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Calculus of Variations

Dirac-geodesics and their heat flows

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Abstract Dirac-geodesics are Dirac-harmonic maps from one dimensional domains. In this paper, we introduce the heat flow for Dirac-geodesics and establish its long-time existence and an asymptotic property of the global solution. We classify Dirac-geodesics on the standard 2-sphere $S^2(1)$ and the hyperbolic plane $\mathbb{H}^2$, and derive existence results on topological spheres and hyperbolic surfaces. These solutions constitute new examples of coupled Dirac-harmonic maps (in the sense that the map part is not simply a harmonic map).

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1 Introduction

Dirac-harmonic maps were introduced in [4,5] as a geometric analytic model corresponding to the supersymmetric nonlinear σ-model of quantum field theory [9,12].

Let us describe the geometric setting. Let (M, g) be a spin manifold with a fixed spin structure, and ΣM the spinor bundle over M, on which we chose a Hermitian metric $\langle\cdot,\cdot\rangle$. The Levi-Civita connection ∇ on ΣM is compatible with $\langle\cdot,\cdot\rangle$. Let (N, h) be a Riemannian manifold, Φ a map from M to N, and $\Phi^{-1}TN$ the pull-back bundle of TN by Φ. On the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$ there is a metric (still denoted by $\langle\cdot,\cdot\rangle$) induced from the metrics on ΣM and $\Phi^{-1}TN$. There is a connection, still denoted by ∇, on $\Sigma M \otimes \Phi^{-1}TN$ naturally induced from those on ΣM and $\Phi^{-1}TN$.

The Dirac operator along the map Φ is defined as

$$\begin{aligned} \not{D}\Psi &:= e_i \cdot \nabla_{e_i}\Psi \\ &= \not{\partial}\psi^\alpha \otimes \theta_\alpha + e_i \cdot \psi^\alpha \otimes \nabla_{e_i}\theta_\alpha, \end{aligned}$$

where we write a cross-section Ψ of $\Sigma M \otimes \Phi^{-1}TN$ locally as $\Psi = \psi^\alpha \otimes \theta_\alpha$, $\{\psi^\alpha\}$ are local cross-sections of ΣM, and $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$, $\{e_i\}$ is a local orthonormal basis on M, $\not{\partial} := e_i \cdot \nabla_{e_i}$ is the usual Dirac operator on M and "$X\cdot$" stands for the Clifford multiplication by the vector field X on M. Here and in the sequel, we use the usual summation convention.

Consider the functional

$$L(\Phi, \Psi) = \frac{1}{2}\int_M \left(\|\mathrm{d}\Phi\|^2 + \left\langle \Psi, \not{D}\Psi \right\rangle\right).$$

The critical points (Φ, Ψ) have to satisfy in M° the following Euler-Lagrange equations for $L(\Phi, \Psi)$ (c.f. [4]):

$$\begin{cases} \tau(\Phi) = \frac{1}{2}\left\langle \psi^\alpha, e_i \cdot \psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i) \equiv \mathcal{R}(\Phi, \Psi), \\ \not{D}\Psi = 0, \end{cases} \tag{1.1}$$

where $R^N(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for $X, Y \in \Gamma(TN)$, stands for the curvature operator of N, and $\tau(\Phi)$ is the tension field of Φ. Solutions of (1.1) are called Dirac-harmonic maps from M *to* N. When M has nonempty boundary ∂M, then we need to impose appropriate boundary conditions for (Φ, Ψ), see e.g. [6,7,22].

When the dimension of the domain manifold M is one, Dirac-harmonic maps are called Dirac-geodesics. The corresponding functional is of the form

$$L(\gamma, \Psi) = \frac{1}{2}\left(\int_M \|\dot{\gamma}\|^2 + \left\langle \Psi, \not{D}\Psi \right\rangle\right), \tag{1.2}$$

where $\dot{\gamma}$ denotes the spatial derivative $\mathrm{d}\gamma/\mathrm{d}s$, M is an interval, say $[0, 1]$ in $\mathbb{R}^1$.

In [11], Isobe introduced a modified functional

$$L_F(\gamma, \Psi) = \frac{1}{2}\left(\int_{S^1} \|\dot{\gamma}\|^2 + \left\langle \Psi, \not{D}\Psi \right\rangle\right) - \int_{S^1} F(\gamma, \Psi),$$

where F is some suitable function. The critical points (γ, Ψ) are called the nonlinear Dirac-geodesics. Existence results were obtained in [11] via an approach from critical point theory, under some conditions on the function $F > 0$ and assumptions on the metric of the target N.

Recently, Branding [2,3] introduced the following regularized functional:

$$L_\varepsilon(\gamma, \Psi) = \frac{1}{2}\left(\int_{S^1} \|\dot{\gamma}\|^2 + \langle \Psi, \not{D}\Psi \rangle + \varepsilon |\not{D}\Psi|^2\right),$$

where $\varepsilon > 0$ is a parameter; critical points of it are called regularized Dirac-geodesics. He proved the global existence and convergence of the heat flow of closed regularized Dirac-geodesic when ε is large. However, the final existence of Dirac-geodesics cannot be obtained by removing the regularization, i.e., by letting $\varepsilon \to 0$.

It is thus a natural question to define a suitable heat flow for Dirac-geodesics and study its global existence and asymptotic behavior. This is the main purpose of the present paper.

Let $\sigma : [0,1] \longrightarrow N$ be a smooth curve. For $\gamma : [0,1] \times [0,T) \longrightarrow N$ and $X(\cdot,t)$, $Y(\cdot,t)$ vector fields along the curve $\gamma(\cdot,t)$, consider the system

$$\begin{cases} \gamma' = \nabla_{\dot{\gamma}}\dot{\gamma} + R(X,Y)\dot{\gamma}, & \text{on } (0,1) \times (0,T), \\ \nabla_{\dot{\gamma}} X = 0, & \text{on } (0,1] \times [0,T), \\ \nabla_{\dot{\gamma}} Y = 0, & \text{on } [0,1) \times [0,T), \end{cases} \tag{1.3}$$

with initial-boundary value conditions

$$\begin{cases} \gamma(s,0) = \sigma(s), & s \in (0,1), \\ \gamma(0,t) = x_0, \quad \gamma(1,t) = y_0, & t \in [0,T), \\ X(0,t) = X_0, & t \in [0,T), \\ Y(0,t) = Y_0, & t \in [0,T), \end{cases} \tag{1.4}$$

where x_0, y_0 are two fixed points in N, $X_0, Y_0 \in T_{x_0}N$ are two fixed tangent vectors, γ' denotes the time derivative $\gamma' = \dfrac{\partial \gamma}{\partial t}$.

The system (1.3) constitutes the heat flow for the Euler-Lagrange equation of the functional (1.2), see Lemma 2.1 in Sect. 2. In fact, (1.3) can be viewed as a parabolic system with extra constraining equations satisfied by the field Ψ, which can be reduced to equations for two parallel vector fields X and Y along the underlying curve γ and hence can be easily solved. The fact that with this elliptic-parabolic system we get a better handle on the existence than other approaches seems to indicate that this is the right parabolic version of the Dirac-geodesic problem. Instead of trying to also turn the first-order Dirac equations for X and Y into parabolic equations, we rather treat them as first order constraints along the second order parabolic flow for γ. Thus, in particular, we can apply elliptic estimates for X and Y along the flow and thereby control the inhomogeneous term in the flow for γ.

The reason why we only consider the flow of Dirac-geodesics (γ, Ψ) defined on an interval $[0,1]$ rather than on the circle S^1 is that, in general, one can not expect that the parallel vector fields X, Y can be defined on the whole S^1. Nevertheless, γ could be a closed curve. For the heat flow of Dirac-harmonic maps from higher dimensional manifolds with boundary, see [6]. We will prove the following global existence result for the Dirac-geodesic heat flow:

Theorem 1.1 *Let N^n be a Riemannian manifold. Then there exists a unique solution of* (1.3) *and* (1.4) *for all $t \in [0, +\infty)$.*

Recall that for the usual geodesic heat flow, Ottarsson [18] proved the long-time existence and uniqueness of a solution for smooth initial data, which has been recently extended by Lin and Wang [17] to $W^{1,2}$ initial data. However, the convergence of the geodesic flow is unexpectedly subtle. Although it is proved in [18] that there is a sequence $\{t_k\}$ with $t_k \to +\infty$

($k \to +\infty$), such that $\gamma(t_k) \to \gamma_\infty$, the convergence of $\gamma(t)$ need not to be true in general, see the example of Topping (c.f. [8,23]). Choi and Parker [8] proved the convergence of the geodesic heat flow for generic metrics, the so-called bumpy metrics on the target manifold N.

Koh [15] proved the global existence of the magnetic geodesic heat flow:

$$\gamma' = \nabla_{\dot\gamma}\dot\gamma + Z(\gamma),$$

where $Z \in \mathrm{Hom}(TM, TM)$ is the so-called Lorenze force, namely, $\Omega := h(\cdot, Z(\cdot))$ is a closed 2-form on the target (N, h). Examples show that the convergence is also not true in general.

If N is the round 2-sphere $S^2(1)$ and $x_0, y_0 \in N$ with $d(x_0, y_0) = \pi$, then one can find initial-boundary data (σ, X_0, Y_0) such that the Dirac-geodesic flow (1.3) and (1.4) cannot converge to a Dirac-geodesic connecting x_0 and y_0 (see Theorem 3.3 and Remark 3.1).

This means that in general one cannot expect the convergence of the global solution of the Dirac-geodesic heat flow (1.3) and (1.4).

A natural problem is then to study the asymptotic behavior of the above global solution. Notice that if N is a Riemann surface, then $X^\flat \wedge Y^\flat = c\omega_\gamma$ for some constant c under the boundary conditions (see Remark 4.1), where $X^\flat$ denotes the 1-form dual to the vector field X and ω is the volume form of N. This special property in the surface case is useful for estimating the kinetic energy, but it does not hold in general in higher dimensions. We will prove the following:

Theorem 1.2 *Let N^2 be a surface with negative Gauss curvature κ. If*

$$|c| < \frac{2\pi}{\sqrt{\kappa^2 + 4\left\|\nabla\sqrt{-\kappa}\right\|^2} - \kappa}, \tag{1.5}$$

then the kinetic energy density $k(\gamma) = \frac{1}{2}\left\|\gamma'\right\|^2$ decays exponentially, i.e.,

$$k(\gamma(s, t-1)) \le Ce^{\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2 + 2\pi|c\kappa| - 2\pi^2\right)t}\int_0^1 k(\sigma)\mathrm{d}s, \quad \forall t > 1,$$

where C is a positive constant dependent only on the geometry of N.

Remark 1.1 We note that it follows from (1.5) that

$$c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2 + \pi|c\kappa| - \pi^2 < 0.$$

The rest of the paper is organized as follows: in Sect. 2 we derive the Euler-Lagrange equations of the function L; in Sect. 3, we discuss Dirac-geodesics on surfaces and classify Dirac-geodesics on the standard 2-sphere $S^2(1)$ (Theorem 3.3) and the hyperbolic plane $\mathbb{H}^2$ (Theorem 3.5), and derive existence results on topological spheres (Theorem 3.4) and hyperbolic surfaces (Theorem 3.6). These solutions constitute new examples of nontrivially coupled Dirac-harmonic maps; see [14] for an explicit example of coupled Dirac-harmonic map from surfaces and [1,5] for constructions and existence of uncoupled Dirac-harmonic maps (in the sense that the map part is an ordinary harmonic map) from surfaces and high dimensional manifolds; in Sect. 4, we prove the global existence of the Dirac-geodesic flow (Theorem 1.1) and the asymptotic property of the solution (Theorem 1.2).

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2 Preliminaries

2.1 Spin bundle $\Sigma\mathbb{R}$

First, let us recall some basic notions from spin geometry. We refer to [10,11,13,16] for additional references. Consider the real line $\mathbb{R}$ with the standard metric and let $\frac{\mathrm{d}}{\mathrm{d}r}$ be the unit tangent vector. The Clifford bundle $\mathrm{Cl}(\mathbb{R})$ is the quotient bundle

$$\mathrm{Cl}(\mathbb{R}) = \sum_{k=0}^{\infty} \otimes^k \mathbb{R}/I(\mathbb{R})$$

where $I(\mathbb{R})$ is the bundle of ideals, i.e., the bundle whose fibre at $r \in \mathbb{R}$ is the two-sided $I(T_r\mathbb{R})$ in $\sum_{k=0}^{\infty} \otimes^k \mathbb{R}$ generated by elements $v \otimes v + \|v\|^2$ for $v \in T_r\mathbb{R}$. It is easy to check that $\mathrm{Cl}(\mathbb{R}) = \mathbb{R} \times \mathbb{C}$, i.e., a trivial bundle with fibre the complex line. Obviously, the principal SO-bundle $P_{\mathrm{SO}}(\mathbb{R})$ of $\mathbb{R}$ is just the real line $\mathbb{R}$, and the principal Spin-bundle of $\mathbb{R}$ becomes to $\mathbb{R} \times \mathbb{Z}_2$. By definition, a spin structure on $\mathbb{R}$ is a lift of $P_{\mathrm{SO}}(\mathbb{R})$ to $P_{\mathrm{Spin}}(\mathbb{R})$. Thus, there are two spin structures on $\mathbb{R}$, the trivial one and the non-trivial one. However, these two spin structures are equivalent to each other.

Notice that $\mathrm{Cl}_1 \cong \mathrm{Cl}_2^0$ (the even parts of Cl_2) via the correspondence $\mathrm{Cl}_1 \ni x = x^0 + x^1 \mapsto x^0 + e_2 \cdot x^1 \in \mathrm{Cl}_2$, where x^0 and x^1 are the even parts and odd parts of x respectively. Identify $\mathbb{R}$ as a subspace of $\mathbb{R}^2$ via the canonical inclusion $\mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{R}^2$. It is well known that Cl_2 is isomorphic to the 2×2-matrix algebra over $\mathbb{C}$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad e_1 \cdot e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Introduce the spinor space $\Delta_2 := \mathbb{C}^2$ and the chiral operator $G := \sqrt{-1}e_1 \cdot e_2$, then Cl_2 acts on the spinor space. Moreover, this chiral operator splits Δ_2 into $\pm$-eigenspaces $\Delta_2^{\pm}$. It is easy to see that $\Delta_2^+ = \mathbb{C}\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix} \cong \mathbb{C}$ and $\Delta_2^- = \mathbb{C}\begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix} \cong \mathbb{C}$. Thus, we get two representation spaces of Cl_1, i.e., $\Delta_2^{\pm}$, and in particular, of Spin_1. Moreover, as a representation of Spin_1, $\Delta_2^{\pm}$ are equivalent to each other. This Δ_2^+ is the spinor space of Spin_1 and we write $\mathcal{S} = \Delta_2^+$. The associated bundle of $P_{\mathrm{Spin}}(\mathbb{R})$ via the representation of Spin_1 is called the spinor bundle and is denoted by $\Sigma\mathbb{R} \cong \mathbb{R} \times \mathcal{S}$. By this convention, we know that the Clifford product on spinors is given through $\mathrm{Cl}(\mathbb{R}) \ni \frac{\mathrm{d}}{\mathrm{d}r} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\Delta_2^+ \cong \mathbb{C}$, this Clifford product is simply given by the complex multiplication by $\sqrt{-1}$.

The connection on the spinor bundle $\Sigma\mathbb{R}$ is the canonical lift of the Levi-Civita connection $\frac{\mathrm{d}}{\mathrm{d}r}$ on $T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ to $\Sigma\mathbb{R} \cong \mathbb{R} \times \mathbb{C}$. The Dirac operator then is $\partial\!\!\!/ = \sqrt{-1}\frac{\mathrm{d}}{\mathrm{d}r}$.

2.2 Dirac-geodesics on Riemannian manifolds

Let N be a Riemannian manifold, and $\gamma : [0, 1] \longrightarrow N$ be a curve and $\Psi \in \Gamma\left(\Sigma[0, 1] \otimes \gamma^{-1}TN\right)$ be a spinor along the curve γ. We identify the spinor Ψ as a complex vector field along the curve γ and introduce $\Psi = X + \sqrt{-1}Y$ where X, Y are two vector fields along the curve γ. By (1.1), the Dirac-harmonic map (γ, Ψ) satisfies the following system

$$\begin{cases} \tau(\gamma) = \mathcal{R}(\gamma, \Psi), \\ D\!\!\!/\Psi = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 (2.1) *is equivalent to the following system*

$$\begin{cases} \nabla_{\dot{\gamma}}\dot{\gamma} + R(X,Y)\dot{\gamma} = 0, \\ \nabla_{\dot{\gamma}} X = 0, \\ \nabla_{\dot{\gamma}} Y = 0, \end{cases} \tag{2.2}$$

where $\dot{\gamma}$ denotes the tangent vector field of γ.

Proof Choose a local orthonormal frame fields $\{e_i\}$ of N and denote the unit tangent vector field over $[0, 1]$ by ∂_t, then a direct computation implies that

$$\begin{aligned} \tau(\gamma) - \mathcal{R}(\gamma, \Psi) &= \nabla_{\dot{\gamma}}\dot{\gamma} - \frac{1}{2}\left\langle \Psi^i, \partial_t \cdot \Psi^j \right\rangle R(e_i, e_j)\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} - \frac{1}{2}\left\langle \Psi^i, \sqrt{-1}\Psi^j \right\rangle R(e_i, e_j)\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\dot{\gamma} + \frac{\sqrt{-1}}{2}\Psi^i \bar{\Psi}^j R(e_i, e_j)\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\dot{\gamma} + \frac{\sqrt{-1}}{2}(X^i + \sqrt{-1}Y^i)(X^j - \sqrt{-1}Y^j)R(e_i, e_j)\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\dot{\gamma} + \frac{\sqrt{-1}}{2}\left(\left(X^i X^j + Y^i Y^j\right) + \sqrt{-1}\left(Y^i X^j - X^i Y^j\right)\right) R(e_i, e_j)\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\dot{\gamma} + R(X, Y)\dot{\gamma}, \end{aligned}$$

and

$$\not{D}\Psi = \partial_t \cdot \nabla_{\partial t}\Psi = \sqrt{-1}\nabla_{\dot{\gamma}}(X + \sqrt{-1}Y) = \sqrt{-1}\nabla_{\dot{\gamma}}X - \nabla_{\dot{\gamma}}Y.$$

□

Definition 2.1 A Dirac-harmonic map (γ, X, Y) as in (2.2) is called a Dirac-geodesic on N. We say that (γ, X, Y) is closed if γ is closed.

Remark 2.1 By a "closed" Dirac-geodesic, we mean that the curve is closed, but the spinor need not close up on S^1. On the other hand, it is also interesting to consider closed Dirac-geodesics defined on S^1, which can be equipped with two different spin structures.

Lemma 2.2 *If (γ, X, Y) is a Dirac-geodesic, then $\|\dot{\gamma}\|$, $\|X\|$, $\|Y\|$, $\langle X, Y\rangle$ are all constant along γ.*

Proof Since X and Y are parallel vector fields along the curve γ, it follows that $\|X\|$, $\|Y\|$ and $\langle X, Y\rangle$ are all constant. On the other hand,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\gamma}\|^2 = \left\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma} \right\rangle = -\langle R(X, Y)\dot{\gamma}, \dot{\gamma}\rangle = 0,$$

which implies that $\|\dot{\gamma}\|$ is a constant. □

Remark 2.2 Suppose $(\tilde{\gamma}, \tilde{X}, \tilde{Y})$ is a Dirac-geodesic defined in $(0, 1)$ with $\left\|\dot{\tilde{\gamma}}\right\| = \varepsilon^{-1} > 0$. Define $\gamma(t) = \tilde{\gamma}(\varepsilon t)$ and $\Psi(t) = \theta\sqrt{\varepsilon}\tilde{\Psi}(\varepsilon t)$ where $\theta \in \mathbb{C}$ is a constant with $\|\theta\| = 1$, then (γ, X, Y) is a Dirac-geodesic with unit-speed defined in $[0, \varepsilon]$.

Suppose $\sigma : [0, 1] \longrightarrow N$ is a C^1-curve so that $\sigma([0, 1])$ is bounded in N, then there exists an open neighborhood N' of $\sigma([0, 1])$ with compact closure so that N' can be (smoothly) isometrically embedded into some Euclidean space $\mathbb{R}^q$. If necessary, by choosing a smaller neighborhood, we may assume that there is a bounded tubular neighborhood $\tilde{N}$ of N' in

$\mathbb{R}^q$. Let $\pi : \tilde{N} \longrightarrow N'$ be the nearest point projection denoted by $\pi = (\pi^1, \pi^2, \ldots, \pi^q) = (\pi^A)_{1\le A\le q}$. By choosing an even smaller N', we may assume that π can be extended smoothly to the whole $\mathbb{R}^q$ so that each π^A is compactly supported. Hence, in particular, $\pi^A, \pi^A_B = \frac{\partial \pi^A}{\partial Z^B}, \pi^A_{BC} = \frac{\partial^2 \pi^A}{\partial Z^B \partial Z^C}, \pi^A_{BCD} = \frac{\partial^3 \pi^A}{\partial Z^B \partial Z^C \partial Z^D}$, etc. are bounded, where $Z = (Z^A)$ are standard coordinates of $\mathbb{R}^q$. Notice that $\mathrm{d}\pi_{N'}$ is an orthogonal projection.

The functional L can be written as

$$L(\gamma, X, Y) = \frac{1}{2}\left(\int_0^1 \left(\dot{\gamma}^A\right)^2 + \dot{X}^A Y^A - X^A \dot{Y}^A\right).$$

Next, we want to derive the Euler-Lagrange equations of L. For any smooth map $\eta : [0,1] \longrightarrow \mathbb{R}^q$ and any smooth real functions ξ^A, ζ^A on $(0,1)$, we consider the variation

$$\gamma_t = \pi(\gamma + t\eta), \quad X_t^A = \pi_N^A(\gamma_t)\left(X^B + t\xi^B\right), \quad Y_t^A = \pi_B^A(\gamma_t)\left(Y^B + t\zeta^B\right).$$

It is easy to check that

$$\gamma_0 = \gamma, \quad X_0 = X, \quad Y_0 = Y,$$
$$\left.\frac{\partial \gamma_t^A}{\partial t}\right|_{t=0} = \pi_B^A(\gamma)\eta^B,$$

and

$$\left.\frac{\partial X_t^A}{\partial t}\right|_{t=0} = \pi_B^A(\gamma)\xi^B + \pi_{BC}^A(\gamma)\pi_D^C(\gamma)X^B\eta^D,$$
$$\left.\frac{\partial Y_t^A}{\partial t}\right|_{t=0} = \pi_B^A(\gamma)\zeta^B + \pi_{BC}^A(\gamma)\pi_D^C(\gamma)Y^B\eta^D.$$

Moreover, if $\gamma \subset N$ and X, Y are two vector fields on N along the curve γ, then

$$\nu_B^A(\gamma)\dot{\gamma}^B = 0, \quad \nu_B^A(\gamma)X^B = 0, \quad \nu_B^A(\gamma)Y^B = 0,$$

where $\nu_B^A := \delta_B^A - \pi_B^A$. The following relationship will be used later:

$$\pi_B^A(\gamma)\pi_C^B(\gamma) = \pi_C^A(\gamma), \quad \pi_{BC}^A(\gamma) = \pi_{CB}^A(\gamma), \quad \pi_B^A(\gamma) = \pi_A^B(\gamma), \quad \pi_{BC}^A(\gamma)\dot{\gamma}^C = \pi_{AC}^B\dot{\gamma}^C.$$

Theorem 2.3 *Using the above notations, the Euler-Lagrange equations for L become*

$$\begin{cases} \ddot{\gamma}^A - \pi_{BC}^A\dot{\gamma}^B\dot{\gamma}^C + \left(\pi_B^A\pi_{BD}^C\pi_{EF}^C Y^D X^E - \pi_B^A\pi_{BD}^C\pi_{EF}^C X^D Y^E\right)\dot{\gamma}^F = 0, \\ \dot{X}^A - \pi_{BC}^A\dot{\gamma}^B X^C = 0, \\ \dot{Y}^A - \pi_{BC}^A\dot{\gamma}^B Y^C = 0. \end{cases}$$

Remark 2.3 Denote

$$\Omega_B^A := \left(\pi_C^A(\gamma)\pi_{BD}^C(\gamma) - \pi_{CD}^A(\gamma)\pi_B^C(\gamma)\right)\dot{\gamma}^D, \quad R_{GDE}^A := \pi_B^A\pi_{BD}^C\pi_F^G\pi_{EF}^C - \pi_B^G\pi_{BD}^C\pi_F^A\pi_{EF}^C,$$

then the Euler-Lagrange equations for L can be rewritten as

$$\begin{cases} \ddot{\gamma}^A + \Omega_B^A\dot{\gamma}^B - R_{BCD}^A(\gamma)\dot{\gamma}^B X^C Y^D = 0, \\ \dot{X}^A + \Omega_B^A X^B = 0, \\ \dot{Y}^A + \Omega_B^A Y^B = 0. \end{cases}$$

Moreover, $\Omega_B^A = -\Omega_A^B$.

Proof of Remark 2.3 First, we check that $\Omega^A_B = -\Omega^B_A$.

$$\Omega^A_B = \left(\pi^A_C \pi^C_{BD} - \pi^A_{CD}\pi^C_B\right)\dot\gamma^D = \pi^C_A \pi^B_{CD}\dot\gamma^D - \pi^C_{AD}\dot\gamma^D \pi^B_C$$
$$= -\left(\pi^B_C \pi^C_{AD} - \pi^B_{CD}\pi^C_A\right)\dot\gamma^D =: -\Omega^B_A.$$

Second,

$$\Omega^A_B\dot\gamma^B = \left(\pi^A_C \pi^C_{BD} - \pi^A_{CD}\pi^C_B\right)\dot\gamma^D\dot\gamma^B = \pi^A_C \pi^C_{BD}\dot\gamma^D\dot\gamma^B - \pi^A_{CD}\pi^C_B\dot\gamma^D\dot\gamma^B = -\pi^A_{BC}\dot\gamma^B\dot\gamma^C.$$

Here we have used $\pi^A_C(\gamma)\pi^C_{BD}(\gamma)\dot\gamma^D\dot\gamma^B = 0$. To see this identity, we begin with the identity $\pi^A_B(\gamma)\pi^B_C(\gamma) = \pi^A_C(\gamma)$, then

$$\pi^A_{BD}\pi^B_C\dot\gamma^D + \pi^A_B\pi^B_{CD}\dot\gamma^D = \pi^A_{CD}\dot\gamma^D.$$

Hence, multiplying both sides by $\dot\gamma^C$, we get that

$$\pi^A_C(\gamma)\pi^C_{BD}(\gamma)\dot\gamma^D\dot\gamma^B = 0.$$

Third, notice that $\pi^A_B(\gamma)X^B = X^A$, we have

$$\pi^A_{BC}\dot\gamma^C X^B + \pi^A_B\dot X^B = \dot X^A,$$

then multiplying both sides by $\pi^D_A(\gamma)$, we get that $\pi^A_B(\gamma)\pi^B_{CD}(\gamma)\dot\gamma^C X^D = 0$. By a similar computation,

$$\Omega^A_B X^B = -\pi^A_{BC}(\gamma)\dot\gamma^B X^C, \quad \Omega^A_B Y^B = -\pi^A_{BC}(\gamma)\dot\gamma^B Y^C.$$

Finally,

$$R^A_{GDE}\dot\gamma^G X^D Y^E = \left(\pi^A_B\pi^C_{BD}\pi^G_F\pi^C_{EF} - \pi^G_B\pi^C_{BD}\pi^A_F\pi^C_{EF}\right)\dot\gamma^G X^D Y^E$$
$$= \pi^A_B\pi^C_{BD}\dot\gamma^F\pi^C_{EF}X^D Y^E - \dot\gamma^B\pi^C_{BD}\pi^A_F\pi^C_{EF}X^D Y^E$$
$$= \left(\pi^A_B\pi^C_{BD}\pi^C_{EF}X^D Y^E - \pi^A_B\pi^C_{BD}\pi^C_{EF}Y^D X^E\right)\dot\gamma^F.$$

□

Proof of Theorem 2.3 Suppose η, ξ, ζ has compact support in (0, 1). Then

$$\left.\frac{\mathrm{d}L(\gamma_t, X_t, Y_t)}{\mathrm{d}t}\right|_{t=0} = \int_0^1 \dot\gamma'^A\dot\gamma^A + \frac{1}{2}\int_0^1\left(\dot X'^A Y^A + \dot X^A Y'^A\right) - \frac{1}{2}\int_0^1\left(X'^A\dot Y^A + X^A\dot Y'^A\right)$$
$$= \int_0^1 \frac{\partial\left(\pi^A_B\eta^B\right)}{\partial s}\dot\gamma^A + \int_0^1 \frac{\partial\left(\pi^A_B(\gamma)\xi^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)X^B\eta^D\right)}{\partial s}Y^A$$
$$\int_0^1 \dot X^A\left(\pi^A_B(\gamma)\zeta^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)Y^B\eta^D\right) - \frac{1}{2}\left.\left(X'^A Y^A + X^A Y'^A\right)\right|_0^1$$
$$= \int_0^1\left(\pi^A_B\dot\eta^B + \pi^A_{BC}\dot\gamma^C\eta^B\right)\dot\gamma^A - \int_0^1\left(\pi^A_B(\gamma)\xi^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)X^B\eta^D\right)\dot Y^A$$
$$\int_0^1 \dot X^A\left(\pi^A_B(\gamma)\zeta^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)Y^B\eta^D\right)$$
$$+ \frac{1}{2}\left.\left(\pi^A_B(\gamma)\xi^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)X^B\eta^D\right)Y^A\right|_0^1$$
$$- \frac{1}{2}\left.\left(\pi^A_B(\gamma)\zeta^B + \pi^A_{BC}(\gamma)\pi^C_D(\gamma)Y^B\eta^D\right)X^A\right|_0^1$$

$$
\begin{aligned}
&= -\int_0^1 \left(\ddot{\gamma}^A - \pi^A_{BC}\dot{\gamma}^B\dot{\gamma}^C + \left(\pi^A_B\pi^C_{BD}\pi^C_{EF}Y^DX^E - \pi^A_B\pi^C_{BD}\pi^C_{EF}X^DY^E\right)\dot{\gamma}^F\right)\eta^A \\
&\quad + \int_0^1 \left(\pi^D_{BC}\pi^C_A Y^B\left(\dot{X}^D - \pi^D_{EF}\dot{\gamma}^E X^F\right) - \pi^D_{BC}\pi^C_A X^B\left(\dot{Y}^D - \pi^D_{EF}\dot{\gamma}^E Y^F\right)\right)\eta^A \\
&\quad + \int_0^1 \left(\dot{X}^A - \pi^A_{BC}\dot{\gamma}^B X^C\right)\zeta^A - \int_0^1 \left(\dot{Y}^A - \pi^A_{BC}\dot{\gamma}^B Y^C\right)\xi^A \\
&\quad + \left(\dot{\gamma}^A\eta^A + \frac{1}{2}Y^A\xi^A - \frac{1}{2}X^A\zeta^B\right)\Big|_0^1 \\
&= -\int_0^1 \left(\ddot{\gamma}^A - \pi^A_{BC}\dot{\gamma}^B\dot{\gamma}^C + \left(\pi^A_B\pi^C_{BD}\pi^C_{EF}Y^DX^E - \pi^A_B\pi^C_{BD}\pi^C_{EF}X^DY^E\right)\dot{\gamma}^F\right)\eta^A \\
&\quad + \int_0^1 \left(\pi^D_{BC}\pi^C_A Y^B\left(\dot{X}^D - \pi^D_{EF}\dot{\gamma}^E X^F\right) - \pi^D_{BC}\pi^C_A X^B\left(\dot{Y}^D - \pi^D_{EF}\dot{\gamma}^E Y^F\right)\right)\eta^A \\
&\quad + \int_0^1 \left(\dot{X}^A - \pi^A_{BC}\dot{\gamma}^B X^C\right)\zeta^A - \int_0^1 \left(\dot{Y}^A - \pi^A_{BC}\dot{\gamma}^B Y^C\right)\xi^A.
\end{aligned}
$$

□

3 Dirac-geodesics on surfaces

Assume $\dim N = 2$, i.e., N is a surface. Put $X^\flat \wedge Y^\flat = c\omega_\gamma$, where ω is the volume form of N and c is a function of t (see Lemma 2.2). Let J_x be the rotation by $\pi/2$ in T_xN measured with the metric and the orientation chosen on N.

Lemma 3.1 *(γ, X, Y) is a Dirac-geodesic on a surface N if and only if*

$$
\begin{cases}
\nabla_{\dot{\gamma}}\dot{\gamma} = c\kappa(\gamma)J_\gamma(\dot{\gamma}), \\
\nabla_{\dot{\gamma}}X = \nabla_{\dot{\gamma}}Y = 0,
\end{cases}
$$

where c is a constant such that $X^\flat \wedge Y^\flat = c\omega_\gamma$ and κ is the Gauss curvature of N.

Proof The proof follows easily from the following identity:

$$R(X, Y)\dot{\gamma} = R(X \wedge Y)\dot{\gamma} = R(c\omega_\gamma)\dot{\gamma} = -c\kappa(\gamma)J_\gamma(\dot{\gamma}).$$

□

Recall that a curve γ satisfying $\nabla_{\dot{\gamma}}\dot{\gamma} = c\kappa(\gamma)J_\gamma(\dot{\gamma})$ is called a ($c\kappa$-)magnetic geodesic and models the motion of a charge in a magnetic field with magnetic form $c\kappa\omega$. Therefore, each Dirac-geodesic on a surface can be viewed as a $c\kappa$-magnetic geodesic coupled with two parallel tangent vector fields along the magnetic geodesic.

According to Remark 2.2, we can choose an orthonormal basis $e_1 = \dot{\gamma}$, e_2 along the curve γ. Denote

$$X(t) = a(\cos(f(t))e_1 + \sin(f(t))e_2), \tag{3.1}$$

$$Y(t) = b(\cos(f(t) + \theta)e_1 + \sin(f(t) + \theta)e_2), \tag{3.2}$$

where $a, b > 0$ and θ are three constants, and $f, g \in C^1[0, \varepsilon]$.

The following theorem gives a geometric description of Dirac-geodesics.

Theorem 3.2 *Let γ be a unit-speed curve with geodesic curvature κ_g on a surface M, a, b, θ constants with $a, b \geq 0$. If κ is the Gauss curvature of M, then (γ, X, Y) is a Dirac-geodesic if and only if*

$$\kappa_g = \kappa ab \sin\theta, \quad \dot{f} = -\kappa ab \sin\theta,$$

where X, Y are given by the formulae (3.1) and (3.2).

Proof Suppose (γ, X, Y) is a Dirac-geodesic, by Lemma 2.2, X, Y are of the form (3.1) and (3.2). By a direct computation, one gets that

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma + R(X,Y)\dot\gamma &= \nabla_{e_1}e_1 + abR(\cos(f)e_1 + \sin(f)e_2, \cos(f+\theta)e_1 + \sin(f+\theta)e_2)e_1\\
&= \langle\nabla_{e_1}e_1, e_2\rangle e_2 + ab\left(\cos(f)\sin(f+\theta) - \sin(f)\cos(f+\theta)\right)R(e_1,e_2)e_1\\
&= \langle\nabla_{e_1}e_1, e_2\rangle e_2 - ab\kappa\sin(\theta)e_2\\
&= \left(\langle\nabla_{e_1}e_1, e_2\rangle - ab\kappa\sin\theta\right)e_2,\\
\nabla_{\dot\gamma}X &= -a\sin(f)\dot{f}e_1 + a\cos(f)\nabla_{e_1}e_1 + a\cos(f)\dot{f}e_2 + a\sin(f)\nabla_{e_1}e_2\\
&= -a\sin(f)\dot{f}e_1 + a\cos(f)\langle\nabla_{e_1}e_1, e_2\rangle e_2 + a\cos(f)\dot{f}e_2\\
&\quad + a\sin(f)\langle\nabla_{e_1}e_2, e_1\rangle e_1\\
&= -a\sin(f)\dot{f}e_1 + a\cos(f)\langle\nabla_{e_1}e_1, e_2\rangle e_2 + a\cos(f)\dot{f}e_2\\
&\quad - a\sin(f)\langle\nabla_{e_1}e_1, e_2\rangle e_1\\
&= \left(-a\sin(f)\dot{f} - a\sin(f)\langle\nabla_{e_1}e_1, e_2\rangle\right)e_1\\
&\quad + \left(a\cos(f)\dot{f} + a\cos(f)\langle\nabla_{e_1}e_1, e_2\rangle\right)e_2\\
&= a\left(-\sin(f)e_1 + \cos(f)e_2\right)\left(\dot{f} + \langle\nabla_{e_1}e_1, e_2\rangle\right),
\end{aligned}$$

and

$$\nabla_{\dot\gamma}Y = b\left(-\sin(f+\theta)e_1 + \cos(f+\theta)e_2\right)\left(\dot{f} + \langle\nabla_{e_1}e_1, e_2\rangle\right).$$

Notice that $\langle\nabla_{e_1}e_1, e_2\rangle$ is just the geodesic curvature κ_g of γ in M, and we finish the proof of the necessity. The sufficiency is obvious. □

3.1 Dirac-geodesics on spheres

First, we consider the unit sphere $S^2(1)$ with the standard metric and let ω be the volume form.

Theorem 3.3 (Dirac-geodesic on the round 2-sphere) *Any Dirac-geodesic (γ, X, Y) with non-constant γ on the round sphere $S^2(1)$ locally can be defined by*

$$\gamma(s) = \left(\sqrt{1-\rho^2}\cos\left(\frac{\lambda s}{\sqrt{1-\rho^2}}\right), \sqrt{1-\rho^2}\sin\left(\frac{\lambda s}{\sqrt{1-\rho^2}}\right), \rho\right),$$

$$X(s) = a\lambda\left(-\sin\left(\frac{\lambda s}{\sqrt{1-\rho^2}} - cs + c_0\right), \cos\left(\frac{\lambda s}{\sqrt{1-\rho^2}} - cs + c_0\right), 0\right),$$

and

$$Y(s) = b\lambda\left(-\sin\left(\frac{\lambda s}{\sqrt{1-\rho^2}} - cs + \theta + c_0\right), \cos\left(\frac{\lambda s}{\sqrt{1-\rho^2}} - cs + \theta + c_0\right), 0\right)$$

where $c = ab\lambda^2 \sin\theta$ and $a, b, \lambda, \theta, c_0$ are constants. Moreover, for $p, q \in S^2(1)$ and constants $c \in \mathbb{R}$, $\lambda > 0$, there is a Dirac-geodesic (γ, X, Y) such that γ connects p, q with speed λ and the oriented area of $X + \sqrt{-1}Y$ is c if and only if the following condition is satisfied:

$$|c| \leq \lambda \cot\left(\frac{\operatorname{dist}(p,q)}{2}\right). \tag{3.3}$$

Proof Equip the sphere $S^2(1)$ with the standard metric, i.e., the pull-back of the metric in $\mathbb{R}^3$. In this case, the Dirac-geodesic equation becomes

$$\begin{cases} \ddot{\gamma} + \lambda^2 \gamma = c\gamma \times \dot{\gamma}, \\ \dot{X} + \langle X, \dot{\gamma}\rangle \gamma = 0, \\ \dot{Y} + \langle Y, \dot{\gamma}\rangle \gamma = 0, \end{cases}$$

where $\lambda = \|\dot{\gamma}\|$ is a constant. First, we claim that γ is a planar curve and a circle with radius $\lambda/\sqrt{\lambda^2 + c^2}$ and centered at $\frac{c}{\sqrt{\lambda^2+c^2}}(\gamma \times \dot{\gamma} + c\gamma)$. In fact,

$$\gamma \times \ddot{\gamma} = c\gamma \times (\gamma \times \dot{\gamma}) = c\left(\langle \gamma, \dot{\gamma}\rangle \gamma - \langle \gamma, \gamma\rangle \dot{\gamma}\right) = -c\dot{\gamma},$$

which means that $\gamma \times \dot{\gamma} + c\gamma$ is a constant since

$$\frac{\mathrm{d}}{\mathrm{d}s}(\gamma \times \dot{\gamma} + c\gamma) = \gamma \times \ddot{\gamma} + c\dot{\gamma} = 0.$$

Moreover, the length of this vector is

$$\|\gamma \times \dot{\gamma} + c\gamma\| = \sqrt{\lambda^2 + c^2}.$$

Suppose $\lambda \neq 0$, i.e., γ is not a constant. Then

$$\left\langle \gamma - \frac{c}{\lambda^2 + c^2}(\gamma \times \dot{\gamma} + c\gamma), \gamma \times \dot{\gamma} + c\gamma \right\rangle = 0.$$

Thus we have proved the claim.

Now by Lemma 2.2, we have that

$$X = a\left(\dot{\gamma}\cos(f(s)) + \gamma \times \dot{\gamma}\sin(f(s))\right),$$

and

$$Y = b\left(\dot{\gamma}\cos(f(s) + \theta) + \gamma \times \dot{\gamma}\sin(f(s) + \theta)\right),$$

where a, b, θ are constants such that $c = ab\lambda^2 \sin\theta$. A direct computation implies that

$$0 = \dot{X} + \langle X, \dot{\gamma}\rangle \gamma = a\left(-\dot{\gamma}\sin(f) + \gamma \times \dot{\gamma}\cos(f)\right)(\dot{f} + c),$$

and

$$0 = \dot{Y} + \langle Y, \dot{\gamma}\rangle \gamma = a\left(-\dot{\gamma}\sin(f + \theta) + \gamma \times \dot{\gamma}\cos(f + \theta)\right)(\dot{f} + c)$$

which implies that $f = -cs + c_0$ for some constant c_0.

For every constant c and two points $p, q \in S^2(1)$, one can check directly that there exists a Dirac-geodesic (γ, X, Y) with $X^\flat \wedge Y^\flat = c\omega_\gamma$ such that $p, q \in \gamma$ if and only if

$$\frac{|c|}{\sqrt{\lambda^2 + c^2}} \leq \cos\left(\frac{\operatorname{dist}(p,q)}{2}\right),$$

i.e.,

$$|c| \leq \lambda \cot\left(\frac{\mathrm{dist}(p,q)}{2}\right).$$

In fact, embedding S^2 into $\mathbb{R}^3$. Suppose γ centered at C and let Q be the midpoint of p and q in $\mathbb{R}^3$, then

$$|OC| \leq |OQ|.$$

This means

$$\frac{|c|}{\sqrt{\lambda^2+c^2}} = |\rho| \leq \cos\left(\frac{\mathrm{dist}(p,q)}{2}\right).$$

□

Remark 3.1 Notice that γ is just the parametrization of a circle up to orientation-preserving isometries and X, Y are two parallel vector fields along the curve γ.

The inequality (3.3) is exactly the fact the distance between p and q is less than the diameter of the the circle γ.

When the inequality (3.3) is strict, there exists only one shortest Dirac-geodesic (γ, X, Y) connecting p, q with speed $\lambda = \|\dot{\gamma}\|$ and $X^\flat \wedge Y^\flat = c\omega_\gamma$. In the case of equality. there exist exactly two shortest Dirac-geodesic (γ, X, Y) with speed λ and $X^\flat \wedge Y^\flat = c\omega_\gamma$ connecting p, q unless $c = 0$. Of course, there exist infinitely many shortest geodesics connecting the north pole and the south pole.

Hence, if $\mathrm{dist}(p,q) < \pi$, there always exist infinitely many constants λ such that (3.3) holds. In other words, there exists an infinite number of Dirac-geodesics (γ, X, Y) with $X^\flat \wedge Y^\flat = c\omega_\gamma$ such that γ connects p, q. However, if $\mathrm{dist}(p,q) = \pi$, then c must be zero and γ must be a geodesic.

Next, for topological spheres S^2, Schneider (c.f. [20]), and Rosenberg-Schneider (c.f. [19]) proved the following existence theorems for closed magnetic geodesics (solutions of $\nabla_{\dot{\gamma}}\gamma = h(\gamma)J_\gamma(\dot{\gamma})$) on S^2.

Theorem A (c.f. [19,20])

(1) *Let h be a positive smooth function on S^2, and $c > 0$ a constant. Suppose that one of the following three assumptions is satisfied:* (i) $c\left(2\pi + (\sup \kappa^-)\,\mathrm{Vol}(S^2)\right) \leq 4(\inf h) inj_{S^2}$, (ii) $\kappa > 0$ *and* $c\sqrt{\sup \kappa} \leq 2(\inf h)$, (iii) $\sup \kappa < 4 \inf \kappa$. *Then there exist at least two simple closed magnetic geodesics γ such that $\|\dot{\gamma}\| = c$.*

(2) *Suppose that S^2 has positive Gauss curvature. There exists a constant $\varepsilon > 0$ such that for all smooth functions $h : S^2 \longrightarrow \mathbb{R}$ satisfying $0 < h \leq c\varepsilon$ for some constant c, there are two embedded distinct simple closed magnetic geodesics γ with $\|\dot{\gamma}\| = c$.*

We have the following

Theorem 3.4 *Suppose the sphere S^2 has positive Gauss curvature κ. Suppose one of the following four assumptions is satisfied:* (1) $\pi \leq 2\,|c|\,(\inf \kappa) inj_{S^2}$; (2) $\sup \sqrt{\kappa} \leq 2\,|c| \inf \kappa$; (3) $\sup \kappa < 4 \inf \kappa$; (4) $|c|\,\kappa \leq \varepsilon$, *where c is some constant and $\varepsilon > 0$ is a suitable constant. Then there are at least two simple closed unit-speed Dirac-geodesics (γ, X, Y) such that $X^\flat \wedge Y^\flat = c\omega_\gamma$ where ω is the volume form of S^2.*

Proof of Theorem 3.4 It is a direct consequence of the theorem mentioned above and Lemma 3.1. In fact, in our case, $h = c\kappa$ and the speed is one. Hence, our conditions become

(1) $\pi \le 2|c|(\inf \kappa) inj_{S^2}$,
(2) $\sqrt{\sup \kappa} \le 2|c| \inf \kappa$,
(3) $\sup \kappa < 4 \inf \kappa$,
(4) for small $\varepsilon > 0$, $|c| \kappa \le \varepsilon$.

Hence there are at least two simple closed unit-speed curves γ satisfying

$$\nabla_{\dot\gamma} \dot\gamma = ck(\gamma) J_\gamma(\dot\gamma).$$

For such a γ, choose some point $x \in \gamma$. Choose two vectors $X_0, Y_0 \in T_x S^2$ with

$$X_0 \wedge Y_0 = c\omega_x$$

Define X, Y to be the parallel vector fields along γ with $X(x) = X_0, Y(x) = Y_0$. Then according to our definition, (γ, X, Y) is a Dirac-geodesic. Moreover, $X^\flat \wedge Y^\flat = c\omega_\gamma$. □

3.2 Dirac-geodesics on the hyperbolic plane

Let $\mathbb{H}^2$ be the standard hyperbolic plane with constant curvature -1, that is, the upper half plane

$$\mathbb{H}^2 = \left\{(x, y) \in \mathbb{R}^2 : y > 0\right\},$$

with the metric

$$\mathrm{d}s^2 = \frac{1}{y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).$$

Next we will derive the local representation of constant geodesic curvature curves in $\mathbb{H}^2$. Let

$$\omega_1 = \frac{\mathrm{d}x}{y}, \quad \omega_2 = \frac{\mathrm{d}y}{y}, \quad e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}.$$

Then a direct computation implies that

$$\omega_{12} = \frac{\mathrm{d}x}{y}.$$

Let $\gamma(s) = (x(s), y(s))$ be a curve in $\mathbb{H}^2$ with geodesic curvature κ_g, then

$$\dot\gamma = \dot x \frac{\partial}{\partial x} + \dot y \frac{\partial}{\partial y} =: \xi^1 e_1 + \xi^2 e_2.$$

In other words,

$$\xi^1 = \frac{\dot x}{y}, \quad \xi^2 = \frac{\dot y}{y}.$$

Now according to the definition of geodesic curvature, we get

$$\begin{cases} \dot\xi^1 = \left(\xi^1 - \kappa_g\sqrt{(\xi^1)^2 + (\xi^2)^2}\right)\xi^2, \\ \dot\xi^2 = -\left(\xi^1 - \kappa_g\sqrt{(\xi^1)^2 + (\xi^2)^2}\right)\xi^1. \end{cases}$$

Then $(\xi^1)^2 + (\xi^2)^2$ is a constant. Without loss of generality, $(\xi^1)^2 + (\xi^2)^2 = 1$. Then

$$\begin{cases} \dot\xi^1 = \left(\xi^1 - \kappa_g\right)\xi^2, \\ \dot\xi^2 = -\left(\xi^1 - \kappa_g\right)\xi^1. \end{cases}$$

Suppose now κ_g is a constant, then

(1) If $\xi^1 = \kappa_g$, then $\dot{\xi}^2 = 0$, i.e., either $y = y_0 > 0$ with $\kappa_g = \pm 1$ or $x = \frac{\kappa_g}{\sqrt{1-\kappa_g^2}} y + C$ with $|\kappa_g| < 1$.

(2) If $\xi^1 \neq \kappa_g$, then from

$$\frac{\mathrm{d}\xi^1}{\xi^1 - \kappa_g} = \xi^2 \mathrm{d}s = \frac{\mathrm{d}y}{y},$$

we get that $\xi^1 = \kappa_g + ay$ $(a \neq 0)$. By the assumption $(\xi^1)^2 + (\xi^2)^2 = 1$, we have

$$(\kappa_g + ay)^2 + \left(\frac{\dot{y}}{y}\right)^2 = 1.$$

Therefore

$$\mathrm{d}s = \frac{\mathrm{d}y}{y\sqrt{1 - (\kappa_g + ay)^2}}.$$

Setting $\kappa_g + ay = \sin t$, we know that

$$\mathrm{d}s = \frac{\mathrm{d}t}{\sin t - \kappa_g}.$$

Hence

$$\xi^2 = \frac{\dot{y}}{y} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \cos t.$$

Hence $\xi^1 = \sin t$. Then

$$x = -\frac{1}{a}\cos t + x_0.$$

Thus,

$$(x - x_0)^2 + \left(y + \frac{\kappa_g}{a}\right)^2 = \frac{1}{a^2}.$$

As a consequence, we have

Theorem 3.5 *In the hyperbolic plane $\mathbb{H}^2$, there exists a contractible closed Dirac-geodesic (γ, X, Y) with speed one and $X^\flat \wedge Y^\flat = c\omega_\gamma$ for a constant c if and only if $|c| > 1$.*

Now suppose (M, g) is a hyperbolic surface with negative Gauss curvature κ. Let $B \subset \mathbb{R}^2$ denote the open ball of radius 1 centered at $0 \in \mathbb{R}^2$. An immersion $\gamma \in C^1(\partial B, M)$ will be called oriented Alexandrov embedded, if there exists an immersion $F \in C^1(\bar{B}, M)$, such that $F|_{\partial B} = \gamma$ and F is orientation preserving in the sense that for all $x \in \partial B$ there holds

$$\left\langle DF_x(x), J_{\gamma(x)}(\dot{\gamma}(x))\right\rangle > 0.$$

Matthias Schneider proved

Theorem B (c.f. [21]) *Let M be a closed oriented surface with negative Euler characteristic $\chi(M)$ and let h be a positive function. Assume that there exists a constant $h_0 > 0$ such that*

$$h \geq \sqrt{h_0} \quad \textit{and} \quad \kappa \geq -h_0.$$

Then for every positive constant $c \in (0, 1)$, there exists an oriented Alexandrov embedded closed magnetic geodesic and the number of such closed magnetic geodesics is at least $-\chi(M)$ provided they are all non-degenerate and $\|\dot{\gamma}\| = c$.

As a direct consequence of Lemma 3.1 and the above theorem, one can get the following

Theorem 3.6 *Let (M, g) be a closed oriented surface with negative Euler characteristic $\chi(M)$ and negative Gauss curvature κ. For every constant $c \neq 0$ with*

$$h_0 \geq |\kappa| \geq \frac{\sqrt{h_0}}{|c|},$$

where $h_0 > 0$ is some constant, there exist at least $-\chi(M)$ non-degenerate and oriented Alexandrov embedded closed unit speed Dirac-geodesics (γ, X, Y) with $X^\flat \wedge Y^\flat = c\omega_\gamma$.

Proof Suppose (γ, X, Y) is a Dirac-geodesic with unit speed and $X^\flat \wedge Y^\flat = c\omega_\gamma$. Then $(\tilde{\gamma}(s) = \gamma(\lambda s), \tilde{X}(s) + \sqrt{-1}\tilde{Y}(s) = \sqrt{\lambda}X(\lambda s) + \sqrt{-1}\sqrt{\lambda}Y(\lambda s))$ is a Dirac-geodesic with speed λ and $\tilde{X} \wedge \tilde{Y} = c\lambda\omega_{\tilde{\gamma}}$. Since

$$h_0 \geq |\kappa| \geq \frac{\sqrt{h_0}}{|c|},$$

we have for $\lambda \in (0, 1)$

$$h_0 \geq |\kappa| \geq \frac{\sqrt{h_0}}{|c|\,\lambda}.$$

Then Theorem B tells us that there exist at least $-\chi(M)$ non-degenerated and oriented Alexandrov embedded closed magnetic curves with $h = c\kappa\lambda$. The rest of the proof is similar to Theorem 3.4. □

4 The Dirac-geodesic heat flow on Riemannian manifolds

In this section, we will consider the Dirac-geodesic flow on Riemannian manifolds.

For $\gamma : [0, 1] \times [0, T) \longrightarrow N$ and $X(\cdot, t)$, $Y(\cdot, t)$ vector fields along the curve $\gamma(\cdot, t)$, we consider the following system

$$\begin{cases} \gamma'^A = \ddot{\gamma}^A + \Omega^A_B \dot{\gamma}^B - R^A_{BCD}(\gamma)\dot{\gamma}^B Y^C X^D, & \text{on } (0,1) \times (0,T), \\ \dot{X}^A + \Omega^A_B X^B = 0, & \text{on } (0,1] \times [0,T), \\ \dot{Y}^A + \Omega^A_B Y^B = 0, & \text{on } (0,1] \times [0,T), \end{cases} \tag{4.1}$$

satisfying the initial conditions

$$\begin{cases} \gamma(s,0) = \sigma(s), & s \in (0,1), \\ \gamma(0,t) = x_0, \quad \gamma(1,t) = y_0, & t \in [0,T), \\ X(0,t) = X_0, & t \in [0,T), \\ Y(0,t) = Y_0, & t \in [0,T), \end{cases} \tag{4.2}$$

where x_0 and y_0 are two fixed points, and X_0, Y_0 are two fixed vectors. We observe

Lemma 4.1 *Suppose the image of γ lies in N', then the Dirac-geodesic heat flow (1.3) is equivalent to the system (4.1).*

Lemma 4.2 *Let (γ, X, Y) be a solution of the system (4.1) with the initial conditions (4.2) satisfying $\sigma \subset N'$ and $x_0, y_0 \in N'$, and $X_0 \in T_{x_0}N'$, $Y_0 \in T_{x_0}N'$. If the image of γ lies in $\tilde{N}$, then $\gamma \subset N'$ and X, Y are vector fields of N' along the curve γ for every time $0 \leq t < T$.*

Proof Denote $\rho(\gamma) = \pi(\gamma) - \gamma$, then a direct computation implies that

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\|\rho(\gamma)\|^2 &= \langle \rho' - \ddot{\rho}, \rho\rangle - \|\dot{\rho}\|^2 \\ &= \left\langle \nu^A_B(\gamma)\left(-\Omega^B_C\dot{\gamma}^C + R^B_{CDE}\dot{\gamma}^C Y^D X^E\right) - \pi^A_{BC}(\gamma)\dot{\gamma}^B\dot{\gamma}^C, \rho^A(\gamma)\right\rangle \\ &\quad - \left\|\nu^A_B(\gamma)\dot{\gamma}^B\right\|^2.\end{aligned}$$

Notice that if $\gamma \subset N'$, then

$$\left\langle \nu^A_B(\gamma)\left(-\Omega^B_C\dot{\gamma}^C + R^B_{CDE}\dot{\gamma}^C Y^D X^E\right) - \pi^A_{BC}(\gamma)\dot{\gamma}^B\dot{\gamma}^C, \rho^A(\gamma)\right\rangle.$$

Hence by using the mean value theorem, we get that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\|\rho(\gamma)\|^2 \le C\,\|\rho(\gamma)\|^2.$$

Thus, if $\sigma \subset N'$ and $x_0, y_0 \in N'$, then γ must be in N' according to the maximum principle.

On the other hand, if $\gamma \in N'$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\nu^A_B(\gamma)X^B\right) &= -\pi^A_{BC}\dot{\gamma}^C X^B + \nu^A_B\dot{X}^B = -\pi^A_{BC}\dot{\gamma}^B X^C + \nu^A_B\left(\pi^B_{DE}\pi^D_C - \pi^B_D\pi^D_{CE}\right)\dot{\gamma}^E X^C \\ &= -\pi^A_{BC}\dot{\gamma}^B X^C + \pi^A_{DE}\pi^D_C\dot{\gamma}^E X^C = -\pi^A_D\pi^D_{BC}\dot{\gamma}^B X^C.\end{aligned}$$

Moreover,

$$-\Omega^A_B\nu^B_C X^C = \left(\pi^A_{DE}\pi^D_B - \pi^A_D\pi^D_{BE}\right)\dot{\gamma}^E\nu^B_F X^F = -\pi^A_D\pi^D_{FE}\dot{\gamma}^E X^F.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\nu^A_B(\gamma)X^B\right) + \Omega^A_B\nu^B_C X^C = 0.$$

Therefore, if $X_0 \in T_{x_0}N'$, then $\nu^A_B(\gamma(0))X^B(0) = 0$ for all A and we get that $\nu^A_B X = 0$ for all A. In other words, X is a vector field along the curve γ. Similarly, Y is a vector field of N' along the curve γ. □

Now we can give the

Proof of Theorem 1.1 First, we shall use Lemmas 4.1 and 4.2 to obtain short time existence. □

Claim *A solution of the system* (4.1) *with the initial conditions* (4.2) *is equivalent to the following system of differential equations for a curve* $\gamma : (0, 1) \times [0, T) \longrightarrow \mathbb{R}^q$ *given by*

$$\gamma'^A = \ddot{\gamma}^A + \Omega^A_B(\gamma)\dot{\gamma}^B - R^A_{BCD}(\gamma)\dot{\gamma}^B Y^C X^D$$

on $(0, 1) \times (0, T)$*, satisfying the initial condition*

$$\begin{cases}\gamma(s, 0) = \sigma(s), & s \in (0, 1),\\ \gamma(0, t) = x_0, \quad \gamma(1, t) = y_0, & t \in [0, T),\end{cases}$$

where X *and* Y *are smooth vector-valued function of* $(X_0, \gamma, \dot{\gamma})$ *and* $(Y_0, \gamma, \dot{\gamma})$ *determined by*

$$\begin{cases}\dot{X}^A + \Omega^A_B X^B = 0, & \text{on } (0, 1] \times [0, T),\\ X(0, t) = X_0, & t \in [0, T),\end{cases}$$

and

$$\begin{cases} \dot{Y}^A + \Omega^A_B Y^B = 0, & \textit{on } (0,1] \times [0,T), \\ Y(0,t) = Y_0, & t \in [0,T), \end{cases}$$

respectively.

Claim (Short time existence) *A solution of the system* (1.3) *with the initial condition* (1.4) *exists at least on some short time interval* $[0, t_0)$ *for some* $t_0 > 0$ *according to Lemmas* 4.1 *and* 4.2*. Moreover, the maximum time* t_0 *is characterized by the condition*

$$\sup_{t<t_0} \|\dot{\gamma}(s,t)\| = \infty.$$

Second, we shall derive a differential equation for the energy density. As a consequence, the energy density grows at most exponentially, implying the long time existence.

Claim (long time existence) *Define the energy density* $e(\gamma)$ *of* γ *by*

$$e(\gamma) = \frac{1}{2}\|\dot{\gamma}\|^2,$$

then

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right) e(\gamma) \le \frac{\left\|X^\flat \wedge Y^\flat\right\|^2 \sup\|R\|^2}{2} e(\gamma).$$

Thus, a solution of (1.3) *and* (1.4) *exists for all time.*

Proof

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right) e(\gamma) &= \left\langle \nabla_{\gamma'}\dot{\gamma} - \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\right\rangle - \left\|\nabla_{\dot{\gamma}}\dot{\gamma}\right\|^2 = \left\langle \nabla_{\dot{\gamma}}\left(\gamma' - \nabla_{\dot{\gamma}}\dot{\gamma}\right), \dot{\gamma}\right\rangle - \left\|\nabla_{\dot{\gamma}}\dot{\gamma}\right\|^2 \\ &= \left\langle R(X,Y)\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}\right\rangle - \left\|\nabla_{\dot{\gamma}}\dot{\gamma}\right\|^2 \le \frac{\left\|X^\flat \wedge Y^\flat\right\|^2 \sup\|R\|^2}{2} e(\gamma).\end{aligned}$$

Notice that at the boundary $\partial[0,1] \times [0,T)$,

$$\frac{\partial e(\gamma)}{\partial s} = \left\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\right\rangle = \left\langle \gamma' - R(X,Y)\dot{\gamma}, \dot{\gamma}\right\rangle = \left\langle \gamma', \dot{\gamma}\right\rangle = 0.$$

Hence,

$$e(\gamma) \le \exp\left(\frac{\left\|X^\flat \wedge Y^\flat\right\|^2 \sup\|R\|^2}{2} t\right) \sup e(\sigma).$$

□

Finally, the uniqueness of this flow is obvious.

To prove Theorem 1.2, we need some preliminary lemmas.

Lemma 4.3 *Let* N^n *be a Riemannian manifold,* (γ, X, Y) *be a global solution of* (1.3) *and* (1.4)*. Then the energy of* γ *is a decreasing function of* t*, precisely,*

$$\frac{\mathrm{d}E(\gamma)}{\mathrm{d}t} = -\int_0^1 \left\|\gamma'\right\|^2.$$

Proof Notice that

$$\int_0^1 \langle \gamma', R(X,Y)\dot\gamma \rangle = \int_0^1 \langle R(\gamma', \dot\gamma)X, Y\rangle = \int_0^1 \langle \nabla_{\gamma'}\nabla_{\dot\gamma}X - \nabla_{\dot\gamma}\nabla_{\gamma'}X, Y\rangle$$
$$= -\int_0^1 \langle \nabla_{\dot\gamma}\nabla_{\gamma'}X, Y\rangle = -\langle \nabla_{\gamma'}X, Y\rangle|_0^1 + \int_0^1 \langle \nabla_{\gamma'}X, \nabla_{\dot\gamma}Y\rangle = 0.$$

As a consequence,

$$\frac{\mathrm{d}E(\gamma)}{\mathrm{d}t} = \int_0^1 \langle \nabla_{\gamma'}\dot\gamma, \dot\gamma\rangle = \int_0^1 \langle \nabla_{\dot\gamma}\gamma', \dot\gamma\rangle = -\int_0^1 \langle \gamma', \nabla_{\dot\gamma}\dot\gamma\rangle + \langle \gamma', \dot\gamma\rangle|_0^1$$
$$= -\int_0^1 \left\|\gamma'\right\|^2 + \int_0^1 \langle \gamma', R(X,Y)\dot\gamma\rangle = -\int_0^1 \left\|\gamma'\right\|^2.$$

□

Based on this lemma, we know that γ is contained in some bounded subset of N. To see this, for every $s, s' \in (0,1)$, we have

$$\mathrm{dist}(\gamma(s,t), \gamma(s',t)) \leq \left|\int_s^{s'} \|\dot\gamma\|\right| \leq \left|s-s'\right|^{1/2}\left(\int_s^{s'} \|\dot\gamma\|^2\right)^{1/2} \leq \left|s-s'\right|^{1/2}(2E(\gamma))^{1/2}$$
$$\leq \left|s-s'\right|^{1/2}(2E(\sigma))^{1/2}.$$

Hence, there exists a sequence $\gamma(\cdot, t_i)$ such that $\gamma(\cdot, t_i)$ absolutely converges to a $C^{1/2}$ curve in C^α for $0 < \alpha < 1/2$ as $t_i \to \infty$.

The kinetic energy density of γ is defined by

$$k(\gamma) = \frac{1}{2}\left\|\gamma'\right\|^2.$$

Remark 4.1 If N is a surface, then there must be a constant c such that

$$R(X,Y)\dot\gamma = R(X \wedge Y)\dot\gamma = -c\kappa^N J_\gamma(\dot\gamma).$$

To see this, first we have $X \wedge Y = c(t)\omega^N(\gamma)$ since X and Y are parallel vector fields along the curve γ. Second, at the fixed point x_0, we know that $c(t)$ does not change the value since X_0 and Y_0 are given.

Now we claim the following inequality

Lemma 4.4 *Assume that N is a Riemann surface with negative Gauss curvature κ, then for any $\varepsilon \in (0,1)$,*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)k(\gamma) \leq \left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2 + \frac{c^2\kappa^2}{2\varepsilon}\right)k(\gamma) - 2(1-\varepsilon)\left\|\nabla\sqrt{k(\gamma)}\right\|^2.$$

Proof

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)k(\gamma) = \langle \nabla_{\gamma'}\gamma' - \nabla_{\dot\gamma}\nabla_{\dot\gamma}\gamma', \gamma'\rangle - \left\|\dot\gamma'\right\|^2$$
$$= \langle \nabla_{\gamma'}\left(\gamma' - \nabla_{\dot\gamma}\dot\gamma\right), \gamma'\rangle - \left\|\dot\gamma'\right\|^2 + R(\dot\gamma, \gamma', \dot\gamma, \gamma')$$

$$= \left\langle \nabla_{\gamma'} \left(R(X \wedge Y)\dot{\gamma} \right), \gamma' \right\rangle - \left\| \dot{\gamma}' \right\|^2 + \kappa^N(\gamma) \left\| \dot{\gamma} \wedge \gamma' \right\|^2$$
$$= \left\langle (\nabla_{\gamma'} R)(X \wedge Y)\dot{\gamma} + R(X \wedge Y)\dot{\gamma}', \gamma' \right\rangle - \left\| \dot{\gamma}' \right\|^2 + \kappa^N(\gamma) \left\| \dot{\gamma} \wedge \gamma' \right\|^2.$$

Suppose now $\kappa^N < 0$, then

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2} \right) k(\gamma) \le 2\,|c| \left\| \nabla^N \sqrt{-\kappa} \right\| \left\| \gamma' \right\| \sqrt{-\kappa} \left\| \dot{\gamma} \wedge \gamma' \right\| - |c|\,\kappa \left\| \dot{\gamma}' \right\| \left\| \gamma' \right\|$$
$$- \left\| \dot{\gamma}' \right\|^2 + \kappa \left\| \dot{\gamma} \wedge \gamma' \right\|^2$$
$$\le \left(2c^2 \left\| \nabla^N \sqrt{-\kappa} \right\|^2 + \frac{c^2 \kappa^2}{2\varepsilon} \right) k(\gamma) - (1-\varepsilon) \left\| \dot{\gamma}' \right\|^2.$$

Noting that

$$\|\nabla k(\gamma)\|^2 = \left\langle \dot{\gamma}', \gamma' \right\rangle^2 \le 2 \left\| \dot{\gamma}' \right\|^2 k(\gamma),$$

namely,

$$\left\| \nabla \sqrt{k(\gamma)} \right\|^2 \le \frac{1}{2} \left\| \dot{\gamma}' \right\|^2,$$

and substituting this into the above inequality, we get the desired conclusion. □

We recall the Poincaré's inequality

$$\pi^2 \int_0^1 \|f\|^2 \le \int_0^1 \|\dot{f}\|^2$$

for smooth functions f with $f(0) = f(1) = 0$. Now we can give the

Proof of Theorem 1.2 Denote

$$C = 2c^2 \left\| \nabla^N \sqrt{-\kappa} \right\|^2 + \frac{c^2 \kappa^2}{2\varepsilon},$$

then we have

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 e^{-Ct} k(\gamma) \mathrm{d}s + 2(1-\varepsilon) \int_0^1 \left\| \nabla \sqrt{e^{-Ct/2} k(\gamma)} \right\|^2 \mathrm{d}s$$
$$\ge \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 e^{-Ct} k(\gamma) \mathrm{d}s + 2(1-\varepsilon)\pi^2 \int_0^1 e^{-Ct} k(\gamma) \mathrm{d}s.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(e^{(2(1-\varepsilon)\pi^2 - C)t} \int_0^1 k(\gamma) \mathrm{d}s \right) \le 0.$$

Therefore, if

$$2c^2 \left\| \nabla^N \sqrt{-\kappa} \right\|^2 + \frac{c^2 \kappa^2}{2\varepsilon} < 2(1-\varepsilon)\pi^2$$

for some $\varepsilon \in (0, 1)$, then the kinetic energy of γ decays exponentially. Obviously, $|c\kappa| < 2\pi$, hence we can choose

$$\varepsilon = \frac{|c\kappa|}{2\pi} \in (0, 1).$$

That is, if we make the assumption

$$c^2 \left\| \nabla^N \sqrt{-\kappa} \right\|^2 + \pi\,|c\kappa| < \pi^2,$$

or equivalently the assumption (1.5), then

$$\int_0^1 k(\gamma)\mathrm{d}s \le e^{\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+2\pi|c\kappa|-2\pi^2\right)t}\int_0^1 k(\sigma)\mathrm{d}s. \tag{4.3}$$

Let $h(x, y, t)$ be the Dirichlet heat kernel of [0, 1]. Applying the differential inequality of $k(\gamma)$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2}\right)k(\gamma) \le \left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+\pi|c\kappa|\right)k(\gamma)$$

we get that

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2}\right)\left(e^{-\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+\pi|c\kappa|\right)t}k(\gamma)\right) \le 0.$$

For every $\tau > 1$, denote $F(s,t) = e^{-\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+\pi|c\kappa|\right)t}k(\gamma(s, t+\tau-1))$, then

$$\begin{aligned} F(s,1) &\le \int_0^1 h(s,x,1)F(x,0)\,\mathrm{d}x \\ &\le \int_0^1 h(s,x,1)k(\gamma(x,\tau-1))\,\mathrm{d}x \\ &\le C\int_0^1 k(\gamma(x,\tau-1))\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

With Lemma 4.3, and (4.3) and (4.4), we have

$$\begin{aligned} k(\gamma(s,\tau-1)) &\le Ce^{2\pi^2-2\pi|c\kappa|}e^{\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+2\pi|c\kappa|-2\pi^2\right)\tau}\int_0^1 k(\sigma)\mathrm{d}s \\ &\le Ce^{\left(2c^2\left\|\nabla^N\sqrt{-\kappa}\right\|^2+2\pi|c\kappa|-2\pi^2\right)\tau}\int_0^1 k(\sigma)\mathrm{d}s. \end{aligned}$$

□

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