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Estimates for solutions of Dirac equations and an application to a geometric elliptic-parabolic problem

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Abstract. We develop estimates for the solutions and derive existence and uniqueness results of various local boundary value problems for Dirac equations that improve all relevant results known in the literature. With these estimates at hand, we derive a general existence, uniqueness and regularity theorem for solutions of Dirac equations with such boundary conditions. We also apply these estimates to a new nonlinear elliptic-parabolic problem, the Dirac-harmonic heat flow on Riemannian spin manifolds. This problem is motivated by the supersymmetric nonlinear σ-model and combines a harmonic heat flow type equation with a Dirac equation that depends nonlinearly on the flow.

Keywords. Dirac equation, existence, uniqueness, chiral boundary condition, Dirac-harmonic map flow

1. Introduction

The Dirac equation is one of the mathematically most important and fruitful structures from physics. As the name indicates, it was first introduced by Dirac [26]. Dirac's original equation is hyperbolic, but the elliptic version, which this paper is concerned with, appears naturally in geometry. Both solutions on closed manifolds and on manifolds with boundary have found important applications. In this paper, we shall systematically investigate boundary value problems and derive results that are sharper and stronger than all relevant results known prior to our work. We shall then provide a new application which depends on our regularity, existence and uniqueness results and which could not have been derived with the results known in the literature.

The mathematical history of boundary value problems for Dirac equations started with the work of Atiyah, Patodi and Singer. In their seminal papers [4–6], they introduced a nonlocal boundary condition for first order elliptic differential operators and established

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an index theorem on compact manifolds with boundary. This constitutes a cornerstone of the theory of first order elliptic boundary value problems.

In recent years, important progress has been achieved on various extensions, generalizations and simplifications of the Atiyah–Patodi–Singer theory and their applications. In particular, in the works of Bismut and Cheeger [12], Booß-Bavnbek and Wojciechowski [14], Brüning and Lesch [16, 17], Bartnik and Chruściel [10], Ballmann, Brüning and Carron [7], Bär and Ballmann [9], etc., regularity theorems, index theorems and Fredholm theorems for such kind of elliptic boundary value problems have been established.

Although the index theorems and Fredholm theorems give us information or criteria for the existence of solutions, in many cases (for instance the proof of the positive energy theorem [28, 31, 44, 52] and Dirac-harmonic maps, see below), for an elliptic boundary problem and given boundary data, one needs more precise results about the existence and uniqueness of solutions, and usually this is based on appropriate global elliptic estimates for the solutions. This is our motivation for studying boundary value problems for Dirac equations.

In this paper, we first consider the existence and uniqueness for Dirac equations under a class of local elliptic boundary value conditions $\mathcal{B}$ (including chiral boundary conditions, MIT bag boundary conditions and J-boundary conditions, see the definitions in Section 2, cf. [9, 32]). A Dirac bundle E over a Riemannian manifold M^m ($m \geq 2$) is a Hermitian metric vector bundle of left Clifford modules over the Clifford bundle $\mathrm{Cl}(M)$, such that multiplication by unit vectors in TM is orthogonal and the covariant derivative is a module derivation. Let ∇_0 be a smooth Dirac connection on E and consider another Dirac connection of the form $\nabla = \nabla_0 + \Gamma$, that is, $\Gamma \in \Omega^1(\mathrm{Ad}(E))$ commutes with Clifford multiplication. We shall work with the *Dirac connection spaces* $\mathfrak{D}^p(E)$ defined by the norm

$$\|\Gamma\|_p := \|\Gamma\|_{L^{2p}(M)} + \|\mathrm{d}\Gamma\|_{L^p(M)}.$$

In particular, Γ need not be smooth.

The Dirac operator associated with the Dirac connection ∇ is defined by

$$\not{D} := e_i \cdot \nabla_{e_i},$$

where $e_i \cdot$ denotes Clifford multiplication, and $\{e_i\}$ is a local orthonormal frame on M. Here and below, we use the usual summation convention. We will establish the existence and uniqueness of solutions of the following Dirac equations:

$$\begin{cases} \not{D}\psi = \varphi & \text{in } M, \\ \mathcal{B}\psi = \mathcal{B}\psi_0 & \text{on } \partial M, \end{cases} \tag{1.1}$$

where $\varphi \in L^p(E)$ and $\mathcal{B}\psi_0 \in W^{1-1/p,p}(E|_{\partial M})$. Here and below, all of the Sobolev spaces of sections of E are associated with the fixed smooth Dirac connection ∇_0. Assuming

$$p^* > 1 \quad \text{if } m = 2, \qquad p^* \geq (3m-2)/4 \quad \text{if } m > 2,$$

we have the following

Theorem 1.1. *Let E be a Dirac bundle over a compact m-dimensional ($m \geq 2$) Riemannian manifold M^m with boundary. Suppose that $\Gamma \in \mathfrak{D}^{p^*}(E)$, then for any $1 < p < p^*$, (1.1) admits a unique solution $\psi \in W^{1,p}(E)$. Moreover, ψ satisfies the estimate*

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\varphi\|_{L^p(E)} + \|\mathcal{B}\psi_0\|_{W^{1-1/p,p}(E|_{\partial M})}), \tag{1.2}$$

where $c = c(p, \|\Gamma\|_{p^*}) > 0$.

This estimate is optimal in dimension 2 in the sense that the exponents cannot be improved. It also improves the known estimates in higher dimensions. (The condition $p^* \geq (3m-2)/4$ for $m > 2$ arises from the unique continuation result of Jerison [35] that we shall need in the proof.) For instance, in the fundamental work of Bartnik and Chruściel [10], only L^2-estimates were developed. While that was sufficient for their Fredholm theory of the Dirac operator, for the nonlinear setting that we shall treat later in this paper, the finer L^p-estimates that we obtain here are necessary. In [10], when applying the Fredholm criteria to get the existence of solutions, one needs additional conditions (the mean curvature) on the boundary. Essentially, these conditions imply the triviality of the kernel of the elliptic operators. In our case, this extra condition is unnecessary.

One key observation in our proof of the above theorem is that for a harmonic spinor $\psi \in W^{1,p}(E)$, the homogeneous boundary condition $\mathcal{B}\psi|_{\partial M} = 0$ is equivalent to the zero Dirichlet condition $\psi|_{\partial M} = 0$ (see Proposition 3.1 and Remark 3.4), which is not the case for general spinors. Thus, the uniqueness problem for (1.1) can be reduced to the triviality of a harmonic spinor with zero Dirichlet boundary value for Dirac operators with a nonsmooth connection $\nabla_0 + \Gamma$, $\Gamma \in \mathfrak{D}^{p^*}$. To derive this uniqueness, in dimension $m = 2$, inspired by the approach of Hörmander [34], we establish an L^2-estimate with some suitable weight for our Dirac operators (see Theorem 3.6); in dimension $m > 2$, we apply the weak unique continuation property (WUCP) of Dirac type operators $D + V$, where D is a Dirac operator with a smooth connection and V is a potential (see [18] for V continuous, [13] for V bounded, and [35] for $V \in L^{(3m-2)/2}$) and use an extension argument for Dirac operators on manifolds with boundary as in [14]. For the case of a smooth connection, see [33, 45]. Finally, by using the uniqueness result for our boundary value problem $(\not{D}, \mathcal{B})$, we can improve the standard elliptic boundary estimate for Dirac operators to our main L^p-estimate (1.2) (see Theorem 3.11 and Remark 3.7), which is uniform in the sense that the constant $c = c(p, \|\Gamma\|_{p^*}) > 0$ depends on the $\|\cdot\|_{p^*}$ norm of Γ but not on Γ itself, a property playing an important role in our later application to some elliptic-parabolic problem.

We can also apply the above result to derive the existence and uniqueness for boundary value problems for Dirac operators along a map. This will also be needed for the Dirac-harmonic map heat flow introduced below. Let M be a compact Riemannian spin manifold with boundary ∂M, N be a compact Riemannian manifold and Φ a smooth map from M to N. Given a fixed spin structure on M, let ΣM be the spin bundle of M. On the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$, one can define the *Dirac operator $\not{D}$ along the map Φ* [22], i.e.,

$$\not{D}\Psi := \not{\partial}\psi^\alpha \otimes \theta_\alpha + e_i \cdot \psi^\alpha \otimes \nabla^{TN}_{\Phi_*(e_i)}\theta_\alpha.$$

Here $\Psi = \psi^\alpha \otimes \theta_\alpha$, $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$, $\{e_i\}$ is a local orthonormal frame of TM, $\partial\!\!\!/ = e_i \cdot \nabla_{e_i}$ is the usual Dirac operator on the spin bundle over M and $X\cdot$ stands for Clifford multiplication by the vector field X on M. We say that Ψ is a *harmonic spinor along the map* Φ if $D\!\!\!/\Psi = 0$. The chiral boundary value problem for Dirac operators along a map was first considered in [24], extending the classical chiral boundary value problem for usual Dirac operators introduced in [28].

Theorem 1.2. *Let M^m $(m \geq 2)$ be a compact Riemannian spin manifold with boundary ∂M, and N be a compact Riemannian manifold. Let $\Phi \in W^{1,2p^*}(M; N)$. Then for every $1 < p < p^*$, $\eta \in L^p(M; \Sigma M \otimes \Phi^{-1}TN)$ and $\mathcal{B}\psi \in W^{1-1/p,p}(\partial M; \Sigma M \otimes \Phi^{-1}TN)$, the boundary value problem for the Dirac equation*

$$\begin{cases} D\!\!\!/\Psi = \eta & \text{in } M, \\ \mathcal{B}\Psi = \mathcal{B}\psi & \text{on } \partial M, \end{cases}$$

admits a unique solution $\Psi \in W^{1,p}(M; \Sigma M \otimes \Phi^{-1}TN)$, where $D\!\!\!/$ is the Dirac operator along the map Φ. Moreover, there exists a constant $c = c(p, \|\Phi\|_{W^{1,2p^*}(M)}) > 0$ such that

$$\|\Psi\|_{W^{1,p}(M)} \leq c(\|\eta\|_{L^p(M)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(\partial M)}).$$

We shall then apply these estimates and existence results to a new elliptic-parabolic problem in geometry that involves Dirac equations. The novelty of this problem consists in the combination of a second order semilinear parabolic equation with a first order elliptic side condition of Dirac type. We see this as a model problem for a heat flow approach to various other first order elliptic problems in geometric analysis. In any case, this is a nonlinear system coupling a Dirac equation with another prototype of a geometric variational problem, that of harmonic maps. The problem is also motivated by the supersymmetric nonlinear σ-model of QFT; see e.g. [25, 36] where the fermionic part is a Dirac spinor. In fact, this is one of the most prominent roles that Dirac equations play in contemporary theoretical physics, as this leads to the action functional of superstring theory.

In order to set up that problem, we first have to recall the notion of *Dirac-harmonic maps*. Consider the functional

$$L(\Phi, \Psi) = \frac{1}{2}\int_M (\|d\Phi\|^2 + (\Psi, D\!\!\!/\Psi)),$$

where $(\,,\,) = \mathrm{Re}\langle\,,\,\rangle$ is the real part of the induced Hermitian inner product $\langle\,,\,\rangle$ on $\Sigma M \otimes \Phi^{-1}TN$.

A *Dirac-harmonic map* (see [21, 22]) is then defined to be a critical point (Φ, Ψ) of L. The Euler–Lagrange equations are

$$\tau(\Phi) = \tfrac{1}{2}(\psi^\alpha, e_i \cdot \psi^\beta)R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i) =: \mathcal{R}(\Phi, \Psi), \tag{1.3}$$

$$D\!\!\!/\Psi = 0, \tag{1.4}$$

where $R^N(X, Y) := [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$ for $X, Y \in \Gamma(TN)$ stands for the curvature operator of N and $\tau(\Phi) := (\nabla_{e_i} d\Phi)(e_i)$ is the tension field of Φ.

The general regularity and existence problems for Dirac-harmonic maps have been considered in [21–24, 47, 51, 54]. The existence of uncoupled Dirac-harmonic maps (in the sense that the map part is harmonic) via the index theory method was obtained in [3]. For the construction of examples of coupled Dirac-harmonic maps (in the sense that the map part is not harmonic), we refer to [2, 39].

Here, we propose and develop an alternative approach to the existence of Dirac-harmonic maps. This will be the parabolic or heat flow approach. (1.3) is a second order elliptic system, and so we can turn it into a parabolic one by letting the solution depend on time t and putting a time derivative on the left hand side. In contrast, (1.4) is first order, and so we cannot convert it into a parabolic equation, but need to carry it as a constraint along the flow. Thus, we introduce the following flow for Dirac-harmonic maps: for $\Phi \in C^{2,1,\alpha}(M \times (0, T]; N)$ and $\Psi \in C^{1,0,\alpha}(M \times [0, T]; \Sigma M \otimes \Phi^{-1}TN)$,

$$\begin{cases} \partial_t \Phi = \tau(\Phi) - \mathcal{R}(\Phi, \Psi) & \text{in } M \times (0, T], \\ \not{D}\Psi = 0 & \text{in } M \times [0, T], \end{cases} \tag{1.5}$$

with the boundary-initial data

$$\begin{cases} \Phi = \phi & \text{in } M \times \{0\} \cup \partial M \times [0, T], \\ \mathcal{B}\Psi = \mathcal{B}\psi & \text{on } \partial M \times [0, T], \end{cases} \tag{1.6}$$

where $\phi \in C^{2,1,\alpha}(M \times \{0\} \cup \partial M \times [0, T]; N)$ and $\psi \in C^{1,0,\alpha}(\partial M \times [0, T]; \Sigma M \times \phi^{-1}TN)$, and $f \in C^{k,l,\alpha}$ means that $f(x, \cdot) \in C^{l+\alpha/2}$ and $f(\cdot, t) \in C^{k+\alpha}$. We call this system (1.5) the *heat flow for Dirac-harmonic maps.*

We consider this problem as a model for a parabolic approach to other problems in geometric analysis that involve first order side conditions. Also, we shall use this problem to demonstrate the power of our estimates for Dirac equations.

We shall apply our elliptic estimates for Dirac equations with boundary conditions to obtain the local existence and uniqueness of the heat flow for Dirac-harmonic maps. The long time existence will be considered elsewhere, as it involves problems of a different nature. For the classical theory of harmonic map heat flow, we refer to e.g. [19, 27, 30, 42, 43, 49, 50].

Theorem 1.3. *Let M^m $(m \geq 2)$ be a compact Riemannian spin manifold with boundary ∂M, and N be a compact Riemannian manifold. Suppose that*

$$\phi \in \bigcap_{T>0} C^{2,1,\alpha}(\bar{M} \times [0, T]; N),$$

and

$$\mathcal{B}\psi \in \bigcap_{T>0} C^{1,0,\alpha}(\partial M \times [0, T]; \Sigma M \otimes \phi^{-1}TN)$$

for some $0 < \alpha < 1$. *Then the problem* (1.5)*,* (1.6) *admits a unique solution*

$$\Phi \in \bigcap_{0<t<\tau<T_1} C^{2,1,\alpha}(\bar{M} \times [t, \tau]) \cap C^0(\bar{M} \times [0, T_1]; N),$$

and

$$\Psi \in \bigcap_{0<t<\tau<T_1} C^{1,0,\alpha}(\bar{M} \times [t,\tau]) \cap C^{2,0,\alpha}(M \times (0,T_1)) \cap C^{1,0,0}(\bar{M} \times [0,T_1]; \Sigma M \otimes \Phi^{-1}TN)$$

for some time $T_1 > 0$. The maximum time T_1 is characterized by the condition

$$\limsup_{t<T_1,\, t\to T_1} \|d\Phi(\cdot,t)\|_{C^0(\bar{M})} = \infty.$$

Remark 1.1. All our results (Theorems 1.1–1.3) hold for the chiral boundary operators $\mathcal{B}^{\pm} := \frac{1}{2}(\mathrm{Id} \pm \mathbf{n}\cdot G)$, the MIT bag boundary operators $\mathcal{B}^{\pm}_{\mathrm{MIT}} := \frac{1}{2}(1 \pm \sqrt{-1}\,\mathbf{n})$, and the J-boundary operators $\mathcal{B}^{\pm}_J := \frac{1}{2}(\mathrm{Id} \pm \mathbf{n}\cdot J)$. We will only give the proofs for the case of the chiral boundary conditions. The proofs for the other cases are similar and hence we will omit them.

We would like to mention that Branding [15] considered regularized Dirac-harmonic maps from closed Riemannian surfaces and studied the corresponding evolution problem.

The paper is organized as follows. In Section 2, we provide the definitions of Dirac bundle etc. and Dirac-harmonic maps. We also derive the Euler–Lagrange equation for Dirac-harmonic maps. In Section 3, we derive some elliptic estimates and the existence and uniqueness of solutions of Dirac equations with chiral boundary value conditions. In Section 4, we will prove Theorem 1.2. In Section 5 we give a proof of the short time existence for the flow of Dirac-harmonic maps, Theorem 1.3. Finally, in Section 6 we discuss a special case of the Dirac equation along a map between Riemannian disks. In this special case, the solution can be given through Cauchy integrals.

Notations. The lower case letter c will designate a generic constant possibly depending on M, N and other parameters, but independent of a particular solution of (5.1) and (5.2), while the capital letter C will designate a constant possibly depending on the solutions.

We list some notations:

- ΣM the spin bundle on M.
- $C^k(M;N)$ the space of all C^k-maps from M to N.
- $C^k(E) = C^k(M;E)$ the space of all C^k-sections of E where E is a vector bundle on M.
- $C^k(\partial M;E)$ the space of all C^k-sections of E restricted to the boundary ∂M.
- $C^{k,l,\alpha}(M \times I;E)$ the space of all sections $\psi(\cdot,t)$ of E such that $\psi \in C^{k,l,\alpha}(M \times I)$.
- $W^{s,p}(E) = W^{s,p}(M;E)$.
- $\mathrm{End}(E)$ the endomorphism bundle of E.
- $\Omega^p(E) = \Gamma(\Lambda^p T^*M \otimes E)$ the space of all E-valued p-forms on M.
- $\Omega^p(\mathfrak{so}_n) = \Omega^p(M) \otimes \mathfrak{so}_n$.
- $\mathrm{Ad}(E)$ a subbundle of $\mathrm{End}(E)$ such that $A = -A^*$ for all $A \in \mathrm{Ad}(E)$.
- $A(D)$ the space of all holomorphic functions on D.
- $\|\cdot\|$ the inner norm, i.e., $\|\psi\|^2 = \langle\psi,\psi\rangle$. We also use the same notation for some special norms, specified in appropriate places.

2. Preliminaries

2.1. Dirac bundles

Definition 2.1 ([40]). Let E be a Hermitian bundle of left Clifford modules over the Clifford bundle $\mathrm{Cl}(M)$ on a Riemannian manifold M^m. Denote Clifford multiplication, the metric and the connection by $\cdot$, $\langle\ ,\ \rangle$, ∇ respectively. We say that E is a *Dirac bundle* if the following properties hold:

D1. Clifford multiplication is parallel, i.e., the covariant derivative on E is a module derivation, i.e.,

$$\nabla_X(Y\cdot\psi)=\nabla_X Y\cdot\psi+Y\cdot\nabla_X\psi,\qquad \forall X,Y\in\Gamma(TM),\ \psi\in\Gamma(E).$$

D2. Clifford multiplication by unit vectors in TM is orthogonal, i.e.,

$$\langle X\cdot\psi,\varphi\rangle=-\langle\psi,X\cdot\varphi\rangle,\qquad \forall X\in TM,\ \psi,\varphi\in E.$$

D3. The connection is a metric connection, i.e.,

$$X\langle\psi,\varphi\rangle=\langle\nabla_X\psi,\varphi\rangle+\langle\psi,\nabla_X\varphi\rangle,\qquad \forall X\in TM,\psi,\varphi\in\Gamma(E).$$

We call such a connection a *Dirac connection*.

Then one can define the *Dirac operator* associated to the Dirac bundle by

$$\not{D}:=\gamma^E\circ\nabla,$$

where γ^E stands for Clifford multiplication on E. In local coordinates, $\not{D}$ is given by

$$\not{D}=\gamma^E(e_i)\nabla_{e_i}=e^i\cdot\nabla_{e_i}$$

where $\{e_i\}$ is a local orthogonal frame of TM. One can check that $\not{D}$ is in general not self-adjoint: we have the *Green formula*

$$\int_M\langle\not{D}\psi,\varphi\rangle=\int_M\langle\psi,\not{D}\varphi\rangle+\int_{\partial M}\langle\mathbf{n}\cdot\psi,\varphi\rangle,\qquad \forall\psi,\varphi\in\Gamma(E).$$

Suppose E is a Dirac bundle on M and $F=E|_{\partial M}$ is the restriction of E to the boundary ∂M. Then F is a Dirac bundle in a natural way:

F1. The metric on F is just the restriction of E on ∂M.
F2. Clifford multiplication of F, denoted by γ, is defined as

$$\gamma(X)\psi:=\mathbf{n}\cdot X\cdot\psi,\qquad \forall X\in T\partial M,\ \psi\in F.$$

F3. The connection $\bar{\nabla}$ of F is defined as

$$\bar{\nabla}_X\psi:=\nabla_X\psi+\tfrac{1}{2}\gamma(A(X))\psi,\qquad \forall X\in\Gamma(T\partial M),\ \psi\in\Gamma(F),$$

where A is the shape operator of ∂M with respect to the unit outward normal field $\mathbf{n}$ along ∂M.

Lemma 2.1. *This construction gives a Dirac bundle F on ∂M.*

Proof. (1) It is obvious that $\gamma \circ A \in \Omega^1(\mathrm{Ad}(F))$. As a consequence, $\bar{\nabla}$ is a metric connection on F. Moreover, $\gamma(X) \in \Gamma(\mathrm{Ad}(F))$.

(2) Let B be the second fundamental form of ∂M in M. For every $X, Y \in \Gamma(T\partial M)$ with $\bar{\nabla}_X Y = 0$ at the point under consideration,

$$\begin{aligned}
\bar{\nabla}_X(\gamma(Y)\psi) &= \nabla_X(\mathbf{n} \cdot Y \cdot \psi) + \tfrac{1}{2}A(X) \cdot Y \cdot \psi \\
&= -A(X) \cdot Y \cdot \psi + \mathbf{n} \cdot B(X,Y) \cdot \psi + \mathbf{n} \cdot Y \cdot \nabla_X \psi + \tfrac{1}{2}A(X) \cdot Y \cdot \psi \\
&= -A(X) \cdot Y \cdot \psi - \langle A(X), Y \rangle \psi + \mathbf{n} \cdot Y \cdot \nabla_X \psi + \tfrac{1}{2}A(X) \cdot Y \cdot \psi \\
&= -A(X) \cdot Y \cdot \psi + \tfrac{1}{2}A(X) \cdot Y \cdot \psi + \tfrac{1}{2}Y \cdot A(X) \cdot \psi + \mathbf{n} \cdot Y \cdot \nabla_X \psi \\
&\quad + \tfrac{1}{2}A(X) \cdot Y \cdot \psi \\
&= \mathbf{n} \cdot Y \cdot \nabla_X \psi + \tfrac{1}{2}Y \cdot A(X) \cdot \psi = \gamma(Y)\bar{\nabla}_X \psi.
\end{aligned}$$

This identity means that γ is parallel. Therefore, F is a Dirac bundle on ∂M. □

The Dirac operator $\bar{\not{D}}$ of F, defined by

$$\bar{\not{D}} := \gamma(e_i)\bar{\nabla}_{e_i},$$

where $\{e_i\}$ is a local orthogonal frame of $T\partial M$, according to the definition, satisfies the relation

$$\bar{\not{D}} = \mathbf{n} \cdot \not{D} + \nabla_{\mathbf{n}} - \frac{m-1}{2}h$$

where h is the mean curvature of ∂M with respect to $\mathbf{n}$. If M is a surface, $-h$ is just the geodesic curvature of ∂M (as a curve) in M.

2.2. *Chiral and MIT bag boundary value conditions*

In this subsection, we introduce the chiral and MIT bag boundary conditions (cf. [9, 32]). We say that G is a *chiral operator* if $G \in \Gamma(\mathrm{End}(E))$ satisfies

$$G^2 = \mathrm{Id}, \qquad G^* = G, \qquad \nabla G = 0, \qquad GX \cdot = -X \cdot G$$

for every $X \in TM$. It is easy to check that

$$\gamma(X)G = G\gamma(X), \qquad \bar{\nabla}G = 0, \qquad \bar{\not{D}}G = G\bar{\not{D}}, \qquad \bar{\not{D}}\mathbf{n} \cdot = -\mathbf{n} \cdot \bar{\not{D}}$$

on the boundary ∂M for all $X \in T\partial M$. The *chiral boundary operator* $\mathcal{B}^{\pm}$ is defined by

$$\mathcal{B}^{\pm} := \tfrac{1}{2}(\mathrm{Id} \pm \mathbf{n} \cdot G).$$

It is obvious that $(\mathcal{B}^{\pm})^* = \mathcal{B}^{\mp}$ and $\bar{\not{D}}\mathcal{B}^{\pm} = \mathcal{B}^{\mp}\bar{\not{D}}$. The chiral boundary operator is elliptic [9] since

$$\mathbf{n} \cdot X \cdot \mathcal{B}^{\pm} = \mathcal{B}^{\mp} \cdot \mathbf{n} \cdot X, \qquad \forall X \in T\partial M.$$

The MIT bag boundary operator is defined by

$$\mathcal{B}^{\pm}_{\mathrm{MIT}} := \frac{1}{2}(\mathrm{Id} \pm \sqrt{-1}\,\mathbf{n}).$$

More generally, when $J \in \Gamma(\mathrm{End}(E))$ satisfies

$$J^2 = -\mathrm{Id}, \quad J^* = -J, \quad \nabla J = 0, \quad JX\cdot = X \cdot J$$

for every $X \in TM$, we can define a boundary operator, called the *J-boundary operator*, by

$$\mathcal{B}^{\pm}_J := \frac{1}{2}(\mathrm{Id} \pm \mathbf{n} \cdot J).$$

If $J = \sqrt{-1}$, then it is easy to see that the MIT bag boundary operator is just the $\sqrt{-1}$-boundary operator. Another example is $J = \sqrt{-1}\,G_1 G_2$ where $[G_1, G_2] = 0$ with G_1, G_2 being chiral operators. In fact, in our setting, the J-operator is just the composition of the chiral operator G with $e_1 \cdot e_2 \cdot$ and vice versa. One can check that $\mathcal{B}^{\pm}_J$ is elliptic since

$$\mathcal{B}^{\pm}_J \cdot \mathbf{n} \cdot X = \mathbf{n} \cdot X \cdot \mathcal{B}^{\mp}_J, \qquad \forall X \in T\partial M.$$

For simplicity, we shall denote by $\mathcal{B}$ one of $\mathcal{B}^{\pm}$, $\mathcal{B}^{\pm}_{\mathrm{MIT}}$ and $\mathcal{B}^{\pm}_J$. For convenience, we will mainly consider the case of chiral boundary conditions and omit the detailed discussions of other boundary conditions.

The following theorem is well known [9, 23, 46, 47].

Theorem 2.2 (see [46, p. 55, Theorem 1.6.2]). *The operator*

$$(\not{D}, \mathcal{B}) : W^{s,p}(E) \to W^{s-1,p}(E) \times W^{s-1/p,p}(E|_{\partial M})$$

is Fredholm for all $s \geq 1$ and $1 < p < \infty$. Moreover its kernel and cokernel are independent of the choice of s and p. Therefore, we have the following elliptic a priori estimate:

$$\|\psi\|_{W^{s,p}(E)} \leq c(\|\not{D}\psi\|_{W^{s-1,p}(E)} + \|\mathcal{B}\psi\|_{W^{s-1/p,p}(E|_{\partial M})} + \|\psi\|_{L^p(E)}),$$

where $c = c(p, s, M, \partial M, \not{D}, \mathcal{B}) > 0$.

Proof. This is a consequence of the fact that $(\not{D}, \mathcal{B})$ is an elliptic operator for $s \geq 1$ and $1 < p < \infty$. □

2.3. *Dirac connection spaces*

Let E be a Dirac bundle. We consider the affine space of those connections ∇ for which E is again a Dirac bundle. Choose a connection ∇_0; then for any other connection ∇ on E,

$$\nabla = \nabla_0 + \Gamma,$$

where $\Gamma \in \Omega^1(\mathrm{End}(E))$.

Lemma 2.3. *Suppose ∇_0 is a Dirac connection. Then $\nabla := \nabla_0 + \Gamma$ is a Dirac connection if and only if*

$$\Gamma \in \Omega^1(\mathrm{Ad}(E)), \qquad [\Gamma, \gamma^E] = 0,$$

where γ^E denotes Clifford multiplication of E.

Proof. We only need to check that γ^E is parallel. For every $X, Y \in \Gamma(TM)$ with $\nabla_X Y = 0$ at the point under consideration, we have

$$\begin{aligned}\nabla_X(\gamma^E(Y)\psi) &= \nabla_{0X}(\gamma^E(Y)\psi) + \Gamma(X)\gamma^E(Y)\psi = \gamma^E(Y)\nabla_{0X}\psi + \gamma^E(Y)\Gamma(X)\psi \\ &= \gamma^E(Y)\nabla_X\psi.\end{aligned}$$

□

Introduce $\not{\Gamma} := \gamma^E \circ \Gamma = \gamma^E(e_i)\Gamma(e_i)$. Then

$$\not{D} = \not{D}_0 + \not{\Gamma},$$

where $\not{D}$, $\not{D}_0$ are the Dirac operators associated to the connections ∇, ∇_0 respectively. From now on, we will consider the modified nonsmooth connection $\nabla_0 + \Gamma$, denoted by ∇. All of the Sobolev spaces are associated with some fixed smooth connection ∇_0. It is well known that this definition of Sobolev spaces is independent of the choice of ∇_0 if M is compact. However, the connection ∇ need not be smooth—we only assume that Γ belongs to some special function space. For example,

$$\mathrm{d}\Gamma \in L^{p^*}(M), \qquad \Gamma \in L^{2p^*}(M),$$

where

$$p^* > 1 \quad \text{if } m = 2, \qquad p^* \geq (3m-2)/4 \quad \text{if } m > 2.$$

Definition 2.2. For $p \geq 1$, define $\mathfrak{D}^p(E)$ to be the completion of the subspace of $\Omega^1(\mathrm{Ad}(E))$ defined by

$$\mathfrak{D}(E) = \{\Gamma \in \Omega^1(\mathrm{Ad}(E)) : [\Gamma, \gamma^E] = 0\}$$

with respect to the norm

$$\|\Gamma\|_p := \|\Gamma\|_{L^{2p}(M)} + \|\mathrm{d}\Gamma\|_{L^p(M)}.$$

We call these spaces the *Dirac connection spaces*.

2.4. Dirac-harmonic maps

Let (M^m, g) be a compact Riemannian spin manifold with (possibly empty) boundary ∂M, and (N^n, h) be a compact Riemannian manifold. Concerning the definition and properties of Riemannian spin manifolds, we refer the reader to [40] for more background material. For any $(\Phi, \Psi) \in C^1(M, N) \times \Gamma(\Sigma M \otimes \Phi^{-1}TN)$, we consider the functional [21]

$$L(\Phi, \Psi) = \frac{1}{2}\int_M (\|\mathrm{d}\Phi\|^2 + (\Psi, \not{D}\Psi)),$$

where $(,) = \mathrm{Re}\langle , \rangle$ is the real part of the Hermitian inner product $\langle , \rangle$.

A *Dirac-harmonic map* (see [21, 22]) is then defined to be a critical point (Φ, Ψ) of L. The Euler–Lagrange equations are

$$\begin{cases} \tau(\Phi) = \frac{1}{2}(\psi^\alpha, e_i \cdot \psi^\beta) R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i) =: \mathcal{R}(\Phi, \Psi), \\ \not{D}\Psi = 0, \end{cases}$$

where $R^N(X, Y) := [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$ for $X, Y \in \Gamma(TN)$ stands for the curvature operator of N and $\tau(\Phi) := (\nabla_{e_i}\mathrm{d}\Phi)(e_i)$ is the tension field of Φ.

Embed N into $\mathbb{R}^q$ isometrically for some integer q. We may assume there is a bounded tubular neighborhood $\tilde{N}$ of N in $\mathbb{R}^q$. Let $\pi : \tilde{N} \to N$ be the nearest point projection. We may assume π can be extended smoothly to the whole $\mathbb{R}^q$ with compact support. Now we can derive the Euler–Lagrange equation for L. Let $\Phi : M \to N$ with $\Phi = (\Phi^A)$, and a spinor $\Psi = \Psi^A \otimes \partial_A \circ \Phi$ along the map Φ with $\Psi = (\Psi^A)$ where Ψ^A are spinors over M, and $\partial_A = \partial/\partial z^A$. Notice that $\mathrm{d}\pi|_N$ is an orthogonal projection and $\mathrm{d}\pi(T^\perp N) = 0$. In fact,

$$\mathrm{d}\pi(X) = X, \quad \forall X \in TN,$$

and

$$\mathrm{d}\pi(\xi) = 0, \quad \forall \xi \in T^\perp N,$$

where $T^\perp N$ is the normal bundle of N in $\mathbb{R}^q$. Hence, after restriction to N, we have

$$\pi^A_B \pi^B_C = \pi^A_C, \quad \pi^A_B = \pi^B_A.$$

It is easy to check that

$$\nu^A_B(\Phi)\nabla\Phi^B = 0, \quad \nu^A_B(\Phi)\Psi^B = 0,$$

where $\nu^A_B := \delta^A_B - \pi^A_B$.

For any smooth map $\eta \in C^\infty_0(M, \mathbb{R}^q)$ and any smooth spinor field $\xi \in C^\infty_0(\Sigma M \otimes \mathbb{R}^q)$, we consider the variation

$$\Phi_t = \pi(\Phi + t\eta), \quad \Psi^A_t = \pi^A_B(\Phi_t)(\Psi^B + t\xi^B).$$

It is easy to check that

$$\Phi_0 = \Phi, \quad \Psi_0 = \Psi$$

and

$$\left.\frac{\partial \Phi_t^A}{\partial t}\right|_{t=0} = \pi_B^A(\Phi)\eta^B, \qquad \left.\frac{\partial \Psi_t^A}{\partial t}\right|_{t=0} = \pi_B^A(\Phi)\xi^B + \pi_{BC}^A(\Phi)\pi_D^C(\Phi)\Psi^B\eta^D,$$

where

$$\pi_B^A = \frac{\partial \pi^A}{\partial z^B}, \qquad \pi_{BC}^A = \frac{\partial^2 \pi^A}{\partial z^B \partial z^C}, \qquad \dots .$$

Moreover,

$$\pi_{BC}^A(\Phi)\pi_D^C(\Phi) = \pi_{AC}^B(\Phi)\pi_D^C(\Phi), \qquad \pi_{BC}^A = \pi_{CB}^A. \tag{2.1}$$

Then we have

Proposition 2.4. *The Euler–Lagrange equations for L are*

$$\Delta\Phi^A = \pi_{BC}^A(\Phi)\langle\nabla\Phi^B, \nabla\Phi^C\rangle + \pi_B^A(\Phi)\pi_{BD}^C(\Phi)\pi_{EF}^C(\Phi)(\Psi^D, \nabla\Phi^E \cdot \Psi^F)$$

and

$$\partial\!\!\!/ \Psi^A = \pi_{BC}^A(\Phi)\nabla\Phi^B \cdot \Psi^C.$$

Remark 2.1. Denote

$$\begin{aligned} \Omega_B^A &:= \nu_C^A(\Phi)\mathrm{d}\nu_B^C(\Phi) - \mathrm{d}\nu_C^A(\Phi)\nu_B^C(\Phi) = [\nu(\Phi), \mathrm{d}\nu(\Phi)]_B^A, \\ R_{GDF}^A &:= \pi_B^A\pi_{BD}^C\pi_E^G\pi_{EF}^C - \pi_B^G\pi_{BD}^C\pi_E^A\pi_{EF}^C, \\ \tilde{\Omega}_G^A &:= \tfrac{1}{2}R_{GDF}^A(\Phi)(\Psi^D, e_i \cdot \Psi^F)\eta^i. \end{aligned}$$

Then $\Omega_B^A = -\Omega_A^B$, $\tilde{\Omega}_B^A = -\tilde{\Omega}_A^B$ and the Euler–Lagrange equations for L can be rewritten as follows [24]:

$$\begin{cases} \Delta\Phi^A = -\langle\Omega_B^A, \mathrm{d}\Phi^B\rangle + \langle\tilde{\Omega}_B^A, \mathrm{d}\Phi^B\rangle, \\ \partial\!\!\!/ \Psi^A = -\Omega_B^A \cdot \Psi^B. \end{cases}$$

Using Clifford multiplication $\cdot$ for the Dirac bundle $\Omega^*(M)$, we can also write the above system as follows:

$$\begin{cases} \Delta\Phi^A = \Omega_B^A \cdot \mathrm{d}\Phi^B + \langle\tilde{\Omega}_B^A, \mathrm{d}\Phi^B\rangle, \\ \partial\!\!\!/ \Psi^A = -\Omega_B^A \cdot \Psi^B. \end{cases}$$

Proof of Remark 2.1. The proof is similar to one in [24]. However, we present it using our notations. Introduce

$$S_D^{AC} := \pi_B^A\pi_{BD}^C.$$

Then

$$R_{GDF}^A = S_D^{AC}S_F^{GC} - S_D^{GC}S_F^{AC}$$

satisfies

$$R_{GDF}^A = -R_{ADF}^G = -R_{GFD}^A.$$

Moreover,

$$\begin{aligned}\langle \tilde{\Omega}^A_B, \mathrm{d}\Phi^B\rangle &= \tfrac{1}{2} R^A_{GDF}(\Phi)(\Psi^D, \nabla\Phi^G \cdot \Psi^F) = S^{AC}_D S^{GC}_F (\Psi^D, \nabla\Phi^E \cdot \Psi^F)\\ &= \pi^A_B \pi^C_{BD} \pi^C_{EF} (\Psi^D, \nabla\Phi^E \cdot \Psi^F).\end{aligned}$$

Now we only need to check that $\Omega^A_B \wedge \mathrm{d}\Phi^B = 0$. Using $\nu^A_B \mathrm{d}\Phi^B = 0$, we have

$$\Omega^A_B \wedge \mathrm{d}\Phi^B = \nu^A_C \mathrm{d}\nu^C_B \wedge \mathrm{d}\Phi^B = 0.$$ □

Proof of Proposition 2.4. Note that both η and ξ have compact support in $\mathring{M}$. Since

$$L(\Phi, \Psi) = \frac{1}{2}\int_M (\|\nabla\Phi^A\|^2 + (\Psi^A, \partial\!\!\!/ \Psi^A)),$$

by using (2.1) we have

$$\begin{aligned}
\frac{\mathrm{d}L(\Phi_t, \Psi_t)}{\mathrm{d}t}\Big|_{t=0} &= \int_M \langle \nabla\Phi^A, \nabla(\pi^A_B \eta^B)\rangle + \frac{1}{2}\int_M (\pi^A_B \xi^B + \pi^A_{BC}\pi^C_D \Psi^B \eta^D, \partial\!\!\!/ \Psi^A)\\
&\quad + \frac{1}{2}\int_M \big(\Psi^A, \partial\!\!\!/ (\pi^A_B \xi^B + \pi^A_{BC}\pi^C_D \Psi^B \eta^D)\big)\\
&= \int_M \langle \nabla\Phi^A, \pi^A_B \nabla\eta^B + \pi^A_{BC}\nabla\Phi^C \eta^B\rangle + \int_M (\pi^A_B \xi^B + \pi^A_{BC}\pi^C_D \Psi^B \eta^D, \partial\!\!\!/ \Psi^A)\\
&\quad + \frac{1}{2}\int_{\partial M} \big(\Psi^A, \mathbf{n}\cdot(\pi^A_B \xi^B + \pi^A_{BC}\pi^C_D \Psi^B \eta^D)\big)\\
&= -\int_M \big(\Delta\Phi^A - \pi^A_{BC}\langle \nabla\Phi^B, \nabla\Phi^C\rangle - \pi^A_B \pi^C_{BD}\pi^C_{EF}(\Psi^D, \nabla\Phi^E \cdot \Psi^F)\big)\eta^A\\
&\quad + \int_M \pi^A_B \pi^C_{BD}(\Psi^D, \partial\!\!\!/ \Psi^C - \pi^C_{EF}\nabla\Phi^E \cdot \Psi^F)\eta^A + \int_M (\partial\!\!\!/ \Psi^A - \pi^A_{BC}\nabla\Phi^B \cdot \Psi^C, \xi^A)\\
&\quad + \int_{\partial M} \nabla_{\mathbf{n}}\Phi^A \eta^A + \tfrac{1}{2}(\Psi^A, \mathbf{n}\cdot\xi^A)\\
&= -\int_M \big(\Delta\Phi^A - \pi^A_{BC}\langle \nabla\Phi^B, \nabla\Phi^C\rangle - \pi^A_B \pi^C_{BD}\pi^C_{EF}(\Psi^D, \nabla\Phi^E \cdot \Psi^F)\big)\eta^A\\
&\quad + \int_M \pi^A_B \pi^C_{BD}(\Psi^D, \partial\!\!\!/ \Psi^C - \pi^C_{EF}\nabla\Phi^E \cdot \Psi^F)\eta^A + \int_M (\partial\!\!\!/ \Psi^A - \pi^A_{BC}\nabla\Phi^B \cdot \Psi^C, \xi^A),
\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector field along ∂M. □

3. Existence and uniqueness of solutions of Dirac equations with chiral boundary conditions

In this section, we suppose that M^m $(m \geq 2)$ is a compact Riemannian spin manifold with boundary ∂M and E is a Dirac bundle on M. We want to derive an existence and uniqueness result for solutions of Dirac equations with chiral boundary conditions. The key observation is that for harmonic spinors, the homogeneous chiral condition is equivalent to the zero Dirichlet boundary condition. With this observation, we can derive a useful L^2-estimate for solutions of Dirac equations with chiral boundary conditions. The assumption that the boundary is nonempty is essential here. Another application of this observation is that one can derive Schauder boundary estimates.

3.1. A property of chiral boundary conditions

Importantly, the chiral boundary condition is conformally invariant. Moreover, it satisfies

Proposition 3.1. *Suppose ∇ is a smooth Dirac connection. Then for every $\psi \in H^1(E)$, we have*

$$\left| \int_{\partial M} (\|\psi\|^2 - 2\|\mathcal{B}\psi\|^2) \right| \leq 2\|\psi\|_{L^2(E)} \|\not{D}\psi\|_{L^2(E)}.$$

Proof. First, we assume that ψ is smooth. Introduce a vector field

$$X := \tfrac{1}{2}(\psi, e_i \cdot G\psi) e_i.$$

Then

$$\langle X, \mathbf{n}\rangle = \tfrac{1}{2}\langle \psi, \mathbf{n} \cdot G\psi\rangle, \qquad \|\mathcal{B}^{\pm}\psi\|^2 = \tfrac{1}{2}\|\psi\|^2 \pm \langle X, \mathbf{n}\rangle, \qquad \operatorname{div} X = -(\not{D}\psi, G\psi).$$

Using these facts and integrating by parts, we get

$$\left| \int_{\partial M} (\|\psi\|^2 - 2\|\mathcal{B}\psi\|^2) \right| = 2\left| \int_M (\not{D}\psi, G\psi) \right| \leq 2\|\not{D}\psi\|_{L^2(E)} \|\psi\|_{L^2(E)}.$$

The general case follows since $\Gamma(E)$ is dense in $H^1(E)$. □

Remark 3.1. This proposition says that the two systems

$$\begin{cases} \not{D}\psi = 0 & \text{in } M, \\ \mathcal{B}\psi = 0 & \text{on } \partial M, \end{cases} \qquad \begin{cases} \not{D}\psi = 0 & \text{in } M, \\ \psi = 0 & \text{on } \partial M, \end{cases}$$

are equivalent. This fact is important for our whole theory of Dirac equations. With this observation, we can then solve Dirac equations with chiral boundary value conditions.

Remark 3.2. Proposition 3.1 also holds for J-boundary operators. The proof is similar to Proposition 3.1 and we omit it here. Moreover, all the results associated to chiral boundary values are also valid for J-boundary values. Again, we omit the proof.

3.2. Regularity of weak solutions and elliptic estimates

Definition 3.1. Suppose that $\not{V} \in L^{p'}_{\mathrm{loc}}(M)$. Let $\psi, \varphi \in L^p_{\mathrm{loc}}(E)$ where $1/p + 1/p' = 1$ with $p \geq 1$. We call φ a *weak solution* of the Dirac equation $\not{D}\psi = \varphi$ if

$$\int_M \langle \varphi, \eta \rangle = \int_M \langle \psi, \not{D}\eta \rangle$$

for all smooth spinors $\eta \in \Gamma_0(E)$ of E with compact support in the interior of M.

Assume

$$\hat{m} > 2 \quad \text{for } m = 2, \qquad \hat{m} = m \quad \text{for } m > 2.$$

First, we have the following regularity results.

Theorem 3.2 (Regularity of weak solutions [9, 10]). *Let M, E be as in Theorem 1.1. Suppose that $\not{V} \in W^{k,\hat{m}}_{\mathrm{loc}}(M)$. Let $\psi \in L^2_{\mathrm{loc}}(E)$ be a weak solution of $\not{D}\psi = \varphi$ with $\varphi \in H^k_{\mathrm{loc}}(E)$. Then $\psi \in H^{k+1}_{\mathrm{loc}}(E)$.*

Theorem 3.3 (L^p-estimate). *Let M, E be as in Theorem 1.1. Suppose that $p \in (1, \infty)$ and $\not{V} \in L^\mu(M)$ where $\mu = m$ if $p < m$ and $\mu > p$ if $p \geq m$. Let $\psi \in W^{1,p}(E)$ be a solution of $\not{D}\psi = \varphi$ with $\varphi \in L^p(E)$. Then there exists a constant $c = c(p, \not{V}) > 0$ with*

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\varphi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})} + \|\psi\|_{L^p(E)}).$$

Proof. For every $\varepsilon > 0$, decompose $\not{V} = \not{V}_\varepsilon + \not{V}_\infty$ with

$$\|\not{V}_\varepsilon\|_{L^\mu(E)} \leq \varepsilon, \qquad \|\not{V}_\infty\|_{L^\infty(E)} \leq c(\varepsilon).$$

Noticing that

$$\not{D}_0\psi = \varphi - \not{V}\psi = \varphi - \not{V}_\varepsilon\psi - \not{V}_\infty\psi,$$

we have

$$\begin{aligned}
\|\not{D}_0\psi\|_{L^p(E)} &\leq \|\varphi\|_{L^p(E)} + \|\not{V}_\varepsilon\psi\|_{L^p(E)} + \|\not{V}_\infty\psi\|_{L^p(E)} \\
&\leq \|\varphi\|_{L^p(E)} + \|\not{V}_\varepsilon\|_{L^\mu(E)}\|\psi\|_{L^{p\mu/(\mu-p)}(E)} + \|\not{V}_\infty\|_{L^\infty(E)}\|\psi\|_{L^p(E)} \\
&\leq \|\varphi\|_{L^p(E)} + \varepsilon\|\psi\|_{W^{1,p}(E)} + c(\varepsilon)\|\psi\|_{L^p(E)}
\end{aligned}$$

since $1/p - 1/\mu = 1/p - 1/m$ for $p < m$, and $1/p - 1/\mu > 1/p - 1/m$ for $\mu > p \geq m$. Hence, by Theorem 2.2, for suitable $\varepsilon > 0$, we get

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\varphi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})} + \|\psi\|_{L^p(E)}). \qquad \square$$

Remark 3.3. If $\mu > m$, we can choose $c = c(p, \|\not{V}\|_{L^\mu(M)}) > 0$.

Proof. We only need to check the case of a smooth spinor, i.e., $\psi \in \Gamma(E)$. If not, suppose that there exist sequences $\psi_n \in \Gamma(E)$ and $\not{\Gamma}_n \in L^\mu(M)$ such that

$$1 = \|\psi_n\|_{W^{1,p}(E)} \geq n(\|\not{D}_n\psi_n\|_{L^p(E)} + \|\mathcal{B}\psi_n\|_{W^{1-1/p,p}(E|_{\partial M})} + \|\psi_n\|_{L^p(E)}),$$

and

$$\|\not{\Gamma}_n\|_{L^\mu(M)} \leq C,$$

where $\not{D}_n = \not{D}_0 + \not{\Gamma}_n$. Then for $\max\{p, m\} < p' < \mu$, $\not{\Gamma}_n$ is a bounded subset in $L^{p'}(E)$ and hence there exists a subsequence, also denoted by $\not{\Gamma}_n$, that converges weakly to $\not{\Gamma} \in L^{p'}(M)$ in the reflexive space $L^{p'}(E)$. We may assume that

$$\psi_n \rightharpoonup \psi \quad \text{in } W^{1,p}(E), \qquad \psi_n \to \psi \quad \text{in } L^{\tilde{p}}(E)$$

according to the Sobolev–Kondrashov embedding theorem where $1/\tilde{p} = 1/p - 1/p' > 1/p - 1/m$. Hence $\psi = 0$ by the choice of ψ_n. Moreover, if we denote $\not{D} = \not{D}_0 + \not{\Gamma}$, then

$$\begin{aligned}\|\not{D}\psi_n\|_{L^p(E)} &\leq \|\not{D}_n\psi_n\|_{L^p(E)} + \|(\not{\Gamma} - \not{\Gamma}_n)\psi_n\|_{L^p(E)} \\ &\leq \|\not{D}_n\psi_n\|_{L^p(E)} + \|\not{\Gamma} - \not{\Gamma}_n\|_{L^{p'}(M)}\|\psi_n\|_{L^{\tilde{p}}(E)}.\end{aligned}$$

In particular, $\not{D}\psi_n$ converges strongly to 0 in $L^p(E)$ and so does $\not{\Gamma}_n\psi_n$. But we already know that

$$\mathcal{B}\psi_n \to 0 \quad \text{in } W^{1-1/p,p}(E|_{\partial M}), \qquad \psi_n \to 0 \quad \text{in } L^p(E).$$

The L^p-estimate Theorem 3.3 implies that

$$1 = \|\psi_n\|_{W^{1,p}(E)} \leq c(p, \not{\Gamma})(\|\not{D}\psi_n\|_{L^p(E)} + \|\mathcal{B}\psi_n\|_{W^{1-1/p,p}(E|_{\partial M})} + \|\psi_n\|_{L^p(E)}) \to 0.$$

Remark 3.4. By a similar computation, Proposition 3.1 holds for the case of a non-smooth connection $\nabla = \nabla_0 + \Gamma$ with $\Gamma \in L^{\hat{m}}(M)$.

3.3. L^2-estimate

In this subsection, we want to prove that the solution of the Dirac equation with chiral boundary values is unique, i.e., the problem

$$\begin{cases} \not{D}\psi = 0 & \text{in } M \\ \mathcal{B}\psi = 0 & \text{on } \partial M, \end{cases}$$

has only the zero solution. In dimension $m = 2$, we recall Hörmander's L^2-estimate method [34] which was originally developed to get the L^2-existence theorem for $\bar{\partial}$-operators on weakly pseudo-convex domains by using Carleman type estimates. Later, Shaw [48] extended this method to $\bar{\partial}_b$-manifolds. Here, we use a similar idea to derive the L^2-estimate for the Dirac equations and use this L^2-estimate to get the uniqueness of solutions of Dirac equations with chiral boundary values. In higher dimensions $m > 2$, we use the Weak Uniqueness Continuation Property (WUCP) for Dirac type operators to get the uniqueness. We shall then use this uniqueness to derive some useful elliptic estimates.

We will need the following Weitzenböck type formula (cf. [37]):

$$\not{D}^2 = -\nabla^2 + \mathcal{R}, \tag{3.1}$$

where the curvature operator $\mathcal{R}$ is given by

$$\mathcal{R} := \tfrac{1}{2} e_i \cdot e_j \cdot R(e_i, e_j)$$

and $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature of the connection $\nabla = \nabla_0 + \Gamma$.

Theorem 3.4 (Weighted Reilly formula). *Let M, E be as in Theorem 1.1 and suppose $\Gamma \in \mathcal{D}^{p*}(E)$. Let f be a smooth function on M. Then for every $\psi \in \Gamma(E)$, we have*

$$\begin{aligned}\int_{\partial M} \exp(f)\Big((\bar{\not{D}}\psi, \psi) + \frac{m-1}{2}(h + \mathbf{n}(f)\Big)\|\psi\|^2) + \frac{m-1}{m}\int_M \exp(f)\|\not{D}\psi\|^2 \\ = \int_M \exp(f)\Big(\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}\|\nabla f\|^2 + \mathcal{R}_\psi\Big)\|\psi\|^2 \\ + \int_M \exp((1-m)f)\Big\|P\Big(\exp\Big(\frac{m}{2}f\Big)\psi\Big)\Big\|^2.\end{aligned}$$

Here

$$\mathcal{R}_\psi \|\psi\|^2 = (\mathcal{R}\psi, \psi).$$

Remark 3.5. (1) If $m = 2$, we have

$$\begin{aligned}\int_{\partial M} \exp(f)\Big((\bar{\not{D}}\psi, \psi) + \frac{1}{2}(h + \mathbf{n}(f))\|\psi\|^2\Big) + \frac{1}{2}\int_M \exp(f)\|\not{D}\psi\|^2 \\ = \int_M \exp(f)\Big(\frac{1}{2}\Delta f + \mathcal{R}_\psi\Big)\|\psi\|^2 + \int_M \exp(-f)\|P(\exp(f)\psi)\|^2.\end{aligned}$$

(2) If $m > 2$, let $f = (1-\tau)\log u$ (with $\tau = m/(m-2)$), then

$$\begin{aligned}\int_{\partial M} u^{1-\tau}(\bar{\not{D}}\psi, \psi) + \frac{m-1}{2}\int_{\partial M} u^{-\tau}\Big(hu - \frac{2}{m-2}\frac{\partial u}{\partial \mathbf{n}}\Big)\|\psi\|^2 + \frac{m-1}{m}\int_M u^{1-\tau}\|\not{D}\psi\|^2 \\ = \int_M u^{-\tau}\Big(-\frac{m-1}{m-2}\Delta u + \mathcal{R}_\psi u\Big)\|\psi\|^2 + \int_M u^{1+\tau}\|P(u^{-\tau}\psi)\|^2.\end{aligned}$$

Proof of Theorem 3.4. We only prove this theorem for the smooth setting. For the general case, this can be done by density. Denote the twistor operator by

$$P_X\psi := \nabla_X\psi + \frac{1}{m} X \cdot \not{D}\psi, \qquad \forall X \in \Gamma(TM),\ \psi \in \Gamma(E).$$

This twistor operator has the property

$$\|\nabla\psi\|^2 = \|P\psi\|^2 + \frac{1}{m}\|\not{D}\psi\|^2. \tag{3.2}$$

In fact, since tr $P = 0$, i.e., $e_i \cdot P(e_i) = \gamma \circ P = 0$, we have

$$\begin{aligned}\|\nabla\psi\|^2 &= \sum_i \left\| P_{e_i}\psi - \frac{1}{m} e_i \cdot \not{D}\psi \right\|^2 = \|P\psi\|^2 - \frac{2}{m}\sum_i (P_{e_i}\psi, e_i \cdot \not{D}\psi) + \frac{1}{m}\|\not{D}\psi\|^2 \\ &= \|P\psi\|^2 + \frac{2}{m}\sum_i (e_i \cdot P_{e_i}\psi, \not{D}\psi) + \frac{1}{m}\|\not{D}\psi\|^2 = \|P\psi\|^2 + \frac{1}{m}\|\not{D}\psi\|^2.\end{aligned}$$

For every smooth function $f \in C^\infty(M)$ and every smooth spinor $\psi \in \Gamma(E)$, by using (3.2) and (3.1) we have

$$\begin{aligned}\tfrac{1}{2}\Delta(\exp(f)\|\psi\|^2) &= \exp(f)\big(\tfrac{1}{2}(\Delta f + \|\nabla f\|^2)\|\psi\|^2 + \tfrac{1}{2}\Delta\|\psi\|^2 + 2(\nabla_{\nabla f}\psi, \psi)\big) \\ &= \exp(f)\big(\tfrac{1}{2}(\Delta f + \|\nabla f\|^2)\|\psi\|^2 + 2(\nabla_{\nabla f}\psi, \psi)) \\ &\quad + \exp(f)\big(\|\nabla\psi\|^2 - (\not{D}^2\psi, \psi) + (\mathcal{R}\psi, \psi)\big) \\ &= \exp(f)\left(\tfrac{1}{2}(\Delta f + \|\nabla f\|^2)\|\psi\|^2 + 2(P_{\nabla f}\psi, \psi) - \frac{2}{m}(\nabla f \cdot \not{D}\psi, \psi)\right) \\ &\quad + \exp(f)\left(\|P\psi\|^2 + \frac{1}{m}\|\not{D}\psi\|^2 - (\not{D}^2\psi, \psi) + (\mathcal{R}\psi, \psi)\right) \\ &= \exp(f)\left(\tfrac{1}{2}\Delta f\|\psi\|^2 + (\mathcal{R}\psi, \psi) + \frac{2-m}{2m}\|\nabla f\|^2\|\psi\|^2\right) - (\not{D}(\exp(f)\not{D}\psi), \psi) \\ &\quad + \frac{1}{m}\exp(f)\|\not{D}\psi\|^2 + \frac{m-2}{m}\exp(f)(\nabla f \cdot \not{D}\psi, \psi) + \exp(-f)\|P(\exp(f)\psi)\|^2.\end{aligned}$$

The last identity follows from the two identities

$$\begin{aligned}\|P(\exp(f)\psi)\|^2 &= \exp(2f)\left\| P_{e_i}\psi + e_i(f)\psi + \frac{1}{m} e_i \cdot \mathrm{d}f \cdot \psi \right\|^2 \\ &= \exp(2f)\left(\|P\psi\|^2 + \left\| e_i(f)\psi + \frac{1}{m} e_i \cdot \mathrm{d}f \cdot \psi \right\|^2 + 2(P_{\nabla f}\psi, \psi)\right) \\ &= \exp(2f)\left(\|P\psi\|^2 + \frac{m-1}{m}\|\psi\|^2\|\nabla f\|^2 + 2(P_{\nabla f}\psi, \psi)\right)\end{aligned}$$

and

$$(\not{D}(\exp(f)\not{D}\psi), \psi) = \exp(f)\big((\not{D}^2\psi, \psi) + (\nabla f \cdot \not{D}\psi, \psi)\big).$$

Integrating by parts, we get

$$\begin{aligned}\int_{\partial M} &\exp(f)\big((\bar{D\!\!\!/}\psi, \psi) + \tfrac{1}{2}((m-1)h + \mathbf{n}(f))\|\psi\|^2\big)\\ &= \int_M \exp(f)\Big(\tfrac{1}{2}\Delta f\|\psi\|^2 + (\mathcal{R}\psi, \psi) + \frac{2-m}{2m}\|\nabla f\|^2\|\psi\|^2\Big)\\ &\quad + \int_M \exp(-f)\|P(\exp(f)\psi)\|^2\\ &\quad - \frac{m-1}{m}\int_M \exp(f)(\|D\!\!\!/\psi\|^2 + \frac{m-2}{m-1}(D\!\!\!/\psi, \nabla f\cdot\psi)).\end{aligned}$$

By using the identity

$$\begin{aligned}\Big\|D\!\!\!/\Big(\exp\Big(\frac{m-2}{2(m-1)}f\Big)\psi\Big)\Big\|^2 &= \exp\Big(\frac{m-2}{m-1}f\Big)\Big\|D\!\!\!/\psi + \frac{m-2}{2(m-1)}\nabla f\cdot\psi\Big\|^2\\ &= \exp\Big(\frac{m-2}{m-1}f\Big)\Big(\|D\!\!\!/\psi\|^2 + \frac{m-2}{m-1}(D\!\!\!/\psi, \nabla f\cdot\psi) + \frac{(m-2)^2}{4(m-1)^2}\|\nabla f\|^2\|\psi\|^2\Big)\end{aligned}$$

we get

$$\begin{aligned}\int_{\partial M} &\exp(f)\big((\bar{D\!\!\!/}\psi, \psi) + \tfrac{1}{2}((m-1)h + \mathbf{n}(f))\|\psi\|^2\big)\\ &= \int_M \exp(f)\Big(\tfrac{1}{2}\Delta f\|\psi\|^2 + (\mathcal{R}\psi, \psi) + \frac{2-m}{4(m-1)}\|\nabla f\|^2\|\psi\|^2\Big)\\ &\quad + \int_M \exp(-f)\|P(\exp(f)\psi)\|^2\\ &\quad - \frac{m-1}{m}\int_M \exp\Big(\frac{1}{m-1}f\Big)\Big\|D\!\!\!/\Big(\exp\Big(\frac{m-2}{2(m-1)}f\Big)\psi\Big)\Big\|^2.\end{aligned}$$

Set $g = f/(m-1)$ and $\sigma = \exp((m-2)f/(2m-2))\psi$. Then

$$\begin{aligned}\int_{\partial M} &\exp(g)\Big((\bar{D\!\!\!/}\sigma, \sigma) + \frac{m-1}{2}(h + \mathbf{n}(g)\Big)\|\sigma\|^2)\\ &= \int_M \exp(g)\Big(\frac{m-1}{2}\Delta g - \frac{(m-1)(m-2)}{4}\|\nabla g\|^2 + \mathcal{R}_\sigma\Big)\|\sigma\|^2\\ &\quad + \int_M \exp((1-m)g)\Big\|P\Big(\exp\Big(\frac{m}{2}g\Big)\sigma\Big)\Big\|^2 - \frac{m-1}{m}\int_M \exp(g)\|D\!\!\!/\sigma\|^2. \qquad \square\end{aligned}$$

It is well known that the curvature operator $\mathcal{R}$ of E can be calculated as

$$\mathcal{R} = \mathcal{R}_0 + \mathrm{d}\Gamma + [\omega_0, \Gamma] + [\Gamma, \omega_0] + [\Gamma, \Gamma],$$

where ω_0 is the associated connection 1-form of ∇_0 (cf. [40]). In particular, $\|\mathcal{R}\| := \|\mathcal{R}\|_{\mathrm{op}} \in L^{p^*}(M)$, where the operator norm $\mathcal{R}_{\mathrm{op}}$ at each point is defined by

$$\|\mathcal{R}\|_{\mathrm{op}} = \sup_{\psi \neq 0} \frac{\|R\psi\|}{\|\psi\|}.$$

We need the following lemma.

Lemma 3.5. *Suppose M is a Riemann surface with boundary and $\Gamma \in \mathfrak{D}^{p^*}(E)$. There is a function $f \in \bigcap_{1<p<p^*} W^{2,p}(M)$ satisfying*

$$\begin{cases} \frac{1}{2}\Delta f - \|\mathcal{R}\| = 0 & \text{in } M, \\ f = 0 & \text{on } \partial M. \end{cases}$$

Now we can state the following L^2-estimate in dimension $m = 2$.

Theorem 3.6 (L^2-estimate). *Let M, E be as in Theorem 1.1. Suppose that $m = 2$ and $\Gamma \in \mathfrak{D}^{p^*}(E)$. Then there exists a function $f \in \bigcap_{1<p<p^*} W^{2,p}(M)$ such that for every spinor $\psi \in H^1(E)$,*

$$\begin{aligned} \int_{\partial M} \exp(f)\big((\bar{D\!\!\!/}\psi, \psi) + \tfrac{1}{2}(h + \mathbf{n}(f))\|\psi\|^2\big) + \frac{1}{2}\int_M \exp(f)\|D\!\!\!/\psi\|^2 \\ \geq \int_M \exp(-f)\|P(\exp(f)\psi)\|^2. \end{aligned} \tag{3.3}$$

Proof. Choose f as in Lemma 3.5. Since $f \in W^{2,p}(M)$ for all $1 < p < p^*$, by using the Sobolev embedding theorem we know that $f \in C^{2-2/p}(M)$. In particular, f is continuous. Moreover, the trace theorem implies that

$$f|_{\partial M} \in W^{2-1/p,p}(\partial M),$$

and hence again the Sobolev theorem $W^{1-1/p,p}(\partial M) \subset L^q(\partial M)$ shows that $\mathbf{n}(f) \in L^q(\partial M)$, where we assume $1 < p < \min\{2, p^*\}$ without loss of generality and

$$\frac{1}{q} = \frac{1}{p} - \left(1 - \frac{1}{p}\right) = \frac{2-p}{p}.$$

Hence

$$\left|\int_{\partial M} e^f \mathbf{n}(f)\|\psi\|^2\right| \leq C\|\mathbf{n}(f)\|_{L^q(\partial M)}\|\psi\|^2_{L^{q'}(\partial M)} \leq C\|\psi\|^2_{H^1(M)}.$$

Here we have used the Sobolev embedding and the trace theorem

$$H^{1/2}(\partial M) \subset L^{q'}(\partial M),$$

where

$$\frac{1}{2}\left(1 - \frac{1}{q}\right) = \frac{1}{q'} > \frac{m-2}{2(m-1)} = 0,$$

which is equivalent to $p > 1$. For the remaining terms, it is easy to get

$$\left|\int_{\partial M} e^f (\bar{\not{D}}\psi, \psi)\right| \leq C\|\psi\|^2_{H^1(M)}, \qquad \int_M e^f \|\not{D}\psi\|^2 \leq C\|\psi\|^2_{H^1(M)},$$
$$\int_M e^{-f} \|P(e^f \psi)\|^2 \leq C\|\psi\|^2_{H^1(M)}.$$

As a consequence, by using Theorem 3.4 and Remark 3.4 we know that (3.3) holds for all $\psi \in H^1(E)$ by density. □

Corollary 3.7 (Uniqueness dimension $m = 2$). *Let M, E be as in Theorem 1.1. Suppose $m = 2$ and $\Gamma \in \mathfrak{D}^{p^*}(E)$. Then a weak solution of*

$$\begin{cases} \not{D}\psi = 0 & \text{in } M, \\ \mathcal{B}\psi = 0 & \text{on } \partial M, \end{cases}$$

is a trivial spinor, i.e., $\psi = 0$.

Proof. It follows that ψ is a strong solution since $\not{\Gamma} \in L^{2p^*}(M)$, i.e., $\psi \in H^1(E)$, according to the elliptic estimates. By using Proposition 3.1 and Remark 3.4, we have $\psi = 0$ on the boundary. Theorem 3.6 then implies that $P(e^f \psi) = 0$ in M. That is, for every tangent vector field X on M, we have

$$\nabla_X \psi + X(f)\psi + \tfrac{1}{2} X \cdot \nabla f \cdot \psi = 0 \tag{3.4}$$

in the weak sense. Notice that

$$(X \cdot \nabla f \cdot \psi, \psi) = (\psi, \nabla f \cdot X \cdot \psi) = -(\psi, X \cdot \nabla f \cdot \psi) - 2X(f)\|\psi\|^2.$$

As a consequence,

$$(X \cdot \nabla f \cdot \psi, \psi) = -X(f)\|\psi\|^2.$$

Therefore, it follows from (3.4) that

$$(\nabla_X \psi, \psi) + \tfrac{1}{2} X(f)\|\psi\|^2 = 0,$$

which means that $\nabla(e^f \|\psi\|^2) = 0$ in M, i.e., $e^f \|\psi\|^2$ is a constant in M. Remembering that we have proved that $\psi = 0$ along the boundary, we then conclude that $\psi = 0$ in the whole manifold M. □

For higher dimensions, first we have the following uniqueness theorem.

Theorem 3.8 (Uniqueness for small perturbation). *Let M, E be as in Theorem 1.1 and $m > 2$. There is a constant $\varepsilon > 0$ such that if $\|\mathcal{R}\|_{L^{m/2}} < \varepsilon$, then there is no nontrivial solution of the boundary value problem*

$$\begin{cases} \not{D}\psi = 0 & \text{in } M, \\ \mathcal{B}\psi = 0 & \text{on } \partial M. \end{cases}$$

Proof. The proof is a direct consequence of the Bochner formula, the Poincaré–Sobolev inequality, Proposition 3.1 and Remark 3.4. First, according to Proposition 3.1, we know that $\psi|_{\partial M} = 0$, and then the Poincaré–Sobolev inequality yields

$$\|\psi\|_{L^{2m/(m-2)}(M)} \leq C_{\mathrm{PS}} \|\nabla\psi\|_{L^2(M)}.$$

Second, the classical Bochner formula (or Theorem 3.4 with weight function $f = 0$) says

$$0 = \int_M (\mathcal{R}\psi, \psi) + \int_M \|\nabla\psi\|^2.$$

Now applying the Hölder and Poincaré–Sobolev inequalities, we get

$$0 \geq \|\nabla\psi\|^2_{L^2(M)} - \|\mathcal{R}\|_{L^{m/2}(M)} \|\psi\|^2_{L^{2m/(m-2)}(M)} \geq \|\nabla\psi\|^2_{L^2(M)} - C_{\mathrm{PS}}\varepsilon \|\nabla\psi\|^2_{L^2(M)}.$$

Hence, if $\varepsilon < C_{\mathrm{PS}}^{-1}$, we get $\nabla\psi = 0$. Therefore, $\psi \equiv 0$ in M. □

In the general case, we still have uniqueness if we require more regularity on Γ, for example, $\Gamma \in \mathfrak{D}^{(3m-2)/4}$. To see this, we recall the Weak Unique Continuation Property (WUCP) for Dirac type operators $D + V$, where D is a Dirac operator with a smooth connection and V is a potential (see [18] for V continuous, [13] for V bounded, and [35] for $V \in L^{(3m-2)/2}$). The operator $D + V$ is said to satisfy the WUCP if any solution $\psi \in H^1(M)$ of $(D + V)\psi = 0$ that ψ vanishes in a nonempty open subset of M, also vanishes in the whole connected component of M. The proofs of the WUCP are based on certain Carleman type estimates. For sharper results on the structure of the zero set of solutions of generalized Dirac equations, we refer to [8].

Theorem 3.9 (WUCP, see [35]). *Let M, E be as in Theorem 1.1 and $m > 2$. Let $D + V$ be a Dirac type operator, where D is a Dirac operator with a smooth connection and $V \in L^{(3m-2)/2}(M)$ is a potential. Then the WUCP holds for $D + V$.*

Thanks to WUCP, we can apply some extension arguments similar to the smooth case considered in [14] to derive the uniqueness theorem.

Theorem 3.10 (Uniqueness in dimension $m > 2$). *Let M, E be as in Theorem 1.1 and $m > 2$. Suppose that $\Gamma \in \mathfrak{D}^{p^*}(E)$ with $p^* \geq (3m-2)/4$. Then there is no nontrivial solution of the boundary value problem*

$$\begin{cases} \not{D}\psi = 0 & \text{in } M, \\ \mathcal{B}\psi = 0 & \text{on } \partial M. \end{cases}$$

Proof. According to Proposition 3.1 and Remark 3.4, we have $\psi = 0$ on the boundary. First, there is a closed double $\tilde{M}$ of M and a Dirac bundle $\tilde{E}$ on $\tilde{M}$ such that $\tilde{E}|_M = E$ and

$$\tilde{\not{D}}_0|_M = \not{D}_0,$$

where $\tilde{\not{D}}_0$ is the associated Dirac operator of $\tilde{E}$ (cf. [14]). Here we write $\not{D} = \not{D}_0 + \not{\Gamma}$ and $\not{D}_0$ is smooth. Extend $\not{\Gamma}$ trivially to some $\tilde{\not{\Gamma}}$ on $\tilde{M}$, i.e.,

$$\tilde{\Gamma} = \begin{cases} \Gamma & \text{in } M, \\ 0 & \text{in } \tilde{M} \setminus M. \end{cases}$$

Then the trivial extension $\tilde{\psi}$ of ψ, i.e.

$$\tilde{\psi} = \begin{cases} \psi & \text{in } M, \\ 0 & \text{in } \tilde{M} \setminus M, \end{cases}$$

is an $H^1(\tilde{M})$-solution of

$$\tilde{\not{D}}_0\tilde{\psi} + \tilde{\not{\Gamma}}\tilde{\psi} = 0.$$

We need only check that $\tilde{\psi}$ is a weak solution. For every smooth spinor φ on $\tilde{M}$, we have

$$\begin{aligned}\int_{\tilde{M}} \langle \tilde{\psi}, \tilde{\not{D}}_0^*\varphi + \tilde{\not{\Gamma}}^*\varphi\rangle &= \int_M \langle \psi, \tilde{\not{D}}_0^*\varphi + \not{\Gamma}^*\varphi\rangle = \int_M \langle \not{D}_0\psi + \not{\Gamma}\psi, \varphi\rangle - \int_{\partial M} \langle \sigma_{\mathbf{n}}(\tilde{\not{D}}_0)\psi, \varphi\rangle \\ &= 0.\end{aligned}$$

Now we can apply the weak UCP of $\tilde{\not{D}}_0 + \tilde{\not{\Gamma}}$ to show that $\tilde{\psi} = 0$ in the whole manifold $\tilde{M}$. Therefore, $\psi \equiv 0$ in M. □

Now we can state the main elliptic L^p-estimate.

Theorem 3.11 (Main L^p-estimate). *Let M, E be as in Theorem 1.1 and $m \geq 2$. Suppose that $\Gamma \in \mathfrak{D}^{p^*}(E)$. Then for $1 < p < p^*$, there exists a constant $c = c(p, \Gamma) > 0$ such that for any $\psi \in W^{1,p}(E)$,*

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\not{D}\psi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})}).$$

Proof. Consider the operator

$$(\not{D}, \mathcal{B}) : W^{1,p}(E) \to L^p(E) \times W^{1-1/p,p}(E|_{\partial M}).$$

Since $\Gamma \in L^{2p^*}(E)$, this is well defined. Moreover, by Theorem 3.3, we have the L^p-estimate

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\not{D}\psi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})} + \|\psi\|_{L^p(E)}),$$

where $c = c(p, \Gamma) > 0$.

Now one can show that the range of $(\not{D}, \mathcal{B})$ is closed and the kernel is trivial. In fact, the kernel is trivial by using Corollary 3.7 (for $m = 2$) and Theorem 3.10 (for $m > 2$). Now we prove that the image is closed. Let $\psi_n \in W^{1,p}(E)$ with

$$\not{D}\psi_n \to \varphi \quad \text{in } L^p(E), \qquad \mathcal{B}\psi_n \to \psi_0 \quad \text{in } W^{1-1/p,p}(E|_{\partial M}).$$

It is clear that $\mathcal{B}\psi_0 = \psi_0$.

First, we assume that $\|\psi_n\|_{L^p(E)} \leq 1$. Then the L^p-estimate Theorem 3.3 implies that ψ_n is bounded in $W^{1,p}(E)$. Hence, there exists a subsequence of ψ_n, denoted also by ψ_n, such that ψ_n converges weakly to ψ in $W^{1,p}(E)$ and strongly in $L^p(E)$. Using Theorem 3.3 again, we see that ψ_n is a Cauchy sequence in $W^{1,p}(E)$. As a consequence, there exists a limit of ψ_n in $W^{1,p}(E)$ and this limit must be ψ. Hence $\varphi = \not{D}\psi$ and $\psi_0 = \mathcal{B}\psi$.

Second, if ψ_n is not bounded in $L^p(E)$, set

$$\tilde{\psi}_n = \frac{\psi_n}{\|\psi_n\|_{L^p(E)}} \in W^{1,p}(E).$$

Then

$$\not{D}\tilde{\psi}_n \to 0 \quad \text{in } L^p(E), \qquad \mathcal{B}\tilde{\psi}_n \to 0 \quad \text{in } W^{1-1/p,p}(E|_{\partial M}).$$

By the same arguments as above, $\tilde{\psi}_n$ has a limit $\tilde{\psi}$ in $W^{1,p}(E)$ such that $\|\tilde{\psi}\|_{L^p(E)} = 1$, $\not{D}\tilde{\psi} = 0$ and $\mathcal{B}\tilde{\psi} = 0$. This is impossible since Corollary 3.7 (for $m = 2$) and Theorem 3.10 (for $m > 2$) implies that $\tilde{\psi} = 0$.

Hence, the closed graph theorem implies that $(\not{D}, \mathcal{B})$ is an isometry between $H^1(E)$ and the range of $(\not{D}, \mathcal{B})$. As a consequence,

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\not{D}\psi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})}),$$

where $c = c(p, \Gamma) > 0$. □

Remark 3.6. If $p = 2$, we can prove this theorem directly by using Theorems 3.3 and 3.6. First, according to Theorem 3.3, there exists a constant $c = c(\Gamma) > 0$ such that

$$\|\psi\|_{H^1(E)} \leq c(\|\not{D}\psi\|_{L^2(E)} + \|\mathcal{B}\psi\|_{H^{1/2}(E|_{\partial M})} + \|\psi\|_{L^2(E)})$$

for all $\psi \in H^1(E)$. Second, Theorem 3.6 implies that

$$\|\psi\|_{L^2(E)} \leq c(\Gamma)(\|\not{D}\psi\|_{L^2(E)} + \|\mathcal{B}\psi\|_{H^{1/2}(E|_{\partial M})}).$$

Combining these two estimates, we complete the proof.

Remark 3.7. We can choose $c = c(p, \|\not{\Gamma}\|_{p^*}) > 0$. See Remark 3.3.

3.4. Existence and uniqueness for solutions of Dirac equations

In this subsection, we shall consider the existence and uniqueness of solutions of the Dirac equation with chiral boundary conditions, to find a solution $\psi \in W^{1,p}(E)$ of

$$\begin{cases} \not{D}\psi = \varphi & \text{in } M, \\ \mathcal{B}\psi = \mathcal{B}\psi_0 & \text{on } \partial M, \end{cases} \tag{3.5}$$

where $\varphi \in L^p(E)$ and $\mathcal{B}\psi_0 \in W^{1-1/p,p}(E|_{\partial M})$.

Several general existence theorems for this system in $H^1(E)$ have been derived under some integral conditions; for example, see [10, p. 53, Theorem 7.3]. which asserts that (3.5) is solvable in $H^1(E)$ if and only if the following integral condition holds:

$$\int_M \langle \varphi, \eta \rangle = 0, \qquad \forall \eta \in \ker(\not{D}, \mathcal{B}^*).$$

Moreover, this solution satisfies the L^2-estimate

$$\|\psi\|_{H^1(E)} \le c(\|\varphi\|_{L^2(E)} + \|\mathcal{B}\psi_0\|_{H^{1/2}(E|_{\partial M})} + \|\psi\|_{L^2(E)}).$$

But the uniqueness may not be true for a general first order elliptic partial differential equation with an elliptic boundary condition.

Notice that in our setting, this integral condition is always satisfied for each φ in $L^2(E)$ since the kernel of $(\not{D}, \mathcal{B}^*)$ is zero according to Corollary 3.7 (for $m = 2$) and Theorem 3.10 (for $m > 2$). In fact, in our setting, we can show the existence and uniqueness Theorem 1.1 with the use of the main L^p-estimate of Theorem 3.11.

Proof of Theorem 1.1. We only need to show the existence. We can use a method that is similar to that for deducing the analogous theorem for second order elliptic partial differential equations with Dirichlet boundary values (see [29, p. 241, Theorem 9.15], for example). For convenience, we will give a detailed proof.

First, we consider the case $p^* > 2$ and $p = 2$. The following argument is typical (see [10, 29] for example). Let us consider the following closed subspace of $H^1(E)$:

$$H^1_{\mathcal{B}}(E) = \{\psi \in H^1(E) : \mathcal{B}\psi = 0\}.$$

Theorem 3.11 gives the a priori estimate

$$\|\psi\|^2_{H^1(E)} = \int_M \|\nabla_0 \psi\|^2 + \|\psi\|^2 \le C \int_M \|\not{D}\psi\|^2, \qquad \forall \psi \in H^1_{\mathcal{B}}(E).$$

In particular, $\int_M \|\not{D}\psi\|^2$ is strictly coercive on $H^1_{\mathcal{B}}(E)$, so the Lax–Milgram theorem gives $\psi \in H^1_{\mathcal{B}}(E)$ satisfying

$$\int_M \langle \varphi, \not{D}\eta \rangle = \int_M \langle \not{D}\psi, \not{D}\eta \rangle, \qquad \eta \in H^1_{\mathcal{B}}(E).$$

Denote $\Phi = \not{D}\psi - \varphi \in L^2(E)$. Then

$$\int_M \langle \Phi, \not{D}\eta \rangle = 0, \qquad \forall \eta \in H^1_{\mathcal{B}}(E).$$

Therefore Φ is a weak solution of

$$\not{D}\Phi = 0, \qquad \mathcal{B}^*\Phi = 0.$$

Since $\mathcal{B}^*$ is elliptic, all the elliptic estimates stated for $\mathcal{B}$ can be stated in a similar way (see Theorem 3.2). In particular, Φ is a strong solution, i.e., $\Phi \in H^1(E)$. By using the L^2-estimate of Theorem 3.6 (for $\mathcal{B}^*$), we know that $\Phi = 0$. Hence $\not{D}\psi = \varphi$ and $\mathcal{B}\psi = 0$.

In the general case, we extend $\mathcal{B}\psi_0$ to a spinor $\tilde{\psi} \in H^1(E)$ such that $\tilde{\psi}|_{\partial M} = \mathcal{B}\psi_0$. Setting $\hat{\psi} = \psi - \tilde{\psi}$, we then have

$$\begin{cases} \not{D}\hat{\psi} = \varphi - \not{D}\tilde{\psi} & \text{in } M, \\ \mathcal{B}\hat{\psi} = 0 & \text{on } \partial M. \end{cases}$$

The previous case shows that there is a solution $\hat{\psi} \in H^1(E)$. Then $\psi = \hat{\psi} + \tilde{\psi}$ is the desired solution of (3.5).

Second, we consider the case $1 < p < p^*$. Let $\varphi_\varepsilon \in \Gamma(E)$ be such that φ_ε converges strongly to φ in $L^p(E)$ as $\varepsilon \to 0$. For each $\varepsilon > 0$, let ψ_ε be the unique solution of

$$\not{D}\psi_\varepsilon = \varphi_\varepsilon, \qquad \mathcal{B}\psi_\varepsilon = 0.$$

The a priori estimate of Theorem 3.11 says that $\psi_\varepsilon \in W^{1,p}(E)$ and is a Cauchy sequence in $W^{1,p}(E)$ since φ_ε converges strongly to φ in $L^p(E)$. We can assume that ψ_ε converges strongly to ψ in $W^{1,p}(E)$. Then $\not{D}\psi = \varphi$ and $\mathcal{B}\psi = 0$. In the nonhomogeneous boundary case, we set $\psi = \hat{\psi} + \tilde{\psi}$ where $\tilde{\psi} \in W^{1,p}(E)$ is an extension of $\mathcal{B}\psi_0$ such that $\tilde{\psi}|_{\partial M} = \mathcal{B}\psi_0$ by the extension theorem. This is possible since $1 < p < \infty$. Then we choose a solution $\hat{\psi} \in W^{1,p}(E)$ such that

$$\begin{cases} \not{D}\hat{\psi} = \varphi - \not{D}\tilde{\psi} & \text{in } M, \\ \mathcal{B}\hat{\psi} = 0 & \text{on } \partial M. \end{cases}$$

Now we get a solution of (3.5). □

4. Dirac equations along a map

We first consider the following system which is slightly more general than (1.1):

$$\begin{cases} \not{D}\psi^A + \Omega^A_B \cdot \psi^B = \eta^A & \text{in } M, \\ \mathcal{B}\psi^A = \mathcal{B}\psi^A_0 & \text{on } \partial M, \end{cases} \tag{4.1}$$

where $A = 1, \ldots, q$, $\eta^A \in L^p(E)$, $\mathcal{B}\psi^A_0 \in W^{1-1/p,p}(E|_{\partial M})$ and $\Omega \in \Omega^1(\mathfrak{so}_n)$, i.e., $\Omega^A_B = -\Omega^B_A$. Under suitable conditions, we can solve this system.

Theorem 4.1. *Let M, E be as in Theorem 1.1. Suppose that $\Gamma \in \mathfrak{D}^{p^*}(E)$, $d\Omega \in L^{p^*}(M)$ and $\Omega \in L^{2p^*}(M)$. Then for $1 < p < p^*$, (4.1) admits a unique solution. Moreover, we have the elliptic estimate*

$$\|\psi\|_{W^{1,p}(E)} \leq c(\|\eta\|_{L^p(E)} + \|\mathcal{B}\psi_0\|_{W^{1-1/p,p}(E|_{\partial M})}),$$

where $c = c(p, \|\Gamma\|_{p^*}, \|\Omega\|_{L^{2p^*}(M)} + \|d\Omega\|_{L^{p^*}(M)}) > 0$.

Proof. To prove this theorem, we construct a new Dirac bundle and a new chirality operator, and then we apply the existence and uniqueness for the usual Dirac equation.

Let $\tilde{E} = \oplus^n E = E \oplus \cdots \oplus E$ (n summands). Then $\tilde{E}$ becomes a Dirac bundle as a Whitney sum bundle. Clifford multiplication is defined as

$$\tilde{\gamma}(X)(\psi^A) := (X \cdot \psi^A), \qquad \forall X \in TM,$$

i.e., $\tilde{\gamma} = \gamma^E \operatorname{Id}$. Here $\gamma^E(X)\psi^A := X \cdot \psi^A$ stands for Clifford multiplication on E. Then the associated $\tilde{\Gamma}$ equals $\Gamma \operatorname{Id}$, i.e.,

$$\tilde{\Gamma}(X)(\psi^A) := (\Gamma(X)\psi^A).$$

Define Γ' by

$$\Gamma'(X)(\psi^A) := (\Omega^A_B(X)\psi^B).$$

It is clear that $\Gamma' \in \Omega^1(\operatorname{Ad}(\tilde{E}))$. We need only check that $[\Gamma', \tilde{\gamma}] = 0$ in order to prove that $\Gamma' \in \mathfrak{D}^{p^*}(\tilde{E})$. In fact,

$$[\Gamma', \tilde{\gamma}](X, Y)(\psi^A) = \big(\Omega^A_B(X)(Y \cdot \psi^B) - Y \cdot \Omega^A_B(X)\psi^B\big) = 0.$$

Thus, we have constructed a new Dirac bundle $\tilde{E}$ with the Dirac operator $\tilde{\not{D}}$ defined by

$$\tilde{\not{D}}(\psi^A) = (\not{D}\psi^A + \Omega^A_B \cdot \psi^B).$$

It is obvious that $\mathrm{d}(\tilde{\Gamma} + \Gamma') \in L^{p^*}(M)$ and $\tilde{\Gamma} + \Gamma' \in L^{2p^*}(M)$.

Introduce an operator $\tilde{G} \in \operatorname{End}(\tilde{E})$,

$$\tilde{G}(\psi^A) := (G\psi^A).$$

It is clear that

$$\tilde{G}^2 = \operatorname{Id}, \qquad \tilde{G}^* = \tilde{G}, \qquad \tilde{G}\tilde{\gamma}(X) = -\tilde{\gamma}(X)\tilde{G}, \qquad \forall X \in TM.$$

Moreover, $\tilde{\nabla}\tilde{G} = 0$. In fact,

$$\tilde{\nabla}_X \tilde{G}(\psi^A) = (\nabla_X G\psi^A + \Omega^A_B(X)G\psi^B) = (G\nabla_X\psi^A + G\Omega^A_B(X)\psi^B) = \tilde{G}\tilde{\nabla}_X(\psi^A).$$

In particular, $\tilde{G}$ is a chirality operator on $\tilde{E}$. Therefore, the associated chirality boundary operator is

$$\tilde{\mathcal{B}}(\psi^A) := (\mathcal{B}\psi^A).$$

Then we can use the theory of the Dirac equation (Theorem 1.1) to finish the proof. □

Now we suppose that M is a Riemannian spin manifold with boundary ∂M and give

Proof of Theorem 1.2. If we embed N into some Euclidean space, then as shown in Section 2, we can rewrite this boundary value problem for the Dirac equation as

$$\begin{cases} \not{\partial}\Psi^A + \Omega^A_B \cdot \Psi^B = \eta^A & \text{in } M, \\ \mathcal{B}\Psi^A = \mathcal{B}\psi^A & \text{on } \partial M, \end{cases}$$

where

$$\Omega_B^A = [\nu(\Phi), \mathrm{d}\nu(\Phi)]_B^A.$$

In particular, $\mathrm{d}\Omega = [\mathrm{d}\nu(\Phi), \mathrm{d}\nu(\Phi)]$. Therefore, if $\Phi \in W^{1,2p^*}(M, N)$, then $\Phi \in C^0(\bar{M}, N)$ by using the Sobolev embedding theorem. As a consequence,

$$\Omega \in L^{2p^*}(M), \qquad \mathrm{d}\Omega \in L^{p^*}(M).$$

Hence, by using Theorem 4.1, we get a unique solution $\Psi \in W^{1,p}(\Sigma M \otimes \Phi^{-1}T\mathbb{R}^q)$ for some larger q. Moreover, there exists a constant $c = c(p, \|\Phi\|_{W^{1,2p^*}(M)}) > 0$ such that

$$\|\Psi\|_{W^{1,p}(M)} \leq c(\|\eta\|_{L^p(M)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(\partial M)}).$$

Now we want to prove that Ψ is a spinor along the map Φ. Introduce $\tilde{\Psi}^A = \nu_B^A \Psi^B$. Then we need only prove that $\tilde{\Psi} = 0$.

Claim.

$$\begin{cases} \partial\!\!\!/ \tilde{\Psi}^A + \Omega_B^A \cdot \tilde{\Psi}^B = 0 & \text{in } M, \\ \mathcal{B}\tilde{\Psi}^A = 0 & \text{on } \partial M. \end{cases}$$

If this claim is true, then using Theorem 4.1 again, we get $\tilde{\Psi} = 0$, which completes the proof of Theorem 1.2.

Now, we prove the claim. In fact, noticing that $\nu_B^A \eta^B = 0$, $\nu_B^A \mathcal{B}\psi^B = 0$ since η is a spinor along the map Φ and $\mathcal{B}\psi$ is the restriction of a spinor along Φ to the boundary ∂M, we have

$$\begin{aligned} \partial\!\!\!/ \tilde{\Psi}^A &= \nu_B^A \partial\!\!\!/ \psi^B + \mathrm{d}\nu_B^A \cdot \Psi^B = -\nu_B^A \mathrm{d}\nu_C^B \cdot \Psi^C + \nu_B^A \mathrm{d}\nu_C^B \nu_D^C \cdot \Psi^D + \nu_B^A \eta^B + \mathrm{d}\nu_B^A \cdot \Psi^B \\ &= \mathrm{d}\nu_B^A \cdot \tilde{\Psi}^B = -\mathrm{d}\pi_B^A \cdot \tilde{\Psi}^B = (\mathrm{d}\pi_C^A \pi_B^C - \pi_C^A \mathrm{d}\pi_B^C) \cdot \tilde{\Psi}^B \\ &= (\mathrm{d}\nu_C^A \nu_B^C - \nu_C^A \mathrm{d}\nu_B^C) \cdot \tilde{\Psi}^B = -\Omega_B^A \cdot \tilde{\Psi}^B, \end{aligned}$$

and

$$\mathcal{B}\tilde{\Psi}^A = \nu_B^A \mathcal{B}\Psi^B = \nu_B^A \mathcal{B}\psi^B = 0. \qquad \square$$

Remark 4.1. Using the same method, one can prove that

$$\begin{cases} D\!\!\!\!/\, \psi = \eta \in L^p(E \otimes \Phi^{-1}V) & \text{in } M, \\ \mathcal{B}\Psi = \mathcal{B}\psi \in W^{1-1/p,p}((E \otimes \Phi^{-1}V)|_{\partial M}) & \text{on } \partial M, \end{cases}$$

admits a unique solution $\Psi \in W^{1,p}(E \otimes \Phi^{-1}V)$, where E is a Dirac bundle on M, V is a Hermitian metric vector bundle on N, $D\!\!\!\!/$ is the associated Dirac operator of $E \otimes \Phi^{-1}V$, and the Dirac connection $\nabla = \nabla_0 + \Gamma$ satisfies the condition $\Gamma \in \mathfrak{D}^{p^*}(E)$ and $\Phi \in W^{1,2p^*}(M; N)$.

Remark 4.2. If Φ is smooth, then $\Sigma M \otimes \Phi^{-1}TN$ is a smooth Dirac bundle; in this case, Theorem 1.2 is just a direct corollary of Theorem 1.1.

Now let us give some further remarks on the *Schauder theory of Dirac equations*. The interior Schauder estimate for the Dirac equation is

Theorem 4.2 (see [1]). *Let M, E be as in Theorem 1.1. Suppose that $\not{V} \in C^{\alpha}(M)$ for some $0 < \alpha < 1$. Then for all $M' \Subset M'' \Subset M$,*

$$\|\psi\|_{1+\alpha;M'} \le c\big(\alpha, \operatorname{dist}(M', \partial M''), \|\not{V}\|_{\alpha;M''}\big)(\|\not{D}\psi\|_{\alpha;M''} + \|\psi\|_{0;M''}).$$

Due to Proposition 3.1, one can also state a boundary Schauder estimate for the Dirac equation.

Theorem 4.3. *Let M, E be as in Theorem 1.1. Suppose that $\not{V} \in C^{\alpha}(\bar{M})$ for some $0 < \alpha < 1$. Then*

$$\|\psi\|_{1+\alpha;M} \le c(\alpha, \|\not{V}\|_{\alpha:M})(\|\not{D}\psi\|_{\alpha;M} + \|\mathcal{B}\psi\|_{1+\alpha;\partial M} + \|\psi\|_{0;M}).$$

Proof. The classical argument for Schauder estimates (see [29]) can be combined with Proposition 3.1. □

By using the main L^p-estimates of Theorem 1.1, we can get

Theorem 4.4. *Let M, E be as in Theorem 1.1. Suppose that $\Gamma \in \mathfrak{D}^1(E)$, $\Gamma \in C^{\alpha}(\bar{M})$ and $\mathrm{d}\Gamma \in C^{\alpha}(\bar{M})$ for some $0 < \alpha < 1$. Then*

$$\|\psi\|_{1+\alpha;M} \le c(\alpha, \|\Gamma\|_{\alpha:M} + \|\mathrm{d}\Gamma\|_{\alpha;M})(\|\not{D}\psi\|_{\alpha;M} + \|\mathcal{B}\psi\|_{1+\alpha;\partial M}).$$

Proof. By using the main L^p-estimates of Theorem 1.1, we know that for large p,

$$\begin{aligned}\|\psi\|_{0;M} &\le c(p, \|\Gamma\|_{p^*})(\|\not{D}\psi\|_{L^p(E)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(E|_{\partial M})}) \\ &\le c(\alpha, \|\Gamma\|_{\alpha:M} + \|\mathrm{d}\Gamma\|_{\alpha;M})(\|\not{D}\psi\|_{\alpha;M} + \|\mathcal{B}\psi\|_{1+\alpha;\partial M}).\end{aligned}$$

Then applying Theorem 4.3, we prove the desired result. □

Similarly to the case of the L^p-estimate, we can prove the following two theorems:

Theorem 4.5. *Let M, E be as in Theorem 1.1. Suppose that $\Gamma \in \mathfrak{D}^1(E)$, $\Gamma \in C^{\alpha}(\bar{M})$ and $\mathrm{d}\Gamma \in C^{\alpha}(\bar{M})$, $\Omega \in C^{\alpha}(\bar{M})$, $\mathrm{d}\Omega \in C^{\alpha}(\bar{M})$ for some $0 < \alpha < 1$. Let $\eta \in C^{\alpha}(\bar{M})$ and $\mathcal{B}\psi_0 \in C^{1,\alpha}(\partial M)$. Then (4.1) admits a unique solution $\psi \in C^{1,\alpha}(\bar{M})$. Moreover,*

$$\|\psi\|_{1+\alpha;M} \le c(\alpha, \|\Gamma\|_{\alpha;M} + \|\mathrm{d}\Gamma\|_{\alpha;M}, \|\Omega\|_{\alpha;M} + \|\mathrm{d}\Omega\|_{\alpha;M})(\|\not{D}\psi\|_{\alpha;M} + \|\mathcal{B}\psi\|_{1+\alpha;\partial M}).$$

Theorem 4.6. *Let M, N be as in Theorem 1.2. Let $\Phi \in C^{1,\alpha}(\bar{M}, N)$ for some $0 < \alpha < 1$. Then the Dirac equation*

$$\begin{cases} \not{D}\Psi = \eta \in C^{\alpha}(\bar{M}; \Sigma M \otimes \Phi^{-1}TN) & \text{in } M, \\ \mathcal{B}\Psi = \mathcal{B}\psi \in C^{1,\alpha}(\partial M; \Sigma M \otimes \Phi^{-1}TN) & \text{on } \partial M, \end{cases}$$

admits a unique solution $\Psi \in C^{1,\alpha}(\bar{M}; \Sigma M \otimes \Phi^{-1}TN)$, where $\not{D}$ is the Dirac operator along Φ. Moreover,

$$\|\Psi\|_{1+\alpha;M} \le c(\alpha, \|\Phi\|_{1+\alpha;M})(\|\eta\|_{\alpha;M} + \|\mathcal{B}\psi\|_{1+\alpha;\partial M}).$$

5. Short time existence of first order Dirac-harmonic map flows

In this section, we assume that M^m ($m \geq 2$) is a compact Riemannian spin manifold with boundary ∂M and choose a fixed spin structure on M.

Let us consider the family of coupled systems of differential equations for a map $\Phi : M \times [0, T] \to \mathbb{R}^q$ with $\Phi = (\Phi^A)$ and for a spinor field $\Psi : M \times [0, T] \to \Sigma M \otimes \Phi^{-1}T\mathbb{R}^q$ with $\Psi = (\Psi^A)$ along Φ:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right)\Phi^A + \Omega^A_B \cdot \mathrm{d}\Phi^B + \langle \tilde{\Omega}^A_B, \mathrm{d}\Phi^B\rangle = 0 & \text{in } M \times (0, T], \\ \partial\!\!\!/ \Psi^A + \Omega^A_B \cdot \Psi^B = 0 & \text{in } M \times [0, T], \end{cases} \tag{5.1}$$

with the initial and boundary conditions

$$\begin{cases} \Phi(x, t) = \phi(x, t), & (x, t) \in \partial M \times [0, T] \cup M \times \{0\}, \\ \mathcal{B}\Psi = \mathcal{B}\psi & \text{on } \partial M \times [0, T], \end{cases} \tag{5.2}$$

where $\mathcal{B}$ is a chirality boundary operator.

The following two lemmas are similar to those for the harmonic map heat flow (cf. [27, 30, 42]).

Lemma 5.1. *Suppose the image of Φ lies in N and Ψ is a spinor along Φ. Then (Φ, Ψ) satisfies the Dirac-harmonic map flow (1.5), i.e.,*

$$\begin{cases} \partial_t \Phi = \tau(\Phi) - \mathcal{R}(\Phi, \Psi), \\ D\!\!\!/ \Psi = 0, \end{cases}$$

if and only if (Φ, Ψ) satisfies (5.1).

Proof. A well-known computation. □

Lemma 5.2. *Suppose that (Φ, Ψ) is a solution of (5.1) which is continuous on $M \times [0, T]$ with $\phi(x, t) \in N$ for all $(x, t) \in \partial M \times [0, T] \cup M \times \{0\}$ and ψ is a spinor along the map $\phi|_{\partial M}$ for all time in $[0, T]$. Suppose $\Phi(x, t) \in \tilde{N}$ on $M \times (0, T]$. Then $\Phi(x, t) \in N$ for all $(x, t) \in M \times [0, T]$ and $\Psi(\cdot, t)$ is a spinor along the map $\Phi(\cdot, t)$ for all $t \in [0, T]$. In fact, $\tilde{\Psi}^A = \nu^A_B \Psi^B$ satisfies the Dirac type equation*

$$\begin{cases} \partial\!\!\!/ \tilde{\Psi}^A + \Omega^A_B \cdot \tilde{\Psi}^B = 0 & \text{in } M, \\ \mathcal{B}\tilde{\Psi} = 0 & \text{on } \partial M. \end{cases}$$

Proof. Define $\rho : \mathbb{R}^q \to \mathbb{R}^q$ by $\rho(z) = z - \pi(z)$ for $z \in \mathbb{R}^q$. Consider

$$\varphi(x, t) = \|\rho(\Phi(x, t))\|^2 = \sum_{A=1}^{q} \|\rho^A(\Phi(x, t))\|^2.$$

We can see that

$$\left(\frac{\partial}{\partial t}-\Delta\right)\varphi(x,t) = -2\|\nabla\rho(\Phi(x,t))\|^2 + 2\langle\partial_t\rho-\Delta\rho,\rho\rangle$$
$$= -2\|\nabla\rho(\Phi(x,t))\|^2 + 2\big\langle\nu^A_B(\partial_t\Phi^B-\Delta\Phi^B)+\pi^A_{BC}\langle\nabla\Phi^B,\nabla\Phi^C\rangle,\rho^A\big\rangle$$
$$= -2\|\nabla\rho(\Phi(x,t))\|^2$$
$$- 2\big\langle\nu^A_B(\Phi)\big(\Omega(\Phi)^B_C\cdot \mathrm{d}\Phi^C + \langle\tilde\Omega(\Phi)^B_C,\mathrm{d}\Phi^C\rangle\big) - \pi^A_{BC}(\Phi)\langle\nabla\Phi^B,\nabla\Phi^C\rangle,\rho(\Phi)^A\big\rangle.$$

Notice that π^A_B is a projection when restricted to N, so after restriction to N we obtain

$$\nu^A_B(\Phi)\big(\Omega(\Phi)^B_C\cdot \mathrm{d}\Phi^C + \langle\tilde\Omega(\Phi)^B_C,\mathrm{d}\Phi^C\rangle\big) - \pi^A_{BC}(\Phi)\langle\nabla\Phi^B,\nabla\Phi^C\rangle = 0.$$

By the mean value theorem,

$$-2\big\langle\nu^A_B(\Phi)\big(\Omega(\Phi)^B_C\cdot \mathrm{d}\Phi^C + \langle\tilde\Omega(\Phi)^B_C,\mathrm{d}\Phi^C\rangle\big) - \pi^A_{BC}(\Phi)\langle\nabla\Phi^B,\nabla\Phi^C\rangle,\rho(\Phi)^A\big\rangle \le c\varphi.$$

Therefore,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\varphi \le c\varphi.$$

Since $\varphi\ge 0$ and $\varphi=0$ on $\partial M\times[0,T]\cup M\times\{0\}$, we deduce that $\varphi=0$ on $M\times[0,T]$. Hence $\Phi(x,t)\in N$ for all $(x,t)\in M\times[0,T]$ according to the maximum principle.

Next we show that Ψ is a spinor along the map Φ. In order to do this, we consider $\tilde\Psi^A=\nu^A_B\Psi^B$. Then

$$\not{\partial}\tilde\Psi^A = \nu^A_B\not{\partial}\Psi^B + \nabla\nu^A_B\cdot\Psi^B = -\nu^A_B\nabla\nu^B_C\cdot\Psi^C + \nabla\nu^A_B\cdot\Psi^B$$
$$= \nabla\nu^A_B\cdot\tilde\Psi^B = -\nabla\pi^A_B\cdot\tilde\Psi^B = (\nabla\pi^A_C\pi^C_B - \pi^A_C\nabla\pi^C_B)\cdot\tilde\Psi^B = -\Omega^A_B\cdot\tilde\Psi^B.$$

Moreover, $\tilde\Psi^A$ satisfies the boundary conditions

$$\mathcal{B}\tilde\Psi^A = 0$$

for all time $t\in[0,T]$. By the uniqueness of solutions of Dirac equations with chiral boundary values (Theorem 4.1), we get $\tilde\Psi=0$, i.e., Ψ is a spinor along Φ. □

To state the short time existence for the Dirac-harmonic map flow, we first recall some basic facts about heat kernels on Riemannian manifolds. An important property is that the heat kernel is almost Euclidean [20, 41]. In other words, if p is a heat kernel, then p and $\mathcal{E}$ are of the same order, locally uniformly in (x,y) as $t\to 0_+$, and a similar statement holds for the first derivatives of p and $\mathcal{E}$, where

$$\mathcal{E}(x,y,t) = (4\pi t)^{-m/2}e^{-\operatorname{dist}(x,y)^2/(4t)}.$$

One can show that the Dirichlet heat kernel $h(x,y,t)$ is also almost Euclidean [20], hence

$$h(x,y,t)\le ct^{-m/2}e^{-\operatorname{dist}(x,y)^2/(4t)},$$

and

$$\|\nabla h(x,y,t)\| \le ct^{-m/2-1} e^{-\operatorname{dist}(x,y)^2/(4t)} \operatorname{dist}(x,y).$$

We summarize these properties in

Lemma 5.3 (see [20, 38]). *For every $\beta > 0$, there exists a constant $c = c(\beta)$ such that*

$$\begin{aligned} h(x,y,t) &\le c(\beta) t^{-m/2+\beta} \operatorname{dist}(x,y)^{-2\beta}, \\ \|\nabla h(x,y,t)\| &\le c(\beta) t^{-m/2-1+\beta} \operatorname{dist}(x,y)^{1-2\beta}, \\ \|\nabla_{\mathbf{n}} h(x,y,t)\| &\le c(\beta) t^{-m/2-1+\beta} \operatorname{dist}(x,y)^{2-2\beta}, \qquad x, y \in \partial M, \end{aligned}$$

as $t \to 0_+$.

Proof. This is a consequence of the inequality

$$x^\beta e^{-x} \le \beta^\beta e^{-\beta}, \qquad \forall x, \beta > 0.$$

The improvement in the exponent of $\operatorname{dist}(x,y)$ in the third inequality is due to the fact that the derivative is in the direction normal to ∂M. □

Now we can prove the main Theorem 1.3. For the short time existence of the harmonic map heat flow, we refer the reader to [27, 30, 42, 43].

Proof of Theorem 1.3. We split the proof into four steps.

Step I: *Short time existence for the flow* (5.1) *and* (5.2). Let $h(x,y,t)$ be the Dirichlet heat kernel of M. Define an operator T by

$$Tu(x,t) = u_0(x,t) - \int_0^t \int_M h(x,y,t-\tau)\big(\Omega(u)\cdot \mathrm{d}u + \langle \tilde{\Omega}(u,\Psi(u)), \mathrm{d}u\rangle\big)(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau,$$

where

$$u_0(x,t) = \int_M h(x,y,t)\phi(y,0)\, \mathrm{d}y - \int_0^t \int_{\partial M} \frac{\partial h}{\partial \mathbf{n}_y}(x,y,t-\tau)\phi(y,\tau)\, \mathrm{d}\sigma(y)\, \mathrm{d}\tau.$$

Here

$$\Omega(u) = [\nu(u), \mathrm{d}\nu(u)],$$

and

$$\tilde{\Omega}(u,\Psi(u)) = \tfrac{1}{2} R^A_{BCD}(u)(\Psi(u)^C, e_i\cdot\Psi(u)^D)\eta^i =: \mathcal{R}(u)(\Psi(u),\Psi(u)),$$

where $\Psi(u)$ is the unique solution of

$$\begin{cases} \partial\!\!\!/ \Psi^A = -\Omega(u)^A_B \cdot \Psi^B & \text{in } M, \\ \mathcal{B}\Psi^A = \mathcal{B}\psi^A & \text{on } \partial M, \end{cases}$$

according to Theorem 4.1.

It is clear that u_0 is the unique solution of

$$\begin{cases} \frac{\partial}{\partial t} u_0 = \Delta u_0 & \text{in } M \times (0, \infty), \\ u_0 = \phi & \text{on } \partial M \times [0, \infty) \cup M \times \{0\}. \end{cases}$$

For every $\varepsilon > 0$ and each $u \in \bigcap_{0<t<\varepsilon} C^{1,0,0}(\bar{M} \times [t, \varepsilon]) \cap C^0(\bar{M}_\varepsilon)$, define the norm

$$\|u\| := \|u\|_{C^0(\bar{M}\times[0,\varepsilon])} + \sup_{t\in[0,\varepsilon]} \|\nabla u(\cdot, t)\|_{C^0(\bar{M})}.$$

Let X_ϕ^ε be the completion of the following subset of $C^0(\bar{M}_\varepsilon)$:

$$\Big\{u \in \bigcap_{0<t<\varepsilon} C^{1,0,0}(\bar{M} \times [t, \varepsilon]) \cap C^0(\bar{M}_\varepsilon) : u = \phi \text{ on } \mathcal{P}M_\varepsilon\Big\}$$

where

$$M_\varepsilon := M \times (0, \varepsilon], \qquad \mathcal{P}M_\varepsilon := \partial M \times [0, \varepsilon] \cup M \times \{0\}.$$

For $u \in X_\phi^\varepsilon$, according to Theorem 4.1 we have, for large p,

$$\begin{aligned} \|\Psi(\cdot, t)\|_{C^\alpha(\bar{M})} &\le c(p)\|\Psi(\cdot, t)\|_{W^{1,p}(M)} \le C(\|u\|)\|\mathcal{B}\Psi_0(\cdot, t)\|_{W^{1-1/p,p}(\partial M)} \\ &\le C(\|u\|)\|\mathcal{B}\Psi_0(\cdot, t)\|_{C^{1,\alpha}(\partial M)} \le C(\|u\|)\|\mathcal{B}\psi\|_{C^{1,0,\alpha}(\partial M_\varepsilon)}. \end{aligned}$$

As a consequence,

$$T : X_\phi^\varepsilon \to X_\phi^\varepsilon$$

is well defined. For $\delta > 0$, let $B_\delta = \{u \in X_\phi^\varepsilon : \|u - u_0\| \le \delta\}$.

According to Lemma 5.3, for every $\beta > 0$,

$$\begin{aligned} h(x, y, t) &\le c(\beta)t^{\beta-m/2} \operatorname{dist}(x, y)^{-2\beta}, \\ \|\nabla h(x, y, t)\| &\le c(\beta)t^{\beta-1-m/2} \operatorname{dist}(x, y)^{1-2\beta}, \\ \|\nabla_{\mathbf{n}} h(x, y, t)\| &\le c(\beta)t^{\beta-1-m/2} \operatorname{dist}(x, y)^{2-2\beta}, \qquad x, y \in \partial M. \end{aligned}$$

1) We now prove that $u_0 \in X_\phi^\varepsilon$. Let $v_0 = u_0 - \phi$. Then

$$\begin{cases} \partial_t v_0 - \Delta v_0 = \Delta\phi - \partial_t\phi =: f & \text{in } M_T, \\ v_0 = 0 & \text{on } \mathcal{P}M_T. \end{cases}$$

Since $\phi \in C^{2,1,\alpha}(\bar{M}_T)$, we know that $f \in C^{0,0,\alpha}(\bar{M}_T)$ and

$$\|f\|_{C^0(\bar{M}_T)} \le c\|\phi\|_{C^{2,1,0}(\bar{M}_T)}, \qquad \|f\|_{C^{0,0,\alpha}(\bar{M}_T)} \le c(\alpha)\|\phi\|_{C^{2,1,\alpha}(\bar{M}_T)}.$$

Moreover, v_0 can be given by the following formula, for $(x, t) \in M_T$:

$$v_0(x, t) = \int_0^t \int_M h(x, y, t - \tau) f(y, \tau)\, \mathrm{d}y\, \mathrm{d}\tau.$$

The Schauder estimates imply that $v_0 \in \bigcap_{0<t<T} C^{2,1,\alpha}(\bar{M} \times [t,T]) \cap C^0(\bar{M}_T)$. The following estimates follow by straightforward computations:

$$|v_0(x,t)| \le \|\phi\|_{C^{2,1,0}(\bar{M}_\varepsilon)}\varepsilon, \qquad |\nabla v_0(x,t)| \le c(\beta)\|\phi\|_{C^{2,1,0}(\bar{M}_\varepsilon)}\varepsilon^{\beta-1},$$

for all $(x,t) \in M_\varepsilon$ and $\beta \in (m/2,(m+1)/2)$. In fact,

$$\begin{aligned}|v_0(x,t)| &\le \int_0^t \int_M h(x,y,t-\tau)|f(y,\tau)|\,\mathrm{d}y\,\mathrm{d}\tau \\ &\le \|f\|_{C^0(\bar{M}_t)} \int_0^t \int_M h(x,y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \le \|f\|_{C^0(\bar{M}_t)}t,\end{aligned}$$

and

$$\begin{aligned}|\nabla v_0(x,t)| &\le \int_0^t \int_M |\nabla_x h(x,y,t-\tau)|\,|f(y,\tau)|\,\mathrm{d}y\,\mathrm{d}\tau \\ &\le c(\beta)\|f\|_{C^0(\bar{M}_t)} \int_0^t \int_M |t-\tau|^{-2+\beta}\,\mathrm{dist}(x,y)^{1-2\beta}\,\mathrm{d}y\,\mathrm{d}\tau \\ &\le c(\beta)\|f\|_{C^0(\bar{M}_t)}t^{\beta-m/2}\end{aligned}$$

for all $\beta \in (m/2,(m+1)/2)$. Therefore $u_0 \in X^\varepsilon_\phi$, and for δ, ε both small we have $u_0 \in \tilde{N}$ if $\phi \in N$.

2) Now we will prove that for ε small, $T(B_\delta) \subset B_\delta$. Let $u \in B_\delta$. Then $\|u\| \le C_1$. Consequently,

$$\|\Omega(u)\|_{C^0(\bar{M}\times[0,\varepsilon])} = \|[\nu(u),\mathrm{d}\nu(u)]\|_{C^0(\bar{M}\times[0,\varepsilon])} \le c \sup_{t\in[0,\varepsilon]} \|\nabla u(\cdot,t)\|_{C^0(\bar{M})} \le c(C_1),$$

and

$$\begin{aligned}\|\tilde{\Omega}(u,\Psi(u))\|_{C^0(\bar{M}\times[0,\varepsilon])} &\le c \sup_{t\in[0,\varepsilon]} \|\Psi(\cdot,t)\|^2_{C^0(\bar{M})} \\ &\le c(C_1) \sup_{t\in[0,\varepsilon]} \|\mathcal{B}\psi(\cdot,t)\|^2_{H^{1-1/p,p}(\partial M)} \\ &\le c(C_1)\|\mathcal{B}\psi\|^2_{C^{1,0,\alpha}(\bar{\partial}M\times[0,\varepsilon])} \le c(C_1)C_2,\end{aligned}$$

for p large enough, where the second inequality has used Theorem 4.1. Hence,

$$\|\Omega(u)\|_{C^0(\bar{M}_\varepsilon)} + \|\tilde{\Omega}(u,\Psi(u))\|_{C^0(\bar{M}_\varepsilon)} \le c(C_1,C_2), \qquad \forall u \in B_\delta.$$

As a consequence,

$$\|Tu - u_0\| \le c(\beta)c(C_1,C_2)\|\mathrm{d}u\|_{C^0(\bar{M}_\varepsilon)}\varepsilon^{\beta-m/2} \qquad \text{for } \beta \in (m/2,(m+1)/2).$$

3) Now we prove that $\|Tu - Tv\| \le \frac{1}{2}\|u - v\|$ for $u, v \in B_\delta$ and ε small. First, we have

$$\begin{aligned} Tu(x,t) - Tv(x,t) \\ &= -\int_0^t \int_M h(x,y,t-\tau)\big(\Omega(u)\cdot \mathrm{d}u + \langle \tilde{\Omega}(u,\Psi(u)), \mathrm{d}u\rangle\big)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \int_0^t \int_M h(x,y,t-\tau)\big(\Omega(v)\cdot \mathrm{d}v + \langle \tilde{\Omega}(v,\Psi(v)), \mathrm{d}v\rangle\big)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

Moreover, noticing that

$$\begin{aligned} \Omega(u) - \Omega(v) &= [\nu(u), \mathrm{d}\nu(u)] - [\nu(v), \mathrm{d}\nu(v)] \\ &= [\nu(u) - \nu(v), \mathrm{d}\nu(u)] + [\nu(v), \mathrm{d}\nu(u) - \mathrm{d}\nu(v)], \end{aligned}$$

and

$$\begin{aligned} \tilde{\Omega}(u,\Psi(u)) - \tilde{\Omega}(v,\Psi(v)) &= \mathcal{R}(u)(\Psi(u),\Psi(u)) - \mathcal{R}(v)(\Psi(v),\Psi(v)) \\ &= (\mathcal{R}(u) - \mathcal{R}(v))(\Psi(u),\Psi(u)) + \mathcal{R}(v)(\Psi(u) - \Psi(v),\Psi(u)) \\ &\quad + \mathcal{R}(v)(\Psi(v),\Psi(u) - \Psi(v)), \end{aligned}$$

we have

$$\|\Omega(u) - \Omega(v)\|_{C^0(\bar{M}_\varepsilon)} \le c(C_1, C_2)\|u - v\|,$$

and

$$\begin{aligned} \|\tilde{\Omega}(u,\Psi(u)) - \tilde{\Omega}(v,\Psi(v))\|_{C^0(\bar{M}_\varepsilon)} &\le c(C_1,C_2)(\|u - v\| + \|\Psi(u) - \Psi(v)\|_{C^0(\bar{M}\times[0,\varepsilon])}) \\ &\le c(C_1,C_2)\|u - v\|. \end{aligned}$$

The last inequality follows from the fact that

$$\begin{cases} \partial\!\!\!/(\Psi(u)^A - \Psi(v)^A) = -\Omega(u)^A_B \cdot (\Psi(u)^B - \Psi(v)^B) \\ \qquad\qquad + (\Omega(u)^A_B - \Omega(v)^A_B)\cdot \Psi(v)^B & \text{in } M, \\ \mathcal{B}(\Psi(u) - \Psi(v)) = 0 & \text{on } \partial M. \end{cases}$$

And Theorem 4.1 implies that for large p,

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{C^\alpha(\bar{M})} &\le c(p)\|\Psi(u) - \Psi(v)\|_{W^{1,p}(M)} \\ &\le c(p,C_1,C_2)\|u - v\|\,\|\Psi(v)\|_{L^p(E)} \le c(p,C_1,C_2)\|u - v\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Omega(u)\cdot \mathrm{d}u + \langle \tilde{\Omega}(u,\Psi(u)), \mathrm{d}u\rangle - \Omega(v)\cdot \mathrm{d}v - \langle \tilde{\Omega}(v,\Psi(v)), \mathrm{d}v\rangle\|_{C^\alpha(\bar{M})} \\ \le c(p,C_1,C_2)\|u - v\|. \end{aligned}$$

Using a similar argument for v_0, one gets

$$\|Tu - Tv\| \le c(\beta)c(C_1,C_2)\varepsilon^{\beta - m/2}\|u - v\|$$

for all $\beta \in (m/2, (m+1)/2)$.

Therefore, there exists a fixed point of T in B_δ, i.e., we have proved the short time existence for (5.1) and (5.2).

Step II: *Regularity.* Let (Φ, Ψ) be the solution of (5.1) and (5.2) constructed above. Theorem 4.1 implies that $\Psi \in L^\infty(\bar{M}_\varepsilon)$ and hence $\Phi \in \bigcap_{0<t<\varepsilon} C^{1,0,\alpha}(\bar{M} \times [t, \varepsilon]) \cap C(\bar{M}_\varepsilon)$ by the L^p-estimate for the heat equation. For all $0 < t, \tau \le \varepsilon$, we have

$$\begin{cases} \not{\partial}(\Psi^A(\cdot, t) - \Psi^A(\cdot, \tau)) = -\Omega^A_B(\cdot, t) \cdot (\Psi^B(\cdot, t) - \Psi^B(\cdot, \tau)) \\ \qquad\qquad + (\Omega^A_B(\cdot, \tau) - \Omega^A_B(\cdot, t)) \cdot \Psi^B(\cdot, \tau) & \text{in } M, \\ \mathcal{B}(\Psi(\cdot, t) - \Psi(\cdot, \tau)) = \mathcal{B}(\psi(\cdot, t) - \psi(\cdot, \tau)) & \text{on } \partial M. \end{cases}$$

Again by using Theorem 4.1, for large p,

$$\|\Psi(\cdot, t) - \Psi(\cdot, \tau)\|_{C^\alpha(\bar{M})} \le c(p, C_1, C_2)|t - \tau|^{\alpha/2}.$$

Thus, $\Psi \in \bigcap_{0<t<\varepsilon} C^{0,0,\alpha}(\bar{M} \times [t, \varepsilon]) \cap C^0(\bar{M}_\varepsilon)$. The Schauder estimate for the heat equation implies that $\Phi \in \bigcap_{0<t<\varepsilon} C^{2,1,\alpha}(\bar{M} \times [t, \varepsilon]) \cap C^0(\bar{M}_\varepsilon)$. The interior Schauder estimate for the Dirac equation implies that $\Psi(\cdot, t) \in C^{2,\alpha}(M)$ for every $t \le \varepsilon$. If one uses the boundary Schauder estimate, one finds that $\Psi(\cdot, t) \in C^{1,\alpha}(\bar{M})$ for $t \le \varepsilon$.

Suppose that

$$\limsup_{t<T_1,\, t\to T_1} \|\mathrm{d}\Phi(\cdot, t)\|_{C^0(\bar{M})} < \infty.$$

The discussion above implies that this flow can be extended to a larger time $T_1' > T_1$, hence T_1 is not the maximum time, a contradiction.

Step III: *Uniqueness.* Finally, we state the uniqueness. Suppose that (Φ_i, Ψ_i) are solutions of (5.1) and (5.2). Let $u = \Phi_1 - \Phi_2$ and $\eta = \Psi_1 - \Psi_2$. Then

$$|\partial_t u - \Delta u| \le C|\nabla u| + C|u| + C|\eta|, \qquad |\not{\partial}\eta + \Omega_1 \cdot \eta| \le C|\nabla u| + C|u|.$$

Hence, applying Theorem 4.1 and using the same computation as for v_0, we get

$$\|\eta(\cdot, t)\|_{C^0(\bar{M})} \le C\|u\|,$$

and

$$\|u\| \le C\|u\|\varepsilon^{\beta - m/2} \qquad \text{for } 0 < \varepsilon \le T_1 \text{ and } \beta \in (m/2, (m+1)/2),$$

where $\|\cdot\|$ is the norm corresponding to M_ε. Thus, if ε is small, then $\|u\| = 0$, i.e., $u = 0$ and hence $\eta = 0$. Then we can prove the uniqueness of the Dirac-harmonic heat flow by iteration.

Step IV: *Completion of the proof.* We have actually proved that $\Phi \in \tilde{N}$ if $\phi \in N$. Therefore, we can use Lemma 5.2. As a consequence, $\Phi(\cdot, t) \in N$ and $\Psi(\cdot, t) \in \Sigma M \times \Phi(\cdot, t)^{-1}TN$ for all $0 \le t < T_1$ since $\mathcal{B}\psi(\cdot, t) \in (\Sigma M \otimes \phi(\cdot, t)^{-1}TN)|_{\partial M}$ for all t. Then applying Lemma 5.1, we complete the proof of the theorem. □

6. Dirac equations along a map between Riemannian disks

In this section, we discuss a Dirac equation along a smooth map $\phi : M = (D, \lambda|dz|^2) \to N = (D, \rho|dw|^2)$ where $D = \{|z| < 1\}$ is the open unit disk on $\mathbb{C}$. Let ΣM be the spin bundle on M. Consider a Dirac bundle $\Sigma M \otimes \phi^{-1}TN$ and split it as (see [40, 53])

$$\Sigma M \otimes \phi^{-1}TN = (\Sigma^+ M \otimes \phi^{-1}T^{1,0}N) \oplus (\Sigma^- M \otimes \phi^{-1}T^{1,0}N) \oplus (\Sigma^+ M \otimes \phi^{-1}T^{0,1}N) \oplus (\Sigma^- M \otimes \phi^{-1}T^{0,1}N),$$

where

$$\Sigma^+ M = \{\psi \in \Sigma M : \partial_{\bar z} \cdot \psi = 0\}, \quad \Sigma^- M = \{\psi \in \Sigma M : \partial_z \cdot \psi = 0\}.$$

We identify Clifford multiplication in the orthogonal bases $\partial_z, \partial_{\bar z}$ with the following matrices (see [24, 40]):

$$\partial_z = \lambda^{1/2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \partial_{\bar z} = \lambda^{1/2}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

And the spinor $\psi \in \Sigma M \otimes \phi^{-1}TN$ can be written as

$$\psi = \begin{pmatrix} f^+ \\ f^- \end{pmatrix} \otimes \partial_\phi + \begin{pmatrix} \tilde f^+ \\ \tilde f^- \end{pmatrix} \otimes \partial_{\bar\phi}$$

with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Sigma^+ M, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \Sigma^- M.$$

The connection on the spin bundle ΣM is then given by the following operators (see [40]):

$$\nabla^{\Sigma M}_{\partial_z} = \frac{\partial}{\partial z} + \frac{1}{4}\frac{\partial \log\lambda}{\partial z}, \quad \nabla^{\Sigma M}_{\partial_{\bar z}} = \frac{\partial}{\partial \bar z} + \frac{1}{4}\frac{\partial \log\lambda}{\partial \bar z}.$$

Therefore, the Dirac operator on $\Sigma M \otimes \phi^{-1}TN$ is

$$\not{D} = 2\lambda^{-1/2}\begin{pmatrix} 0 & -\frac{\partial}{\partial z} - \frac{1}{4}\frac{\partial\log\lambda}{\partial z} - \frac{\partial\log\rho}{\partial\phi}\frac{\partial\phi}{\partial z} \\ \frac{\partial}{\partial\bar z} + \frac{1}{4}\frac{\partial\log\lambda}{\partial\bar z} + \frac{\partial\log\rho}{\partial\phi}\frac{\partial\phi}{\partial\bar z} & 0 \end{pmatrix}$$
$$\oplus\, 2\lambda^{-1/2}\begin{pmatrix} 0 & -\frac{\partial}{\partial z} - \frac{1}{4}\frac{\partial\log\lambda}{\partial z} - \frac{\partial\log\rho}{\partial\bar\phi}\frac{\partial\bar\phi}{\partial z} \\ \frac{\partial}{\partial\bar z} + \frac{1}{4}\frac{\partial\log\lambda}{\partial\bar z} + \frac{\partial\log\rho}{\partial\bar\phi}\frac{\partial\bar\phi}{\partial\bar z} & 0 \end{pmatrix}.$$

The chiral boundary operator [23, 24, 32] is given by

$$\mathcal{B}^{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm z^{-1} \\ \pm z & 1 \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} 1 & \pm z^{-1} \\ \pm z & 1 \end{pmatrix}.$$

Now we consider the Dirac equation

$$\begin{cases} \not{D}\psi = 0 & \text{in } D, \\ \mathcal{B}^{\pm}\psi = \mathcal{B}^{\pm}\psi_0 & \text{on } \partial D. \end{cases}$$

As discussed above, $\not{D}\psi = 0$ is equivalent to the systems

$$f^+_{\bar z} + \big(\tfrac14(\log\lambda)_{\bar z} + (\log\rho)_\phi \phi_{\bar z}\big) f^+ = 0, \qquad f^-_z + \big(\tfrac14(\log\lambda)_z + (\log\rho)_\phi \phi_z\big) f^- = 0,$$

and

$$\tilde f^+_{\bar z} + \big(\tfrac14(\log\lambda)_{\bar z} + (\log\rho)_{\bar\phi} \bar\phi_{\bar z}\big) \tilde f^+ = 0, \qquad \tilde f^-_z + \big(\tfrac14(\log\lambda)_z + (\log\rho)_{\bar\phi} \bar\phi_z\big) \tilde f^- = 0,$$

where the spinor ψ has the form

$$\psi = \begin{pmatrix} f^+ \\ f^- \end{pmatrix} \otimes \partial_\phi + \begin{pmatrix} \tilde f^+ \\ \tilde f^- \end{pmatrix} \otimes \partial_{\bar\phi}.$$

Let g be a solution of the Riemann–Hilbert problem

$$\begin{cases} g_{\bar z} = \frac14(\log\lambda)_{\bar z} + (\log\rho)_\phi \phi_{\bar z} & \text{in } D, \\ \operatorname{Re} g = \log(\lambda^{1/4}\rho^{1/2}) & \text{on } \partial D. \end{cases}$$

All solutions can be given by the following formula [11, p. 71, Theorem 21]:

$$\begin{aligned} g(z) = i \operatorname{Im} g(0) + \log\lambda^{1/4}(z) + \frac{1}{2\pi i}\int_{\partial D} \frac{\log(\rho(\phi(\zeta))^{1/2})}{\zeta}\, \frac{\zeta + z}{\zeta - z}\, d\zeta \\ + \frac{1}{4\pi i}\int_D \left(\frac{(\log\rho)_\phi \phi_{\bar\zeta}(\zeta)}{\zeta}\, \frac{\zeta+z}{\zeta-z} + \frac{(\log\rho)_{\bar\phi}\bar\phi_\zeta(\zeta)}{\bar\zeta}\, \frac{1+z\bar\zeta}{1-z\bar\zeta} \right) d\zeta \wedge d\bar\zeta \end{aligned}$$

for all $z \in D$. Then

$$f^+_{\bar z} + g_{\bar z} f^+ = 0, \qquad \overline{f^-_z} + \big((\log\rho + \log\lambda^{1/2})_{\bar z} - g_{\bar z}\big)\overline{f^-} = 0,$$

and

$$\tilde f^+_{\bar z} + \big((\log\rho + \log\lambda^{1/2})_{\bar z} - g_{\bar z}\big)\tilde f^+ = 0, \qquad \overline{\tilde f^-_{\bar z}} + g_{\bar z}\overline{\tilde f^-} = 0.$$

Therefore, there exist four holomorphic functions $A^+, A^-, \tilde A^+, \tilde A^-$ such that

$$f^+ = e^{-g}A^+, \qquad f^- = \lambda^{-1/2}\rho^{-1}e^{\bar g}\overline{A^-}, \qquad \tilde f^+ = \lambda^{-1/2}\rho^{-1}e^{g}\tilde A^+, \qquad \tilde f^- = e^{-\bar g}\overline{\tilde A^-}.$$

The chirality boundary condition $\mathcal{B}^{\pm}\psi = \mathcal{B}^{\pm}\psi_0$ is now equivalent to

$$\begin{aligned} A^+ \pm z^{-1}\overline{A^-} &= \lambda^{1/4}\rho^{1/2}e^{i\operatorname{Im} g}(f_0^+ \pm z^{-1} f_0^-), \\ \tilde A^+ \pm z^{-1}\overline{\tilde A^-} &= \lambda^{1/4}\rho^{1/2}e^{-i\operatorname{Im} g}(\tilde f_0^+ \pm z^{-1}\tilde f_0^-), \end{aligned}$$

for $z \in \partial D$, where

$$\psi_0 = \begin{pmatrix} f_0^+ \\ f_0^- \end{pmatrix} \otimes \partial_\phi + \begin{pmatrix} \tilde{f}_0^+ \\ \tilde{f}_0^- \end{pmatrix} \otimes \partial_{\bar{\phi}}.$$

Since the index of z^{-1} is -1, the solutions $A^+, A^-, \tilde{A}^+, \tilde{A}^-$ must be unique according to Theorem A.1 (see Appendix A). In particular, $f^+, f^-, \tilde{f}^+, \tilde{f}^-$ are independent of the choice of g. In fact, any other choice of g is $g + ic$ where c is some real number. Then the solutions $A^+, A^-, \tilde{A}^+, \tilde{A}^-$ must be replaced by $e^{ic}A^+, e^{-ic}A^-, e^{-ic}\tilde{A}^+, e^{ic}\tilde{A}^-$ respectively. As a consequence, $f^+, f^-, \tilde{f}^+, \tilde{f}^-$ do not change.

Next, we construct these solutions by using Theorem A.1. Denote

$$F(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{\lambda(\zeta)^{1/4}\rho(\phi(\zeta))^{1/2}e^{i\,\mathrm{Im}\,g(\zeta)}(f_0^+(\zeta) \pm \zeta^{-1} f_0^-(\zeta))}{\zeta - z}\,\mathrm{d}\zeta, \qquad z \notin \partial D,$$
$$\tilde{F}(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{\lambda(\zeta)^{1/4}\rho(\phi(\zeta))^{1/2}e^{-i\,\mathrm{Im}\,g(\zeta)}(\tilde{f}_0^+(\zeta) \pm \zeta^{-1} \tilde{f}_0^-(\zeta))}{\zeta - z}\,\mathrm{d}\zeta, \qquad z \notin \partial D.$$

Then, for $z \in D$,

$$A^+(z) = F(z), \qquad A^-(z) = \pm z^{-1}\overline{F(1/\bar{z})}, \qquad \tilde{A}^+(z) = \tilde{F}(z), \qquad \tilde{A}^-(z) = \pm z^{-1}\overline{\tilde{F}(1/\bar{z})}$$

are the solutions.

To summarize the previous discussion and using Theorem A.2 (see Appendix A), we have

Theorem 6.1. *Suppose that $\phi \in C^{1+\alpha}(\bar{D})$ and $\mathcal{B}^\pm\psi_0 \in C^{1+\alpha}(\partial D)$ for some $\alpha \in (0,1)$. Then there exists a unique solution $\psi \in C^{1+\alpha}(\bar{D})$ of*

$$\begin{cases} \not{D}\psi = 0 & \text{in } D, \\ \mathcal{B}^\pm\psi = \mathcal{B}^\pm\psi_0 & \text{on } \partial D. \end{cases}$$

Moreover, there exists a constant $c = c(\alpha)$ such that

$$\|\psi\|_{1+\alpha;D} \le c\|\mathcal{B}^\pm\psi_0\|_{1+\alpha;\partial D}\|\phi\|_{1+\alpha;D}.$$

Remark 6.1. When the domain is $D = \{|z| < 1\}$, the MIT bag boundary operator is given by

$$\mathcal{B}^\pm_{\mathrm{MIT}} = \frac{1}{2}\begin{pmatrix} 1 & \mp iz^{-1} \\ \pm iz & 1 \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} 1 & \mp iz^{-1} \\ \pm iz & 1 \end{pmatrix}.$$

The index of $\mp iz^{-1}$ is -1 and therefore we can use Theorems A.1 and A.2. In particular, the above theorem is also true for MIT bag boundary value conditions.

Appendix A. The boundary value problem for the $\bar{\partial}$-equation

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plain $\mathbb{C}$. We say that (A^+, A^-) is a *holomorphic function pair* on D if A^+, A^- are two holomorphic functions on D. Consider the transformation

$$\tilde{A}^+(z) := A^+(z), \qquad \forall z \in D^+ := D,$$
$$\tilde{A}^-(z) := \overline{A^-(1/\bar{z})}, \quad \forall z \in D^- := \bar{\mathbb{C}} \setminus \bar{D}.$$

Then $\tilde{A}^+$ and $\tilde{A}^-$ are holomorphic in D^+ and D^- respectively. Moreover, if A^+, A^- satisfy the boundary condition

$$A^+ - \varphi\overline{A^-} = f \quad \text{on } \partial D,$$

then $\tilde{A}^+, \tilde{A}^-$ satisfy the boundary condition

$$\tilde{A}^+ - \varphi\tilde{A}^- = f \quad \text{on } \partial D.$$

Theorem A.1 (see [11, Theorems 5, 14, 15]). *Suppose that $\varphi \in C^\alpha(\partial D)$ with $\varphi(\zeta) \neq 0$ for all $\zeta \in \partial D$, where $0 < \alpha < 1$. Let $f \in C^\alpha(\partial D)$ and κ be the index of φ. If $\kappa \geq 0$, then there exist exactly $\kappa + 1$ linearly independent holomorphic function pairs (A^+, A^-) on D such that*

$$A^+ - \varphi\overline{A^-} = 0 \quad \text{on } \partial D.$$

If $\kappa = -1$, then there exists a unique holomorphic function pair (A^+, A^-) such that

$$A^+ - \varphi\overline{A^-} = f \quad \text{on } \partial D.$$

Moreover, $A^\pm \in C^\alpha(\bar{D}) \cap A(D)$. In fact, if we set

$$\gamma(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{\log(\zeta\varphi(\zeta))}{\zeta - z}\,\mathrm{d}\zeta, \qquad z \notin \partial D,$$
$$\psi(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)e^{-\gamma(\zeta)}}{\zeta - z}\,\mathrm{d}\zeta, \qquad z \notin \partial D,$$

then

$$A^+(z) = e^{\gamma(z)}\psi(z), \qquad A^-(z) = z^{-1}\overline{e^{\gamma(1/\bar{z})}\psi(1/\bar{z})}, \qquad z \in D.$$

Moreover, there exists a constant $c = c(\alpha)$ such that

$$\|A^\pm\|_{\alpha;D} \leq c\|\varphi\|_{\alpha;D}\|f\|_{\alpha;D}.$$

We also need the following Schauder estimate.

Theorem A.2 (see [11, Theorems 5, 29]). *Suppose that* $f \in C^{\alpha}(\bar{D})$ *and* $h \in C^{1+\alpha}(\partial D)$ *for some* $0 < \alpha < 1$. *Then every solution of*

$$\begin{cases} g_{\bar{z}} = f & \text{in } D, \\ \operatorname{Re} g = h & \text{on } \partial D. \end{cases}$$

is of class $C^{1+\alpha}(\bar{D})$. *Moreover, there exists a constant* $c = c(\alpha)$ *such that*

$$\|g\|_{1+\alpha;D} \leq c(\|f\|_{\alpha;D} + \|h\|_{1+\alpha;\partial D} + |\operatorname{Im} g(0)|).$$

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