\left(\left|\sum_{i_1}\omega_{i_1 i_2\dots i_p}A_{i_1 k}^\alpha\right|^2 + \left|\sum_{i_1}\omega_{i_1 i_2\dots i_{p-1}k}A_{i_1 i_p}^\alpha\right|^2 + \cdots\right. \end{aligned}$$

$$
\begin{aligned}
&+\left|\sum_{i_1}\omega_{i_1 k i_3\dots i_p}A^{\alpha}_{i_1 i_2}\right|^2\Bigg)\\
&=\frac{p^2}{p!}\sum_{i_2,\dots,i_p}\sum_{\alpha=1}^{m}\sum_{k\notin\{i_2,\dots,i_p\}}\left|\sum_{i_1}\omega_{i_1 i_2\dots i_p}A^{\alpha}_{i_1 k}\right|^2\\
&\le\frac{p^2}{p!}\sum_{i_2,\dots,i_p}\left|\sum_{\alpha=1}^{m}(A^{\alpha})^2\right|_2\sum_{i_1}\omega^2_{i_1 i_2\dots i_p}\\
&=p^2\left|\sum_{\alpha=1}^{m}(A^{\alpha})^2\right|_2|\omega|^2. \quad \square
\end{aligned}
$$

Proof of Theorem 1.7. Let $\omega\in\Omega^p(M)$ with $|\omega|=1$. First,

$$
\begin{aligned}
&\left|\mathring{S}(\omega)-\frac{n-2p}{2}H\omega\right|^2-\frac{n^2}{4}|H|^2\\
=&\left|\mathring{S}(\omega)\right|^2-(n-2p)\left\langle\mathring{S}(\omega),H\omega\right\rangle-p(n-p)|H|^2\\
=&\left|\mathring{S}(\omega)-\frac{n-2}{2}pH\omega\right|^2+(p-1)n\left\langle\mathring{S}(\omega),H\omega\right\rangle-\frac{(n-2)^2}{4}p^2|H|^2-p(n-p)|H|^2.
\end{aligned}
$$

Moreover, a direct calculation yields

$$
\begin{aligned}
&\left|\mathring{S}(\omega)-\frac{n-2p}{2}H\omega\right|^2-\frac{n^2}{4}|H|^2\\
=&\left|\mathring{S}(*\omega)-\frac{n-2(n-p)}{2}H*\omega\right|^2-\frac{n^2}{4}|H|^2\\
=&\left|\mathring{S}(*\omega)-\frac{n-2}{2}(n-p)H*\omega\right|^2+(n-p-1)n\left\langle\mathring{S}(*\omega),H*\omega\right\rangle\\
&-\frac{(n-2)^2}{4}(n-p)^2|H|^2-p(n-p)|H|^2\\
=&\left|\mathring{S}(\omega)+\frac{n-2}{2}(n-p)H\omega\right|^2-(n-p-1)n\left\langle\mathring{S}(\omega),H\omega\right\rangle\\
&-\frac{(n-2)^2}{4}(n-p)^2|H|^2-p(n-p)|H|^2.
\end{aligned}
$$

Therefore,

$$
\left|\mathring{S}(\omega)-\frac{n-2p}{2}H\omega\right|^2-\frac{n^2}{4}|H|^2
$$

$$= \frac{n-p-1}{n-2}\left(\left|\mathring{S}(\omega) - \frac{n-2}{2}pH\omega\right|^2 - \frac{(n-2)^2}{4}p^2|H|^2\right)$$
$$+ \frac{p-1}{n-2}\left(\left|\mathring{S}(\omega) + \frac{n-2}{2}(n-p)H\omega\right|^2 - \frac{(n-2)^2}{4}(n-p)^2|H|^2\right) - p(n-p)|H|^2 .$$

Notice that acting on p-forms, $\mathring{S} - \frac{n-2}{2}pH$ can be viewed as a linearly extension operator of $\left(\mathring{A}^\alpha - \frac{n-2}{2}H^\alpha \,\mathrm{Id}\right) \otimes \nu_\alpha$. In particular, applying Lemma 4.1, we obtain

$$\left|\mathring{S}(\omega) - \frac{n-2}{2}pH\omega\right|^2 \le p^2 \left|\sum_{\alpha=1}^m \left(\mathring{A}^\alpha - \frac{n-2}{2}H^\alpha \,\mathrm{Id}\right)^2\right|_2 .$$

Similarly,

$$\left|\mathring{S}(\omega) + \frac{n-2}{2}(n-p)H\omega\right|^2 = \left|\mathring{S}(*\omega) - \frac{n-2}{2}(n-p)H*\omega\right|^2$$
$$\le (n-p)^2 \left|\sum_{\alpha=1}^m \left(\mathring{A}^\alpha - \frac{n-2}{2}H^\alpha \,\mathrm{Id}\right)^2\right|_2 .$$

Consequently,

$$\left|\mathring{S}|_{\Lambda^p} - \frac{n-2p}{2}H\right|_{op}^2 - \frac{n^2}{4}|H|^2$$
$$\le \frac{(n-p-1)p^2 + (p-1)(n-p)^2}{n-2}\left(\left|\sum_{\alpha=1}^m \left(\mathring{A}^\alpha - \frac{n-2}{2}H^\alpha \,\mathrm{Id}\right)^2\right|_2 - \frac{(n-2)^2}{4}|H|^2\right)$$
$$= \frac{(n+2)p(n-p) - n^2}{n-2}\left(\left|\sum_{\alpha=1}^m \left(\mathring{A}^\alpha - \frac{n-2}{2}H^\alpha \,\mathrm{Id}\right)^2\right|_2 - \frac{(n-2)^2}{4}|H|^2\right).$$

By Gauss equation, we know that

$$Ric_{ii} = \sum_{j=1}^n \bar{R}_{ijij} - \left|\mathring{B}_{ii}\right|^2 + (n-2)\left\langle \mathring{B}_{ii}, H\right\rangle - \sum_{j\neq i}\left|\mathring{B}_{ij}\right|^2 + (n-1)|H|^2$$
$$= \sum_{j=1}^n \bar{R}_{ijij} - \left|\mathring{B}_{ii} - \frac{n-2}{2}H\right|^2 - \sum_{j\neq i}\left|\mathring{B}_{ij}\right|^2 + \frac{n^2}{4}|H|^2 .$$

By choosing $\{e_i\}$ so that

$$\sum_{\alpha=1}^{m}\left(\mathring{A}^{\alpha}-\frac{n-2}{2}H^{\alpha}\,\mathrm{Id}\right)^{2}$$

is a diagonalized matrix, without loss of generality, assume the largest eigenvalue is $\left|\mathring{B}_{11}-\frac{n-2}{2}H\right|^{2}$ we obtain

$$\begin{aligned}
Ric_{\min}\le Ric_{11}&=\sum_{j=1}^{n}\bar{R}_{1j1j}-\left|\mathring{B}_{11}-\frac{n-2}{2}H\right|^{2}+\frac{n^{2}}{4}|H|^{2}\\
&=\sum_{j=1}^{n}\bar{R}_{1j1j}-\left|\sum_{\alpha=1}^{m}\left(\mathring{A}^{\alpha}-\frac{n-2}{2}H^{\alpha}\,\mathrm{Id}\right)^{2}\right|_{2}+\frac{n^{2}}{4}|H|^{2}\\
&\le(n-1)c^{*}-\left|\sum_{\alpha=1}^{m}\left(\mathring{A}^{\alpha}-\frac{n-2}{2}H^{\alpha}\,\mathrm{Id}\right)^{2}\right|_{2}+\frac{n^{2}}{4}|H|^{2}.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\left|\mathring{S}|_{\Lambda^{p}}-\frac{n-2p}{2}H\right|_{op}^{2}-\frac{n^{2}}{4}|H|^{2}\\
&\quad\le\frac{(n+2)p(n-p)-n^{2}}{n-2}\Big((n-1)(c^{*}+|H|^{2})-Ric_{\min}\Big)-p(n-p)|H|^{2}.
\end{aligned}$$

Hence, by assumption

$$\begin{aligned}
W^{[p]}&\ge p(n-p)c_{*}-\frac{(n+2)p(n-p)-n^{2}}{n-2}\Big((n-1)(c^{*}+|H|^{2})-Ric_{\min}\Big)+p(n-p)|H|^{2}\\
&=\frac{(n+2)p(n-p)-n^{2}}{n-2}\Bigg(Ric_{\min}-(n-1)\left(c^{*}+|H|^{2}\right)\\
&\qquad+\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^{2}}\left(c_{*}+|H|^{2}\right)\Bigg).
\end{aligned}$$

Therefore, according to Lemma 3.1, we have our conclusion. □

Proof of weak Ejiri type Theorem 1.8. By Theorem 1.7 and a similar argument as the proof of Theorem 1.6. □

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Appendix A. An example

The following example shows that the conditions mentioned in this paper are sharp.

Example A.1. Consider the following Clifford torus

$$\mathbb{S}^p\left(\frac{\mu}{\sqrt{1+\mu^2}}\right)\times\mathbb{S}^{n-p}\left(\frac{1}{\sqrt{1+\mu^2}}\right)\subset\mathbb{S}^{n+1}(1)\subset\mathbb{R}^{n+2},\quad p=1,2,\dots,n-1,\quad \mu>0.$$

It is obvious that $b_p\geq \max\{p,n-p\}$ and $\lambda_{1,p}=\lambda_{1,n-p}=0$.

Let $\phi=(x,y)$ be the position vector, then the first fundamental form is given by

$$\mathrm{d}s^2=\mathrm{d}x\mathrm{d}x+\mathrm{d}y\mathrm{d}y.$$

A unit norm vector field is $\nu=(-\mu^{-1}x,\mu y)$. Hence, the second fundamental form B is

$$\begin{aligned}B&=-\Big\langle \mathrm{d}(x,y),\mathrm{d}(-\mu^{-1}x,\mu y)\Big\rangle\\&=\mu^{-1}\mathrm{d}x\mathrm{d}x-\mu\mathrm{d}y\mathrm{d}y.\end{aligned}$$

Consequently the principal curvatures are μ^{-1} and $-\mu$ with multiplicity p and $n-p$ respectively. In particular,

$$\begin{aligned}H&=\frac{1}{n}\Big(p\mu^{-1}-(n-p)\mu\Big),\\|B|^2&=p\mu^{-2}+(n-p)\mu^2,\\\left|\mathring{B}\right|^2&=\frac{p(n-p)}{n}\Big(\mu^{-1}+\mu\Big)^2.\end{aligned}$$

Moreover, the sectional curvature satisfies

$$K_{ij}=\begin{cases}1+\mu^{-2}, & 1\leq i<j\leq p;\\1+\mu^2, & p+1\leq i\neq j\leq n;\\0, & 1\leq i\leq p,\, p+1\leq j\leq n.\end{cases}$$

The Ricci curvature satisfies

$$Ric_{ii}=\begin{cases}(p-1)(1+\mu^{-2}), & 1\leq i\leq p;\\(n-p-1)(1+\mu^2), & p+1\leq i\leq n.\end{cases}$$

A direct computation gives the following: if $(n-2p)(p\mu^{-1}-(n-p)\mu)\leq 0$, then

$$\frac{\left|\mathring{B}\right|}{\sqrt{n}}+\frac{|n-2p|\,|H|}{2\sqrt{p(n-p)}}-\sqrt{1+\frac{n^2\,|H|^2}{4p(n-p)}}$$

$$
\begin{aligned}
&= \frac{\sqrt{p(n-p)}(\mu^{-1}+\mu)}{n} + \frac{|n-2p|\left|p\mu^{-1}-(n-p)\mu\right|}{2n\sqrt{p(n-p)}} - \frac{(p\mu^{-1}+(n-p)\mu)}{2\sqrt{p(n-p)}} \\
&= \frac{2p(n-p)(\mu^{-1}+\mu)+|n-2p|\left|p\mu^{-1}-(n-p)\mu\right|-(np\mu^{-1}+n(n-p)\mu)}{2n\sqrt{p(n-p)}} \\
&= \frac{2p(n-p)(\mu^{-1}+\mu)-(n-2p)(p\mu^{-1}-(n-p)\mu)-(np\mu^{-1}+n(n-p)\mu)}{2n\sqrt{p(n-p)}} \\
&= 0.
\end{aligned}
$$

Hence, if $(n-2p)(p\mu^{-1}-(n-p)\mu) \leq 0$, we obtain that $|B|^2 = \alpha(1, p, n, H)$.

When $p = 1$ or $p = n-1$, we have

$$
Ric_{min} - \left(n-1-\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\right)(1+|H|^2) = 0.
$$

When $1 < p < n-1$, taking $\mu = \sqrt{\frac{p-1}{n-p-1}}$, we have $Ric_{ii} \equiv n-2$ for all $1 \leq i \leq n$ which implies $Ric = (n-2)g$. Therefore,

$$
\begin{aligned}
&Ric_{min} - \left(n-1-\frac{(n-2)p(n-p)}{(n+2)p(n-p)-n^2}\right)(1+|H|^2) \\
&= Ric_{min} - \frac{n^2(p-1)(n-p-1)}{(n+2)p(n-p)-n^2} \cdot \left(1+\left(\frac{p\mu^{-1}-(n-p)\mu}{n}\right)^2\right) \\
&= n-2-\frac{n^2(p-1)(n-p-1)}{(n+2)p(n-p)-n^2} \cdot \frac{(n-2)((n+2)p(n-p)-n^2)}{n^2(p-1)(n-p-1)} \\
&= n-2-(n-2) \\
&= 0.
\end{aligned}
$$

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