ON THE VOLUME OF LOCALLY CONFORMALLY FLAT 4-DIMENSIONAL CLOSED HYPERSURFACE

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ABSTRACT. Let M be a 5-dimensional Riemannian manifold with $Sec_M \in [0,1]$ and Σ be a locally conformally flat closed hypersurface in M with mean curvature function H. We prove that there exists $\varepsilon_0 > 0$ such that

$$(1) \qquad \int_{\Sigma} (1+H^2)^2 \geq \frac{4\pi^2}{3} \chi(\Sigma),$$

provided $|H| \leq \varepsilon_0$, where $\chi(\Sigma)$ is the Euler number of Σ. In particular, if Σ is a locally conformally flat minimal hypersphere in M, then $Vol(\Sigma) \geq 8\pi^2/3$, which partially answers a question proposed by Mazet and Rosenberg. Moreover, we show that if M is (some special but large class) rotationally symmetric, then the inequality (1) holds for all H.

1. INTRODUCTION

Let M be a 2-sphere with a smooth Riemannian metric such that the curvature is between 0 and 1. It is known (see [5] or [8]) that the length of an embedded closed geodesic in M is at least 2π, which is the length of the standard circle in the Euclidean plane. When M is a Riemannian 3-manifold with sectional curvature between 0 and 1, one can easily apply the Gauss equation and the Gauss-Bonnet theorem to obtain that an embedded minimal sphere Σ in M has area at least 4π, that is,

$$4\pi = \int_{\Sigma} Sec_{\Sigma} = \int_{\Sigma} R_{1212} = \int_{\Sigma} \left(\overline{R}_{1212} - \frac{1}{2}|A|^2 \right) \leq \int_{\Sigma} \overline{R}_{1212} \leq Vol\,(\Sigma),$$

where R and $\overline{R}$ denote the curvature tensors of Σ and M, and A denotes the second fundamental form of Σ in M.

In [6], Mazet and Rosenberg study the equality case and get a rigidity theorem for M. The authors also put forward two very interesting questions: one of them is, if M is an $(n+1)$-Riemannian manifold with $Sec_M \in [0,1]$, does an embedded minimal hypersphere (i.e., minimal hypersurface diffeomorphic to the standard Euclidean n-sphere $\mathbb{S}^n$) have volume at least the volume of $\mathbb{S}^n$? In 1974, Hoffman

and Spruck [4] studied the isoperimetric inequality and showed that if M is a simply connected Riemannian $(n + 1)$-manifold with $Sec_M \in [1/4, 1]$, then any closed minimal hypersurface has at least the volume of $\mathbb{S}^n$. Therefore, if the answer to Mazet and Rosenberg's question is true, it can be seen as a generalization (with topological restriction) of Hoffman and Spruck's result. We would like to point out that if $Sec_M \in [0, 1]$, the topological restriction on Σ is necessary. Actually, given $\varepsilon > 0$, let Σ be a flat n-torus with $Vol(\Sigma) \leq \varepsilon$ (which can be done by passing a dilation). Then Σ is a totally geodesic closed hypersurface embedded in $\Sigma \times \mathbb{R}$ whose sectional curvature is 0.

Note that in the case of $n = 2$, every surface admits isothermal coordinates and therefore is locally conformally flat. It seems natural to add the condition "locally conformally flat" on the hypersurface in the high dimensional case. In this paper, we focus our attention on the case of $n = 4$, pose the assumption that Σ is locally conformally flat, and partially answer the question proposed by Mazet and Rosenberg. Actually, we get a more general result as follows.

Theorem 1.1. *Let M be a 5-dimensional Riemannian manifold with $Sec_M \in [0, 1]$, and Σ be an embedded locally conformally flat closed hypersurface in M with mean curvature function H. Then we have*

$$(2) \qquad \int_{\Sigma} \left((1 + H^2)^2 + |H| f(|H|) \right) \geq \frac{4\pi^2}{3} \chi(\Sigma) \,,$$

where f is a nonnegative function defined in Section 2, and $\chi(\Sigma)$ is the Euler number of Σ.

Moreover, there exists $\varepsilon_0 > 0$ such that if $|H| \leq \varepsilon_0$, we obtain

$$(3) \qquad \int_{\Sigma} \left(1 + H^2 \right)^2 \geq \frac{4\pi^2}{3} \chi(\Sigma).$$

The equality holds if and only if the mean curvature H is constant, and Σ is totally umbilic and isometric to $\mathbb{S}^4 \left(\frac{1}{1+H^2} \right)$.

As an immediate corollary of Theorem 1.1, the following result partially answers the question proposed by Mazet and Rosenberg.

Theorem 1.2. *Let M be a 5-dimensional Riemannian manifold with $Sec_M \in [0, 1]$, and Σ be an embedded locally conformally flat minimal hypersphere in M. Then*

$$Vol(\Sigma) \geq \frac{8\pi^2}{3} = Vol\left(\mathbb{S}^4 \right).$$

The equality holds if and only if Σ is totally geodesic and isometric to $\mathbb{S}^4$.

This paper is organized as follows. In Section 2, we list some notation and known formulas and give the proof of Theorem 1.1. In Section 3, we deal with a special case when M is rotationally symmetric and get the lower bound volume for all H; see Theorem 3.2.

2. Preliminaries and proof of Theorem 1.1

Let $(M, \bar{g})$ be an $(n + 1)$ dimensional Riemannian manifold, and (Σ, g) be a hypersurface isometrically immersed in M. If there is no ambiguity, $\langle \cdot \, , \, \cdot \rangle$ will

denote both $\bar{g}$ and g. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection induced by metric $\bar{g}$ and g respectively. Let R be the curvature tensor on Σ defined by, for all $X, Y, Z, W \in \mathfrak{X}(T\Sigma)$,

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle,$$

where $R(X, Y) = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X,Y]}$. Also let $\overline{R}$ be the curvature tensor on M which is defined similarly.

Let $e_1, \cdots, e_n$ be a local orthonormal frame on Σ. For all $1 \leq i, j, k, l \leq n$, write

$$R_{ijkl} = R(e_i, e_j, e_k, e_l), \quad \overline{R}_{ijkl} = \overline{R}(e_i, e_j, e_k, e_l).$$

The sectional curvature will be

$$Sec_M(e_i \wedge e_j) = R_{ijij}, \quad Sec_{\overline{M}}(e_i \wedge e_j) = \overline{R}_{ijij}.$$

Let A be the second fundamental form of Σ in M, and $h_{ij} = \langle A(e_i), e_j \rangle$ be the coefficients of A. Then the Gauss equation can be written as

$$R_{ijkl} = \overline{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.$$

We also denote by $\mathbb{S}^n$ the standard unit n sphere in $(n+1)$ Euclidean space and by $\mathbb{S}^n(r)$ the round n-sphere with radius r. Now we will prove Theorem 1.1.

Proof of Theorem 1.1. The Gauss-Bonnet-Chern formula for a closed 4-manifold Σ is (see [2] or [3])

$$(4) \qquad 4\pi^2 \chi(\Sigma) = \int_\Sigma \left(\frac{S^2}{12} - \frac{|Ric|^2}{4} + \frac{|W|^2}{8} \right),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, S is the scalar curvature, Ric is the Ricci tensor and W is the Weyl tensor. It is well known that when the dimension is greater than 3, locally conformally flatness is equivalent to Weyl tensor vanishing. Therefore, to prove the first part of the theorem, it is sufficient to prove, pointwisely,

Claim.

$$Q := \frac{S^2}{12} - \frac{|Ric|^2}{4} \leq 3(1 + H^2)^2 + 3|H|f(|H|).$$

Next we will consider our problem at one point $p \in \Sigma$ (in the calculations, we omit the letter "p" for simplicity). Throughout this proof, i, j, k, l will range from 1 to 4 if there is no special explanation.

Firstly, by the Gauss equation, we obtain

$$(5) \quad S^2 = \left(\sum_{i,j} R_{ijij} \right)^2 = \left(\sum_{i,j} \overline{R}_{ijij} + 16H^2 - |A|^2 \right)^2 = \left(\sigma + 12H^2 - |\mathring{A}|^2 \right)^2$$

$$= \sigma^2 + 144H^4 + |\mathring{A}|^4 + 24\sigma H^2 - 2\sigma|\mathring{A}|^2 - 24H^2|\mathring{A}|^2,$$

where

$$\sigma := \sum_{i,j} \overline{R}_{ijij} \quad \text{and} \quad \mathring{A} := A - HI;$$

i.e., $\mathring{A}$ is the traceless part of A.

For simplicity, let e_1, e_2, e_3, e_4 be the principal directions at the point p, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the corresponding principal curvatures, so we have

$$|Ric|^2 = \sum_{i,j} \left(\sum_k R_{ikjk} \right)^2 = \sum_{i,j} \left(\sum_k \overline{R}_{ikjk} + 4\delta_{ij}\lambda_i H - \delta_{ij}\lambda_i\lambda_j \right)^2.$$

For simplifying $|Ric|^2$, we need to introduce some notation as follows:

$$a_{ij} := \sum_k \overline{R}_{ikjk}, \quad \mathring{a}_{ij} := \sum_k \overline{R}_{ikjk} - \frac{\sigma}{4}\delta_{ij}.$$

Note that σ is the trace of (a_{ij}) and $(\mathring{a}_{ij})$ is the traceless part of (a_{ij}). Using this notation, we get

$$
\begin{aligned}
(6) \qquad |Ric|^2 &= \sum_{i,j} (a_{ij} + 4\delta_{ij}\lambda_i H - \delta_{ij}\lambda_i\lambda_j)^2 \\
&= \sum_{i,j} a_{ij}^2 + 16H^2|A|^2 + \sum_i \lambda_i^4 \\
&\quad + 8H\sum_i \lambda_i a_{ii} - 2\sum_i \left(\lambda_i^2 a_{ii}\right) - 8H\sum_i \lambda_i^3 \\
&= \frac{\sigma^2}{4} + |\mathring{a}|^2 + 16H^2|\mathring{A}|^2 + 64H^4 + \sum_i \lambda_i^4 \\
&\quad - 2\sum_i \left(\lambda_i^2 - 4H\lambda_i\right) a_{ii} - 8H\sum_i \lambda_i^3,
\end{aligned}
$$

where $|\mathring{a}|^2 = \sum_{i,j} \mathring{a}_{ij}^2$. Next we set $\mu_i = \lambda_i - H$ which is the eigenvalue of $\mathring{A}$. Then by a direct computation, we have

$$
\begin{aligned}
(7) \qquad \sum_i \lambda_i^4 &= \sum_i \mu_i^4 + 12H^4 - 6H^2|A|^2 + 4H\sum_i \lambda_i^3 \\
&= \sum_i \mu_i^4 - 12H^4 - 6H^2|\mathring{A}|^2 + 4H\sum_i \lambda_i^3
\end{aligned}
$$

and

$$(8) \qquad \sum_i \lambda_i^3 = \sum_i \mu_i^3 + 4H^3 + 3H|\mathring{A}|^2.$$

Combining (5), (6), (7) and (8) we obtain

$$
\begin{aligned}
(9) \qquad Q = \frac{1}{12} \Bigg(&\frac{\sigma^2}{4} + 6\sigma H^2 + 36H^4 + |\mathring{A}|^4 - 3\sum_i \mu_i^4 + 6\sum_i \mu_i^2 \left(a_{ii} - \frac{\sigma}{3}\right) \\
&-12H\sum_i \mu_i a_{ii} - 18H^2|\mathring{A}|^2 + 12H\sum_i \mu_i^3 - 3|\mathring{a}|^2 \Bigg).
\end{aligned}
$$

We will divide our proof of the Claim into two main cases. We will see that locally conformally flatness will play a key role in the estimate of Q.

(i) At the point p, $|\mathring{A}|^2(p) \leq 12 + 24H^2(p)$.

By the Gauss equation,

$$S = \sigma + 12H^2 - |\mathring{A}|^2,$$

where σ is defined in (5). Since $0 \le \sigma \le 12$, on the one hand,

$$S = \sigma + 12H^2 - |\mathring{A}|^2 \le 12 + 12H^2.$$

On the other hand,

$$S = \sigma + 12H^2 - |\mathring{A}|^2 \ge 12H^2 - (12 + 24H^2) = -12(H^2 + 1).$$

The above two inequalities yield $S^2 \le 144(1 + H^2)^2$. Therefore,

$$Q = \frac{S^2}{12} - \frac{|Ric|^2}{4} = \frac{S^2}{48} - \frac{|E|^2}{4} \le \frac{S^2}{48} \le 3(H^2 + 1)^2,$$

where E is the traceless part of the Ricci tensor, namely the Einstein tensor.

(ii) At the point p, $|\mathring{A}|^2(p) \ge 12 + 24H^2(p)$.

The proof of this case is more difficult than of case (i). To prove the Claim, we need to estimate Q by using the equality (9).

First note that for a fixed i, the term $a_{ii} - \frac{\sigma}{3}$ is bounded above by 1. We take $i = 1$ for example:

$$(10) \qquad a_{11} - \frac{\sigma}{3} = \sum_k \overline{R}_{1k1k} - \frac{\sigma}{3} = \sum_k \overline{R}_{1k1k} - \frac{1}{3} \sum_{i,j} \overline{R}_{ijij}$$

$$= \frac{1}{3} \left(\overline{R}_{1212} + \overline{R}_{1313} + \overline{R}_{1414} \right) - \frac{2}{3} \left(\overline{R}_{2323} + \overline{R}_{2424} + \overline{R}_{3434} \right) \le 1$$

where we have used the curvature condition that $0 \le \overline{R}_{ijij} \le 1$ for all $i \ne j$.

By a direct computation, we have

$$(11) \qquad \sum_i \mu_i^4 = \sum_{i=1}^3 \mu_i^4 + \left(\sum_{i=1}^3 \mu_i \right)^4$$

$$= \frac{1}{2} \left(\sum_{i=1}^3 \mu_i^2 + \left(\sum_{i=1}^3 \mu_i \right)^2 \right)^2 + 4\mu_1\mu_2\mu_3 \left(\sum_{i=1}^3 \mu_i \right)$$

$$= \frac{1}{2} |\mathring{A}|^4 - 4 \prod_i \mu_i := \frac{1}{2} |\mathring{A}|^4 - 4\mathcal{K},$$

where $\mathcal{K} = \prod_i \mu_i$ is the Gauss-Kronecker curvature of $\mathring{A}$.

Observing that $\sum_i \mu_i = 0$, we get

$$(12) \qquad -12H \sum_i \mu_i a_{ii} - 3|\mathring{a}|^2$$

$$= -12H \sum_i \mu_i \mathring{a}_{ii} - 3|\mathring{a}|^2$$

$$= -3 \sum_i \left(\mathring{a}_{ii}^2 - 4H\mu_i\mathring{a}_{ii} + 4H^2\mu_i^2 \right) + 12H^2 |\mathring{A}|^2 - 3 \sum_{i \ne j} \mathring{a}_{ij}^2$$

$$\le 12H^2 |\mathring{A}|^2.$$

Combining (9), (10), (11), (12) and the fact that $0 \leq \sigma \leq 12$, we have

$$(13) \qquad Q \leq 3 \left(1 + H^2\right)^2$$
$$+ \frac{1}{12} \left(-\frac{1}{2}|\mathring{A}|^4 + 12H \sum_i \mu_i^3 + 6(1 - H^2)|\mathring{A}|^2 + 12\mathcal{K} \right).$$

Next we will take the Weyl tensor into consideration. The Weyl tensor defined in a coordinate chart is given by (see e.g. [1], p. 117)

$$(14) \qquad W_{ijkl} = R_{ijkl} - \frac{1}{2} \left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} \right)$$
$$+ \frac{S}{6} \left(g_{jl}g_{ik} - g_{jk}g_{il} \right),$$

where $R_{ij} = \sum_k R_{ikjk}$ is the Ricci tensor. Therefore, when $i \neq j$, we have

$$W_{ijij} = R_{ijij} - \frac{1}{2} \left(R_{ii} + R_{jj} \right) + \frac{S}{6}.$$

Now we fix $i = 1$ and $j = 2$ for example and get

$$\frac{S}{6} - W_{1212}$$
$$= \frac{1}{2} \left(R_{11} + R_{22} \right) - R_{1212}$$
$$= \frac{1}{2} \left(R_{1313} + R_{1414} + R_{2323} + R_{2424} \right)$$
$$= \frac{1}{2} \left(R_{1212} + R_{1313} + R_{1414} + R_{2323} + R_{2424} + R_{3434} \right) - \frac{1}{2} \left(R_{1212} + R_{3434} \right)$$
$$= \frac{S}{4} - \frac{1}{2} \left(R_{1212} + R_{3434} \right).$$

As a consequence, for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, we obtain

$$(15) \qquad \frac{S}{6} = R_{ijij} + R_{klkl} - 2W_{ijij}$$
$$= \overline{R}_{ijij} + \overline{R}_{klkl} - 2W_{ijij} + \lambda_i\lambda_j + \lambda_k\lambda_l$$
$$= \overline{R}_{ijij} + \overline{R}_{klkl} - 2W_{ijij} + 2H^2$$
$$+ (\lambda_i - H)(\lambda_j - H) + (\lambda_k - H)(\lambda_l - H)$$
$$= \overline{R}_{ijij} + \overline{R}_{klkl} - 2W_{ijij} + 2H^2 + \mu_i\mu_j + \mu_k\mu_l.$$

Note that the above formula has no summation on i, j, k, l.

In what follows, without loss of generality, we assume, at the point p, that

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4.$$

Next we will split our proof of case (ii) into three parts according to the values of $\mathcal{K}$ (defined in (11)) and μ_i.

(a) $\mathcal{K}(p) \geq 0$, $\mu_1 \geq \mu_2 \geq 0 \geq \mu_3 \geq \mu_4$.

Let $i = 1$ and $j = 2$ in (15), and since locally conformally flatness implies $W \equiv 0$, we get

$$\frac{S}{6} = \overline{R}_{ijij} + \overline{R}_{klkl} + 2H^2 + \mu_1\mu_2 + \mu_3\mu_4 \geq 2H^2.$$

Consequently, $S \geq 12H^2 \geq -12(1+H^2)$. The remaining proof of this part is similar to case (i).

(b) $\mathcal{K}(p) < 0$ and $\mu_1 \geq \mu_2 \geq \mu_3 > 0 > \mu_4$.

In this part, a direct computation gives

$$(16) \qquad \sum_i \mu_i^3 = \sum_{i=1}^{3} \mu_i - \left(\sum_{i=1}^{3} \mu_i \right)^3 \leq 0.$$

Without loss of generality, we assume that $H(p) \geq 0$ (otherwise the term "$12H \sum_i \lambda_i^3$" in (13) will be nonnegative. This case can be dealt with a similar method as the next part (c)). Therefore, combining (13), (16) and the assumption $\mathcal{K} \leq 0$, we obtain

$$(17) \qquad Q \leq 3 \left(1 + H^2 \right)^2 - \frac{1}{24} |\mathring{A}|^2 \left(|\mathring{A}|^2 - 12(1 - H^2) \right).$$

Note that in case (ii)

$$|\mathring{A}|^2 \geq 12 + 24H^2 \geq 12(1 - H^2).$$

Thus the second term in the right hand side of (17) is nonpositive, and consequently we have $Q \leq 3(1 + H^2)^2$.

(c) $\mathcal{K}(p) < 0$ and $\mu_1 > 0 > \mu_2 \geq \mu_3 \geq \mu_4$.

In this part, inequality (13) is not enough for our estimate, and we will go back to equality (9) and estimate term by term.

Firstly, $\sum_i \mu_i^3 \geq 0$, and we will use the following inequality (see [9, Lemma 1]):

$$(18) \qquad \sum_i \mu^3 \leq \frac{1}{\sqrt{3}} |\mathring{A}|^3.$$

Secondly, for the term $6 \sum_i \mu_i^2 \left(a_{ii} - \frac{\sigma}{3}\right)$, under the assumption of this part, we will use a more accurate (than (10)) estimate as follows:

(19)
$$3 \sum_i \mu_i^2 \left(a_{ii} - \frac{\sigma}{3}\right)$$

$$= \mu_1^2 \left(\overline{R}_{1212} + \overline{R}_{1313} + \overline{R}_{1414} - 2\left(\overline{R}_{2323} + \overline{R}_{2424} + \overline{R}_{3434}\right)\right)$$
$$+ \mu_2^2 \left(\overline{R}_{2121} + \overline{R}_{2323} + \overline{R}_{2424} - 2\left(\overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{3434}\right)\right)$$
$$+ \mu_3^2 \left(\overline{R}_{3131} + \overline{R}_{3232} + \overline{R}_{3434} - 2\left(\overline{R}_{1212} + \overline{R}_{1414} + \overline{R}_{2424}\right)\right)$$
$$+ \mu_4^2 \left(\overline{R}_{4141} + \overline{R}_{4242} + \overline{R}_{4343} - 2\left(\overline{R}_{1212} + \overline{R}_{1313} + \overline{R}_{2323}\right)\right)$$
$$= (\mu_1^2 + \mu_2^2 - 2(\mu_3^2 + \mu_4^2))\overline{R}_{1212} + (\mu_1^2 + \mu_3^2 - 2(\mu_2^2 + \mu_4^2))\overline{R}_{1313}$$
$$+ (\mu_1^2 + \mu_4^2 - 2(\mu_2^2 + \mu_3^2))\overline{R}_{1414} + (\mu_2^2 + \mu_3^2 - 2(\mu_1^2 + \mu_4^2))\overline{R}_{2323}$$
$$+ (\mu_2^2 + \mu_4^2 - 2(\mu_1^2 + \mu_3^2))\overline{R}_{2424} + (\mu_3^2 + \mu_4^2 - 2(\mu_1^2 + \mu_2^2))\overline{R}_{3434}$$
$$\leq (\mu_1^2 + \mu_2^2 - 2(\mu_3^2 + \mu_4^2))\overline{R}_{1212} + (\mu_1^2 + \mu_3^2 - 2(\mu_2^2 + \mu_4^2))\overline{R}_{1313}$$
$$+ (\mu_1^2 + \mu_4^2 - 2(\mu_2^2 + \mu_3^2))\overline{R}_{1414}$$
$$= (-2\mu_1\mu_2 - (\mu_3 - \mu_4)^2)\overline{R}_{1212} + (-2\mu_1\mu_3 - (\mu_2 - \mu_4)^2)\overline{R}_{1313}$$
$$+ (-2\mu_1\mu_4 - (\mu_2 - \mu_3)^2)\overline{R}_{1414}$$
$$\leq -2\mu_1(\mu_2 + \mu_3 + \mu_4) = 2\mu_1^2 \leq \frac{3}{2}|\mathring{A}|^2,$$

where we have used the facts that

$$\mu_1 > 0 > \mu_2 \geq \mu_3 \geq \mu_4, \quad \sum_i \mu_i = 0,$$

and the inequality

$$|\mathring{A}|^2 = \sum_i \mu_i^2 \geq \mu_1^2 + \frac{(\mu_2 + \mu_3 + \mu_4)^2}{3} = \frac{4}{3}\mu_1^2.$$

Combining (9), (11), (12), (18), (19) and the fact that $0 \leq \sigma \leq 12$, we obtain

(20)
$$Q \leq 3\left(1 + H^2\right)^2$$
$$+ \frac{1}{12}\left(-\frac{1}{2}|\mathring{A}|^4 + 4\sqrt{3}|H||\mathring{A}|^3 + 3(1 - 2H^2)|\mathring{A}|^2\right)$$
$$:= 3\left(1 + H^2\right)^2 + F(|\mathring{A}|),$$

where

(21)
$$F(|\mathring{A}|) = \frac{1}{12}\left(-\frac{1}{2}|\mathring{A}|^4 + 4\sqrt{3}|H||\mathring{A}|^3 + 3(1 - 2H^2)|\mathring{A}|^2\right)$$
$$= \frac{1}{12}\left(-\frac{1}{2}(|\mathring{A}|^4 - 6|\mathring{A}|^2) + |H|(4\sqrt{3}|\mathring{A}|^3 - 6|H||\mathring{A}|^2)\right).$$

It is easy to see that $F(x)$ attains its maximum at $x_0 = 3\sqrt{3}|H| + \sqrt{3 + 21H^2}$ and decreases when $x \geq x_0$.

Keep in mind that this part is one of the three parts of case (ii), which assumes that

$$|\mathring{A}|^2 \geq 12 + 24H^2.$$

Therefore, if $x_0 \leq \sqrt{12 + 24H^2}$, we have

$$(22) \quad F(|\mathring{A}|) \leq F(\sqrt{12 + 24H^2})$$

$$= -(12H^2 + 6)(2H^2 + 1) + |H| \left. \left(\frac{\sqrt{3}}{3} x^3 - \frac{1}{2}|H|x^2 \right) \right|_{x=\sqrt{12+24H^2}}$$

$$\leq |H| f_1(|H|),$$

where

$$f_1(|H|) = \left. \left(\frac{\sqrt{3}}{3} x^3 - \frac{1}{2}|H|x^2 \right) \right|_{x=\sqrt{12+24H^2}} \geq 0.$$

If $x_0 \geq \sqrt{12 + 24H^2}$, we obtain

$$(23) \quad F(|\mathring{A}|) \leq F(x_0)$$

$$= \frac{1}{12} \left(-\frac{1}{2}(x^4 - 6x^2)|_{x=x_0} + |H|(4\sqrt{3}x^3 - 6|H|x^2)|_{x=x_0} \right)$$

$$\leq \frac{1}{12} \left(-\frac{1}{2}(x^4 - 6x^2)|_{x=\sqrt{12+24H^2}} \right) + |H| f_2(|H|)$$

$$\leq -(12H^2 + 6)(2H^2 + 1) + |H| f_2(|H|)$$

$$\leq |H| f_2(|H|),$$

where

$$f_2(|H|) = \left. \left(\frac{\sqrt{3}}{3} x^3 - \frac{1}{2}|H|x^2 \right) \right|_{x=x_0} \geq 0.$$

Combining (20), (21), (22) and (23), we get

$$(24) \quad Q \leq 3(1 + H^2)^2 + 3|H| f(|H|),$$

where $f(|H|)$ is a function of $|H|$ defined by

$$(25) \quad 3f(|H|) := \begin{cases} f_1(|H|), & x_0 \leq \sqrt{12 + 24H^2}, \\ f_2(|H|), & x_0 \geq \sqrt{12 + 24H^2}. \end{cases}$$

To sum up the above two cases, we have proved the Claim, and inequality (2) follows immediately.

Next we will show that if $|H|$ is small, inequality (3) holds. Checking all the cases in the proof of the Claim, we find inequality (3) holds except for the case (ii)(c). Thus, it is enough to show that if $|H|$ is small, inequality (3) holds in the case (ii)(c). By (20), it is sufficient to show that $F(|\mathring{A}|) \leq 0$ when $|H|$ is small. Observe that $F(|\mathring{A}|)$ can be decomposed as

$$(26) \quad F(|\mathring{A}|) = -\frac{|\mathring{A}|^2}{24} \left(|\mathring{A}| - \eta_1 \right) \left(|\mathring{A}| - \eta_2 \right),$$

where $\eta_1 = 4\sqrt{3}|H| - \sqrt{6 + 36H^2}$, $\eta_2 = 4\sqrt{3}|H| + \sqrt{6 + 36H^2}$. Remember that in case (ii), $|\mathring{A}| \geq \sqrt{12 + 24H^2}$. It is easy to see that if $|H|$ is small, say $|H| \leq \varepsilon_0$ for some constant ε_0, then

$$|\mathring{A}| \geq \sqrt{12 + 24H^2} \geq \eta_2 > \eta_1,$$

which implies $F(|\mathring{A}|) \leq 0$.

Checking the above arguments step by step, we find the equality holds in (3) if and only if

$$\sigma = \sum_{i,j} \overline{R}_{ijij} = 12, \quad \mathring{A} \equiv 0,$$

which implies that Σ is totally umbilic and

$$\overline{R}_{ijij} = 1, \quad \text{for all} \quad i \neq j.$$

Therefore, by the Gauss equation, we get, for all $i \neq j$,

$$(27) \qquad R_{ijij} = \overline{R}_{ijij} + \lambda_i \lambda_j = 1 + (\mu_i + H)(\mu_j + H) = 1 + H^2,$$

which means the sectional curvature of Σ at one point p is the same for all tangent planes $\pi \in T_p\Sigma$. By Schur's lemma, Sec_Σ is constant. Hence, by (27), H is constant and $Sec_\Sigma \equiv 1 + H^2$. Therefore, Σ is isometric to $\mathbb{S}^4(\frac{1}{1+H^2})$. $\qquad \square$

Remark. The condition "$|H| \leq \varepsilon_0$" is just a technical condition. The constant ε_0 can be taken to be $\sqrt{\frac{368\sqrt{3}-598}{46}}$. But this is not the best number. Actually, after a long calculation similar to (19), we can get a better estimate than (12) and finally improve ε_0. We believe the condition "$|H| \leq \varepsilon_0$" is not necessary for inequality (3). Actually, in the next section, we study a special case when M is rotationally symmetric and show that inequality (3) holds for all H.

3. A SPECIAL CASE

In this section, we will deal with a special case, the ambient manifold is rotationally symmetric, i.e., $M = \mathbb{R} \times_\varphi \mathbb{S}^n$ with the metric

$$(28) \qquad g = dt^2 + \varphi^2(t)ds_n^2,$$

where $\varphi(t)$ is a smooth positive function, and ds_n^2 is the standard metric of $\mathbb{S}^n$. Denote by ∂_t the unit vector in the $\mathbb{R}$ direction, and assume X, Y are two vectors tangent to $\mathbb{S}^n$. Then the curvature tensor is given by (see [7, Section 4.2.3])

$$(29) \qquad \overline{R}(X \wedge \partial_t) = -\frac{\ddot{\varphi}}{\varphi} X \wedge \partial_t, \quad \overline{R}(X \wedge Y) = \frac{1 - \dot{\varphi}^2}{\varphi^2} X \wedge Y.$$

For simplicity we write

$$(30) \qquad \kappa_1 := -\frac{\ddot{\varphi}}{\varphi}, \quad \kappa_2 := \frac{1 - \dot{\varphi}^2}{\varphi^2}.$$

Let Σ be a hypersurface in M, T be the tangential (with Σ) part of ∂_t, $e_1, \cdots, e_n$ be the local orthonormal frame on Σ. Write $T_i = g(T, e_i)$. Decomposing each e_i into two parts,

$$(31) \qquad e_i = e_i' + g(e_i, \partial_t) = e_i' + T_i,$$

where e_i' is tangent to $\mathbb{S}^n$. A direct computation, by using (29), (31) and the multilinearity of the curvature tensor, gives

(32)
$$\overline{R}_{ijkl} = \kappa_2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (\kappa_1 - \kappa_2)\left(T_iT_k\delta_{jl} + T_jT_l\delta_{ik} - T_iT_l\delta_{jk} - T_jT_k\delta_{il}\right).$$

Therefore, for $i \neq j$, we have

(33)
$$R_{ijij} = \overline{R}_{ijij} + \lambda_i\lambda_j = \kappa_2 + (\kappa_1 - \kappa_2)\left(T_i^2 + T_j^2\right) + \lambda_i\lambda_j.$$

We need the following lemma, which was first proved by Cartan (we appreciate Professor Marcos Dajczer pointing this fact out to us). For completeness, we give a direct proof here.

Lemma 3.1. *Let M be an $(n+1)$ $(n \geq 4)$ dimensional rotationally symmetric Riemannian manifold with metric (28), and let Σ be a hypersurface in M. Then Σ is locally conformally flat if and only if at each point $p \in \Sigma$ there are at most two distinct principal curvatures, one of them having multiplicity $n - 1$.*

Proof. We will adopt the notation in Section 2. In this proof, i, j, k, l will range from 1 to n. By using the Weyl tensor formula ([1, p. 117]), we have

(34)
$$\begin{aligned}
W_{ijij} =& R_{ijij} - \frac{1}{n-2}\left(R_{ii} + R_{jj}\right) + \frac{S}{(n-1)(n-2)}\\
=& \kappa_2 + (\kappa_1 - \kappa_2)\left(T_i^2 + T_j^2\right) + \lambda_i\lambda_j\\
& - \frac{1}{n-2}\sum_{k\neq i}\left(\kappa_2 + (\kappa_1 - \kappa_2)\left(T_i^2 + T_k^2\right) + \lambda_i\lambda_k\right)\\
& - \frac{1}{n-2}\sum_{k\neq j}\left(\kappa_2 + (\kappa_1 - \kappa_2)\left(T_j^2 + T_k^2\right) + \lambda_j\lambda_k\right)\\
& + \frac{\sum_{k\neq l}\left(\kappa_2 + (\kappa_1 - \kappa_2)\left(T_k^2 + T_l^2\right)\right) + n^2H^2 - |A|^2}{(n-1)(n-2)}\\
=& \lambda_i\lambda_j - \frac{1}{n-2}\left(\sum_{k\neq i}\lambda_i\lambda_k + \sum_{k\neq j}\lambda_j\lambda_k\right) + \frac{n^2H^2 - |A|^2}{(n-1)(n-2)}.
\end{aligned}$$

Using the relations
$$\lambda_k = \mu_k + H, \quad |A|^2 = |\mathring{A}|^2 + nH^2,$$

we substitute μ_k for λ_k in the above equality and obtain
$$W_{ijij} = \frac{(\mu_i + \mu_j)^2 + (n-4)\mu_i\mu_j}{n-2} - \frac{|\mathring{A}|^2}{(n-1)(n-2)}.$$

Therefore, Σ is

locally conformally flat $\Longleftrightarrow W \equiv 0$

(at each point) $\Longleftrightarrow (\mu_i + \mu_j)^2 + (n-4)\mu_i\mu_j = \dfrac{|\mathring{A}|^2}{(n-1)}, \; \forall i \neq j$

$(\sum_i \mu_i = 0, |\mathring{A}|^2 = \sum_i \mu_i^2) \Longleftrightarrow \mu_i = \mu_j \;\text{ or }\; \mu_i = -(n-1)\mu_j, \forall i \neq j.$

Thus $\{\mu_1, \cdots, \mu_n\} = \{\mu, -(n-1)\mu\}$, and μ has multiplicity $n-1$. Consequently,

$$\{\lambda_1, \cdots, \lambda_n\} = \{\mu + H, -(n-1)\mu + H\},$$

and $\mu + H$ has multiplicity $n-1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With the aid of the above lemma, we can prove the following theorem.

Theorem 3.2. *Let M be a rotationally symmetric Riemannian 5-manifold with $0 \leq \kappa_1 \leq \kappa_2 \leq 1$ (κ_1 and κ_2 are defined in (30)), and let Σ be a locally conformally flat closed hypersurface embedded in M with mean curvature H. Then*

$$\int_\Sigma (1 + H^2)^2 \geq \frac{4\pi^2}{3} \chi(\Sigma).$$

The equality holds if and only if H is constant, and Σ is totally umbilic and isometric to $\mathbb{S}^4\left(\frac{1}{1+H^2}\right)$.

Proof. We will adopt the same notation as in the proof of Theorem 1.1. By Lemma 3.1, we have $|\mu_1| = 3|\mu_i|$, $i = 2, 3, 4$. Therefore,

$$\sum_i \mu_i^4 = \frac{7}{12}|\mathring{A}|^4.$$

Direct computations by using (32) yield

$$\sigma = 12\kappa_2 + 6(\kappa_1 - \kappa_2)|T|^2,$$
$$a_{ii} = 3\kappa_2 + (\kappa_1 - \kappa_2)(2T_i^2 + |T|^2),$$
$$|\mathring{a}|^2 = 3(\kappa_1 - \kappa_2)^2|T|^4.$$

Inserting the above equalities into (9), we obtain

$$(35) \qquad Q = 3(\kappa_2 + H^2)^2 + \frac{1}{12}\left(-\frac{3}{4}|\mathring{A}|^4 + 12H\sum_i \mu_i^3 - 6(\kappa_2 + 3H^2)|\mathring{A}|^2\right)$$

$$+ \frac{(\kappa_1 - \kappa_2)}{2}\left(\left(6 + |\mathring{A}|^2 + 4H^2\right)|T|^2 + 2\sum_i T_i^2(\mu_i - H)^2\right)$$

$$\leq 3(\kappa_2 + H^2)^2 + \frac{1}{12}\left(-\frac{3}{4}|\mathring{A}|^4 + 12H\sum_i \mu_i^3 - 6(\kappa_2 + 3H^2)|\mathring{A}|^2\right)$$

$$\leq 3(\kappa_2 + H^2)^2 + \frac{1}{12}\left(-\frac{3}{4}|\mathring{A}|^4 + 4\sqrt{3}|H|\,|\mathring{A}|^3 - 6(\kappa_2 + 3H^2)|\mathring{A}|^2\right)$$

$$\leq 3(\kappa_2 + H^2)^2 \leq 3(1 + H^2)^2.$$

The remainder of the proof is similar to the proof of Theorem 1.1. $\qquad\qquad\square$

Remark. The assumption "$0 \leq \kappa_1 \leq \kappa_2 \leq 1$" is reasonable for many manifolds. For example:
- if we take $\varphi(t) = \sin(t)$, then $M = \mathbb{S}^5$ and $\kappa_1 = \kappa_2 \equiv 1$;
- if we take $\varphi(t) \equiv 1$, then $M = \mathbb{S}^4 \times \mathbb{R}$ and $0 \equiv \kappa_1 < \kappa_2 \equiv 1$.

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