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Gradient estimates and Liouville theorems for Dirac-harmonic maps

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ABSTRACT

In this paper, we derive gradient estimates for Dirac-harmonic maps from complete Riemannian spin manifolds into regular balls in Riemannian manifolds. With these estimates, we can prove Liouville theorems for Dirac-harmonic maps under curvature or energy conditions.

1. Introduction

Dirac-harmonic maps have been introduced in [1,2]. They couple a harmonic map type field with a spinor field [3]. This model originated in the supersymmetric σ-model of quantum field theory, the only difference being that in the supersymmetric σ-model the (anticommuting) spinor fields take values in a Grassmannian algebra, making the model supersymmetric, while in Dirac-harmonic maps, the spinors are commuting as in spin geometry, keeping the model within the category of the geometric calculus of variations.

Let us recall the terminology and setting for Dirac-harmonic maps. Let (M^m, g) be a Riemannian spin manifold of dimension $m \geq 2$ with a fixed spin structure, and ΣM the spinor bundle over M, on which we chose a Hermitian metric $\langle \cdot, \cdot \rangle$. The Levi-Civita connection ∇ on ΣM is compatible with $\langle \cdot, \cdot \rangle$. Let (N^n, h) be a Riemannian manifold of dimension n, Φ a map from M to N, and $\Phi^{-1}TN$ the pull-back bundle of TN by Φ. On the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$ there is a metric (still denoted by $\langle \cdot, \cdot \rangle$) induced from the metrics on ΣM and $\Phi^{-1}TN$. There is also a connection, still denoted by ∇, on $\Sigma M \otimes \Phi^{-1}TN$ naturally induced from those on ΣM and $\Phi^{-1}TN$.

Locally, we can write a cross-section Ψ of $\Sigma M \otimes \Phi^{-1}TN$ as $\Psi = \psi^\alpha \otimes \theta_\alpha$, where $\{\psi^\alpha\}$ are local cross-sections of ΣM, $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$. Here and in the sequel, we use the usual summation convention.

The *Dirac operator along the map* Φ is defined as

$$\begin{aligned} \not{D}\Psi &:= e_i \cdot \nabla_{e_i}\Psi \\ &= \not{\partial}\psi^\alpha \otimes \theta_\alpha + \psi^\alpha \otimes \nabla_{e_i}\theta_\alpha, \end{aligned}$$

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where $\{e_i\}$ is a local orthonormal basis on M, $\partial\!\!\!/ := e_i \cdot \nabla_{e_i}$ is the usual Dirac operator on M and "$X\cdot$" stands for the Clifford multiplication by the vector field X on M.

Consider the functional

$$L(\Phi, \Psi) = \frac{1}{2}\int_M \left(\|\mathrm{d}\Phi\|^2 + \langle \Psi, D\!\!\!/\Psi\rangle\right).$$

The critical points (Φ, Ψ) satisfy the Euler–Lagrange equations for $L(\Phi, \Psi)$ are (cf. [1])

$$\begin{cases} \tau(\Phi) = \frac{1}{2}\langle \psi^\alpha, e_i \cdot \psi^\beta\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i), \\ D\!\!\!/\Psi = 0, \end{cases} \tag{1.1}$$

where $R^N(X, Y) := [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$, $\forall X, Y \in \Gamma(TN)$ stands for the curvature operator of N, and $\tau(\Phi) := (\nabla^{T^*M\otimes\Phi^{-1}TN}_{e_i} \mathrm{d}\Phi)(e_i)$ is the tension field of Φ. Therefore, solutions of (1.1) are called *Dirac-harmonic maps from M to N*.

Dirac-harmonic maps have been investigated under various aspects, see the recent article [4] and the references therein. In [4], a maximum principle of Jäger–Kaul type [5] was established for Dirac-harmonic maps from compact Riemannian spin manifolds with mean convex boundaries and positive scalar curvatures into certain geodesic balls of the target manifolds, based on which a general existence and uniqueness theorem for boundary value problems was proved through the continuity method. Most recently, the space of Dirac-harmonic maps was analyzed by B. Ammann and N. Ginoux in [6] by using tools from index theory, and the existence of uncoupled solutions (i.e., Φ is a harmonic map) was proved.

Most of the previous works deal with Dirac-harmonic maps from compact manifolds. It is the main aim of the present paper to derive properties of Dirac-harmonic maps on complete noncompact manifolds M.

In the classical works of S.T. Yau [7] and others on harmonic functions on noncompact manifolds, the gradient estimate method plays a key role. On one hand, these estimates may directly give rise to Liouville type results; on the other hand, they may also lead to fundamental analytic properties such as Harnack inequalities, and furthermore, they are very useful for establishing existence results. This method has been extended to the case of harmonic maps. In [8], S.Y. Cheng established gradient estimates and derived the Liouville theorem for harmonic maps from a noncompact manifold M into a nonpositively curved manifold N. In [9] H.I. Choi proved a similar result for harmonic maps into a *regular ball*, namely, a geodesic ball $B_{y_0}(R)$ with radius R that lies within the cut locus of its center $y_0 \in N$ and satisfies $R < \pi/2\sqrt{K_N}$, where the sectional curvature of N is bounded above by $K_N > 0$. The gradient estimates turn out to be a powerful tool for proving existence results of harmonic maps and their heat flows on noncompact manifolds. For example, in [10], J.Y. Li used it to improve the result of P. Li and L.F. Tam [11] with a different method.

In this paper, we will first derive a gradient estimate for Dirac-harmonic maps from complete Riemannian spin manifolds into regular balls in the target manifolds, which generalizes the result for harmonic maps in [9]. As an application, we then prove a Liouville theorem for Dirac-harmonic maps under curvature conditions. We also obtain Liouville theorems under energy conditions.

When the target has nonpositive curvature, the size of the target ball is arbitrary (topological issues can be avoided by lifting to universal covers). In the presence of positive target curvature, however, we know since [12] that a restriction on the radius of the target ball is needed in order to obtain estimates. The optimal size of such a ball corresponds to an open hemisphere in the case of the standard sphere, as shown in [12]. Remarkably, we can achieve the same optimal condition on the radius $R < \pi/2\sqrt{K_N}$ as in [9] for Dirac-harmonic maps as in the original work for harmonic maps.

We can now state our gradient estimate.

Theorem 1 (*Gradient Estimate*)**.** *Suppose the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq -\kappa$ for some nonnegative constant κ, the sectional curvature sec_N and the curvature tensor R^N of N satisfy $-b_2 \leq \mathrm{sec}_N \leq b_1$ and $\left\|\nabla R^N\right\| \leq b_3$ respectively, where b_i are constants with $b_2 \geq b_1 > 0$, $b_3 \geq 0$. Denote*

$$b = b_2^3 + b_2^4 + b_3^2.$$

If (Φ, Ψ) is Dirac-harmonic and $\Phi : M^m \longrightarrow B_{y_0}(R) \subset N^n$, $R < \pi/(2\sqrt{b_1})$, then, for any $x_0 \in M$ and any positive constant a, we have

$$\sup_{B_{x_0}(a/2)} \|\mathrm{d}\Phi\| \leq \frac{C(m, n)}{\sqrt{b_1}\cos^2(\sqrt{b_1}R)}\left(\frac{1 + \sqrt{\kappa}a}{a} + \sqrt{\frac{b}{b_1}\sup_{B_{x_0}(a)}\|\Psi\|^2}\right), \tag{1.2}$$

where $C(m, n) > 0$ is a constant depending only on the dimensions m and n.

Remark 1. Under the hypothesis of Theorem 1, if Φ is a harmonic map and we choose $\Psi \equiv 0$, then in fact we can obtain the following global estimate for $\mathrm{d}\Phi$:

$$\sup_M \|\mathrm{d}\Phi\| \leq \frac{\sqrt{\min\{m, n\}\,\kappa}}{\sqrt{b_1}\cos(\sqrt{b_1}R)}.$$

As the upper bound in our estimate is given by an explicit expression in terms of the geometric quantities involved such as the bounds on the curvatures of M and N or the radius of the regular ball, it becomes clear that and how these geometric quantities control the behavior of the map.

We can then apply this gradient estimate to obtain a Liouville type theorem for Dirac-harmonic maps.

Theorem 2 (*Liouville Theorem*)**.** *Assume that M is complete with nonnegative Ricci curvature and the scalar curvature is bounded below by a positive constant ϵ, suppose the sectional curvature $\sec_N$ and curvature tensor R^N of N satisfy $-b_2 \le \sec_N \le b_1$ and $\left\|\nabla R^N\right\| \le b_3$ respectively, where b_i are constants with $b_2 \ge b_1 > 0$, $b_3 \ge 0$. Then there is a constant $\delta > 0$ such that for any Dirac-harmonic (Φ, Ψ) satisfying $\Phi(M) \subset B_{y_0}(R) \subset N$, $R < \pi/(2\sqrt{b_1})$ and $\|\Psi\| < \delta$, we have $\Phi \equiv$ constant and $\Psi \equiv 0$.*

Remark 2. (1) The constant δ can be chosen as

$$\delta = \frac{C(m,n)(b_1^3\epsilon)^{1/4}\cos(\sqrt{b_1}R)}{\sqrt{b}},$$

for some suitable constant $C(m, n) > 0$, where $b = b_2^3 + b_2^4 + b_3^2$.

(2) The condition on the positive lower bound for the scalar curvature of M cannot be removed. For instance, there are Dirac-harmonic maps $(\Phi_0, \Psi) : \mathbb{R}^m \to N$, with Φ_0 constant maps and the components of Ψ nontrivial harmonic spinors, namely, $\partial\!\!\!/\psi^\alpha \equiv 0, \alpha = 1, \dots, n$.

When the domain manifold M is compact, H.C. Sealey studied harmonic maps with small energy in [13], and derived some Liouville theorems for such harmonic maps. For Dirac-harmonic maps, we also have the following Liouville theorem which includes the result of Sealey by letting $\Psi \equiv 0$.

Theorem 3 (*Liouville Theorem*)**.** *Let M be a compact spin manifold. Suppose $\mathrm{Ric}_M \ge a$ for some positive constant a, and $-b_2 \le \sec_N \le b_1$ for some positive constants b_1, b_2 such that $b_2 \ge b_1$. Let (Φ, Ψ) be a Dirac-harmonic map such that $\max\operatorname{rank}\Phi \le q$. If for some $\delta > 0$,*

$$\frac{q-1}{q} b_1 \|\mathrm{d}\Phi\|^2 + \frac{m-1+\delta}{4(m+\delta)} (n-1)^2 q b_2^2 \|\Psi\|^4 \le a, \tag{1.3}$$

and the equality is not valid at least at one point, then Φ must be constant and $\Psi \equiv 0$.

In particular, if

$$b_1 \|\mathrm{d}\Phi\|^2 + \frac{\min\{m,n\}}{4} (n-1)^2 b_2^2 \|\Psi\|^4 \le a, \tag{1.4}$$

then Φ must be constant and $\Psi \equiv 0$.

For complete noncompact manifolds M, we can prove the following Liouville theorem for Dirac-harmonic maps under an energy hypothesis, which extends a result of R. Schoen and S.T. Yau in [14].

Theorem 4 (*Liouville Theorem*)**.** *Let M be a complete noncompact spin manifold. Suppose the Ricci curvature of M is bounded below by a nonnegative function a, the sectional curvature of N is bounded above by a nonnegative function b_1 and bounded below by a nonpositive function $-b_2, b_2 \ge b_1$. Let (Φ, Ψ) be a Dirac-harmonic map such that $\max\operatorname{rank}\Phi \le q$. If for some constant $\delta \in (0, 1)$,*

$$\frac{q-1}{q} b_1 \|\mathrm{d}\Phi\|^2 + \frac{1+\delta}{4\delta} (n-1)^2 q b_2^2 \|\Psi\|^4 \le a, \tag{1.5}$$

and

$$\int_M \|\mathrm{d}\Phi\|^2 + \|\Psi\|^4 < \infty, \tag{1.6}$$

then Φ must be constant and $\Psi \equiv 0$.

The paper is organized as follows: in Section 2, we establish basic estimates for Dirac-harmonic maps including Kato–Yau inequalities and give the proof of Theorem 1. In Section 3 we prove Liouville theorems for Dirac-harmonic maps, Theorems 2–4.

2. Gradient estimates for Dirac-harmonic maps

2.1. Preliminaries

We first recall the following Weitzenböck formula [15].

Proposition 1. *For a smooth map $\Phi : M \to N$,*

$$\frac{1}{2}\Delta \|\mathrm{d}\Phi\|^2 - \mathrm{div}(\langle\tau(\Phi), \Phi_*(e_i)\rangle e_i) = \|\nabla \mathrm{d}\Phi\|^2 - \|\tau(\Phi)\|^2 + \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle - R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)), \tag{2.1}$$

where $\{e_i\}$ is a local orthonormal frame on M. □

It follows from (2.1) that

$$\frac{1}{2}\Delta \|\mathrm{d}\Phi\|^2 = \|\nabla \mathrm{d}\Phi\|^2 + \langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle + \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle - R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)). \tag{2.2}$$

By direct computations, one has the following.

Lemma 1. *If (Φ, Ψ) is Dirac-harmonic, then*

$$\begin{aligned}\langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle = &-\frac{1}{2}\langle \Psi, e_i\cdot\psi^\alpha \otimes (\nabla_{\Phi_*(e_j)}R^N)(\Phi_*(e_i), \Phi_*(e_j))\theta_\alpha\rangle \\ &+\frac{1}{2}\langle \Psi, e_j\cdot\psi^\alpha \otimes R^N(\Phi_*(e_i), (\nabla_{e_i}\mathrm{d}\Phi)(e_j))\theta_\alpha\rangle \\ &-\langle \nabla_{e_j}\Psi, e_i\cdot\psi^\alpha \otimes R^N(\Phi_*(e_i), \Phi_*(e_j))\theta_\alpha\rangle. \quad \square\end{aligned} \tag{2.3}$$

It is then easy to derive the following estimates:

$$\|\tau(\Phi)\| \le \frac{1}{2}(n-1)\sqrt{\min\{m,n\}}\left\|R^N\right\| \|\Psi\|^2 \|\mathrm{d}\Phi\|, \tag{2.4}$$

and

$$\begin{aligned}\left|\langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle\right| \le &\frac{1}{2}(n-1)(\min\{m,n\}-1)\left\|\nabla R^N\right\| \|\Psi\|^2 \|\mathrm{d}\Phi\|^3 \\ &+\frac{1}{2}(n-1)\sqrt{\min\{m,n\}}\left\|R^N\right\| \|\Psi\|^2 \|\mathrm{d}\Phi\| \|\nabla \mathrm{d}\Phi\| \\ &+(\min\{m,n\}-1)\sqrt{n}\left\|R^N\right\| \|\Psi\| \|\nabla\Psi\| \|\mathrm{d}\Phi\|^2.\end{aligned} \tag{2.5}$$

In fact, firstly choose $\{e_i\}$ such that $\langle \Phi_*(e_i), \Phi_*(e_j)\rangle = \lambda_i^2\delta_{ij}$ where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_q > 0 = \lambda_{q+1} = \cdots = \lambda_m$. Secondly, choose θ_α be a local orthonormal frame on N along the map Φ, such that $\Phi_*(e_i) = \lambda_i\theta_i$ for $i = 1, 2, \ldots, q$, then $\|\Psi\|^2 = \sum_\alpha \|\psi^\alpha\|^2$. By the definition of Dirac-harmonic map (1.1), one can get that

$$\begin{aligned}\|\tau(\Phi)\| &= \frac{1}{2}\left|\sum_{i,\alpha,\beta}\langle\psi^\alpha, e_i\cdot\psi^\beta\rangle R^N(\theta_\alpha,\theta_\beta)\Phi_*(e_i)\right| \\ &\le \frac{1}{2}\left|\sum_{0<i\le q,\alpha\ne\beta}\|\psi^\alpha\|\left\|\psi^\beta\right\|\left\|R^N\right\|\lambda_i\right| \\ &\le \frac{1}{2}q(n-1)\left\|R^N\right\| \|\mathrm{d}\Phi\|\sum_\alpha\|\psi^\alpha\|^2 \\ &\le \frac{1}{2}(n-1)\sqrt{\min\{m,n\}}\left\|R^N\right\| \|\Psi\|^2 \|\mathrm{d}\Phi\|.\end{aligned}$$

Similarly, one can get estimate (2.5).

In order to estimate $\|\nabla \mathrm{d}\Phi\|^2$ and $\|\nabla\Psi\|^2$, we need to establish some Kato–Yau inequalities. We first recall that for any Riemannian vector bundle E and any cross-section Ψ of E,

$$\|\nabla\Psi\| \ge \|\nabla\|\Psi\|\|, \tag{2.6}$$

provided that $\Psi \ne 0$. We can prove the following Kato–Yau inequalities for Dirac-harmonic maps which generalize both the result for harmonic maps in [14] and the result for harmonic spinors in [16].

Proposition 2 (*Kato–Yau Inequalities*)**.** *Let E be any Dirac bundle on M with dimension m. Then for any cross-section $\Psi \in \Gamma(E)$ and $\delta > 0$, we have*

$$\|\nabla\Psi\|^2 \ge \left(1+\frac{1}{m-1+\delta}\right)\|\nabla\|\Psi\|\|^2 - \frac{1}{\delta}\|\not{D}\Psi\|^2, \tag{2.7}$$

provided that $\Psi \neq 0$. *More generally, for any* $\epsilon \geq 1/m$, *we have*

$$\|\nabla\Psi\|^2 \geq \frac{1-\epsilon}{m-1}\,\|\not{D}\Psi\|^2 + \left(1+\frac{1-1/\epsilon}{m-1}\right)\|\nabla\,\|\Psi\|\,\|^2\,, \tag{2.8}$$

provided $\Psi \neq 0$.

In particular, when (Φ,Ψ) *is a Dirac-harmonic map, we have*

$$\|\nabla \mathrm{d}\Phi\|^2 \geq \left(1+\frac{1}{m-1+\delta}\right)\|\nabla\ \|\mathrm{d}\Phi\|\,\|^2 - \frac{1}{4\delta}(n-1)^2\min\{m,n\}\left\|R^N\right\|^2\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\,, \tag{2.9}$$

and

$$\|\nabla\Psi\|^2 \geq \left(1+\frac{1}{m-1}\right)\|\nabla\,\|\Psi\|\,\|^2\,, \tag{2.10}$$

provided $\mathrm{d}\Phi \neq 0$ *and* $\Psi \neq 0$.

Proof. If $\nabla\ \|\Psi\| = 0$, (2.7) is obvious and (2.8) holds since

$$\|\nabla\Psi\|^2 \geq \frac{1}{m}\,\|\not{D}\Psi\|^2\,,$$

and $\epsilon \geq 1/m$. Now suppose that $\Psi \neq 0$ and $\nabla\ \|\Psi\| \neq 0$ at the considered point. Then we can choose an orthonormal frame $\{e_i\}$ such that at the considered point

$$e_1 = \|\nabla\,\|\Psi\|\,\|^{-1}\,\nabla\ \|\Psi\|$$

and

$$\|\nabla\,\|\Psi\|\,\| = \nabla_{e_1}\|\Psi\| \leq \left\|\nabla_{e_1}\Psi\right\|.$$

Note that the Cauchy–Schwarz inequality implies that for it follows from every positive number ϵ, the flowing inequality

$$2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$$

holds for all real numbers a, b. Hence, for any $\epsilon \geq 1/m > 0$, at the considered point, applying this Cauchy–Schwarz inequality and the well-known triangle inequality, one gets that

$$\begin{aligned}
\|\nabla\Psi\|^2 &= \frac{1}{m}\|\not{D}\Psi\|^2 + \sum_j\left\|\nabla_{e_j}\Psi + \frac{1}{m}e_j\cdot\not{D}\Psi\right\|^2\\
&= \frac{1}{m}\|\not{D}\Psi\|^2 + \left\|e_1\cdot\nabla_{e_1}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2 + \sum_{j>1}\left\|e_j\cdot\nabla_{e_j}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2\\
&\geq \frac{1}{m}\|\not{D}\Psi\|^2 + \left\|e_1\cdot\nabla_{e_1}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2 + \frac{1}{m-1}\left|\sum_{j>1}\left\|e_j\cdot\nabla_{e_j}\Psi - \frac{1}{m}\not{D}\Psi\right\|\right|^2\\
&\geq \frac{1}{m}\|\not{D}\Psi\|^2 + \left\|e_1\cdot\nabla_{e_1}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2 + \frac{1}{m-1}\left\|\sum_{j>1}\left(e_j\cdot\nabla_{e_j}\Psi - \frac{1}{m}\not{D}\Psi\right)\right\|^2\\
&= \frac{1}{m}\|\not{D}\Psi\|^2 + \left\|e_1\cdot\nabla_{e_1}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2 + \frac{1}{m-1}\left\|\frac{1}{m}\not{D}\Psi - e_1\cdot\nabla_{e_1}\Psi\right\|^2\\
&= \frac{1}{m}\|\not{D}\Psi\|^2 + \frac{m}{m-1}\left\|e_1\cdot\nabla_{e_1}\Psi - \frac{1}{m}\not{D}\Psi\right\|^2\\
&= \frac{1}{m-1}\|\not{D}\Psi\|^2 + \frac{m}{m-1}\left\|\nabla_{e_1}\Psi\right\|^2 - \frac{2}{m-1}\Re\langle\not{D}\Psi, e_1\cdot\nabla_{e_1}\Psi\rangle\\
&\geq \frac{1-\epsilon}{m-1}\|\not{D}\Psi\|^2 + \left(1+\frac{1-1/\epsilon}{m-1}\right)\left\|\nabla_{e_1}\Psi\right\|^2\\
&\geq \frac{1-\epsilon}{m-1}\|\not{D}\Psi\|^2 + \left(1+\frac{1-1/\epsilon}{m-1}\right)\|\nabla\,\|\Psi\|\,\|^2\,,
\end{aligned}$$

where the first inequality follows by the mean value inequality. Choose $\epsilon > 1 \geq 1/m$ such that

$$\delta = \frac{m-1}{\epsilon - 1},$$

then

$$\|\nabla\Psi\|^2 \geq \left(1 + \frac{1}{m-1+\delta}\right)\|\nabla \|\Psi\|\|^2 - \frac{1}{\delta}\|\not{D}\Psi\|^2.$$

Now we consider $\mathrm{d}\Phi$ as a cross-section of the Dirac bundle $\bigwedge^* T^*M \otimes \Phi^{-1}TN$. The Clifford multiplication is defined by

$$X\cdot(\eta^\alpha \otimes E_\alpha) = X^\sharp \wedge \eta^\alpha \otimes E_\alpha - \iota_X \eta^\alpha \otimes E_\alpha,$$

and the associated Dirac operator $\not{D}$ is defined by $\not{D} = D + D^*$, where $D = \eta^i \wedge \nabla_{e_i}$ and D^* is the dual of D. In particular, $\not{D}\mathrm{d}\Phi = D^*\mathrm{d}\Phi = -\tau(\Phi)$. Thus, when (Φ, Ψ) is a Dirac-harmonic map, it follows from (2.4) and (2.7) that

$$\begin{aligned}\|\nabla \mathrm{d}\Phi\|^2 &\geq \left(1 + \frac{1}{m-1+\delta}\right)\|\nabla\|\mathrm{d}\Phi\|\|^2 - \frac{1}{\delta}\|\tau(\Phi)\|^2 \\ &\geq \left(1 + \frac{1}{m-1+\delta}\right)\|\nabla\|\mathrm{d}\Phi\|\|^2 - \frac{1}{4\delta}(n-1)^2\min\{m,n\}\left\|R^N\right\|^2\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2 \end{aligned} \tag{2.11}$$

provided $\mathrm{d}\Phi \neq 0$ and $\Psi \neq 0$. (2.10) follows from (2.7) by using the Dirac-harmonicity and letting $\delta \to 0$. □

Remark 3. One can prove (2.9) directly. Indeed, let $\mathrm{d}\Phi = \phi_i^\alpha \eta^i \otimes \theta_\alpha$ where $\{\eta^i\}$ is the dual of $\{e_i\}$, and choose e_1 as in the proof of Proposition 2, then for any $\epsilon > 0$,

$$\begin{aligned}\|\nabla \mathrm{d}\Phi\|^2 &= \sum_{\alpha,i,j}(\phi_{ij}^\alpha)^2 \geq \sum_\alpha(\phi_{11}^\alpha)^2 + 2\sum_{\alpha,j>1}(\phi_{1j}^\alpha)^2 + \sum_{\alpha,j>1}(\phi_{jj}^\alpha)^2 \\ &\geq \sum_\alpha(\phi_{11}^\alpha)^2 + 2\sum_{\alpha,j>1}(\phi_{1j}^\alpha)^2 + \frac{1}{m-1}\sum_\alpha\left(\sum_{j>1}\phi_{jj}^\alpha\right)^2 \\ &= \sum_\alpha(\phi_{11}^\alpha)^2 + 2\sum_{\alpha,j>1}(\phi_{1j}^\alpha)^2 + \frac{1}{m-1}\sum_\alpha\left(\tau(\Phi)^\alpha - \phi_{11}^\alpha\right)^2 \\ &\geq \frac{m}{m-1}\sum_{\alpha,j}(\phi_{1j}^\alpha)^2 + \frac{1}{m-1}\|\tau(\Phi)\|^2 - \frac{2}{m-1}\phi_{11}^\alpha\tau(\Phi)^\alpha \\ &\geq \frac{1-\epsilon}{m-1}\|\tau(\Phi)\|^2 + \left(1 + \frac{1-1/\epsilon}{m-1}\right)\sum_{\alpha,j}(\phi_{1j}^\alpha)^2. \end{aligned}$$

Choosing $\epsilon > 1$ such that $\delta = (m-1)/(\epsilon-1) > 0$ and noting that $\sum_{\alpha,j}(\phi_{1j}^\alpha)^2 \geq \|\nabla\|\mathrm{d}\Phi\|\|^2$, we deduce that

$$\|\nabla \mathrm{d}\Phi\|^2 \geq \left(1 + \frac{1}{m-1+\delta}\right)\|\nabla\|\mathrm{d}\Phi\|\|^2 - \frac{1}{\delta}\|\tau(\Phi)\|^2;$$

consequently, (2.9) follows.

2.2. Gradient estimates

Now we consider the gradient estimates for Dirac-harmonic maps. By using (2.2) and (2.5), one gets that

$$\begin{aligned}\frac{1}{2}\Delta\|\mathrm{d}\Phi\|^2 \geq\ &\|\nabla \mathrm{d}\Phi\|^2 - b_3C(m,n)\|\Psi\|^2\ \|\mathrm{d}\Phi\|^3 - b_2C(m,n)\|\Psi\|^2\ \|\mathrm{d}\Phi\|\ \|\nabla \mathrm{d}\Phi\| \\ &- b_2C(m,n)\|\Psi\|\ \|\nabla\Psi\|\ \|\mathrm{d}\Phi\|^2 - \kappa\|\mathrm{d}\Phi\|^2 - b_1\left(1 - \frac{1}{p}\right)\|\mathrm{d}\Phi\|^4\end{aligned}$$

for some constant $C(m,n) > 0$ depending only on m and n. Applying the Cauchy–Schwarz inequality, one gets that, for any $\delta_1 > 0$,

$$\begin{aligned}\frac{1}{2}\Delta\|\mathrm{d}\Phi\|^2 \geq\ &(1-\delta_1)\|\nabla \mathrm{d}\Phi\|^2 - \frac{b_2^2}{\delta_1}C(m,n)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2 \\ &- \frac{b_3^2}{b_1}C(m,n)2p(p+1)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2 - \frac{b_1}{2p(p+1)}\|\mathrm{d}\Phi\|^4\end{aligned}$$

$$
\begin{aligned}
&-\frac{b_2^2}{b_1}C(m,n)2p(p+1)\|\Psi\|^2\ \|\nabla\Psi\|^2-\frac{b_1}{2p(p+1)}\|\mathrm{d}\Phi\|^4-\kappa\|\mathrm{d}\Phi\|^2-b_1\left(1-\frac{1}{p}\right)\|\mathrm{d}\Phi\|^4\\
\geq\ &(1-\delta_1)\|\nabla\mathrm{d}\Phi\|^2-\left(\frac{b_2^2}{\delta_1}+\frac{b_3^2}{b_1}\right)C(m,n)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\\
&-\frac{b_2^2}{b_1}C(m,n)\|\Psi\|^2\ \|\nabla\Psi\|^2-\kappa\|\mathrm{d}\Phi\|^2-\left(1-\frac{1}{p+1}\right)b_1\|\mathrm{d}\Phi\|^4,
\end{aligned}
$$

for some $C(m,n)>0$. Let $\delta=1$ and choose δ_1 such that

$$
(1-\delta_1)\left(1+\frac{1}{m-1+\delta}\right)=1+\frac{1}{m-1+2\delta},
$$

i.e.,

$$
\delta_1=\frac{1}{(m+1)^2},
$$

then according to the Kato–Yau inequality for the map (2.9) when $\delta=1$, one gets that

$$
\begin{aligned}
\frac{1}{2}\Delta\|\mathrm{d}\Phi\|^2\geq\ &\left(1+\frac{1}{m+1}\right)\|\nabla\|\mathrm{d}\Phi\|\|^2-\frac{m^2+2m}{4(m+1)^2}(n-1)^2\min\{m,n\}\,b_2^2\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\\
&-\left(b_2^2+\frac{b_3^2}{b_1}\right)C(m,n)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\\
&-\frac{b_2^2}{b_1}C(m,n)\|\Psi\|^2\ \|\nabla\Psi\|^2-\kappa\|\mathrm{d}\Phi\|^2-\left(1-\frac{1}{p+1}\right)b_1\|\mathrm{d}\Phi\|^4\\
\geq\ &\left(1+\frac{1}{m+1}\right)\|\nabla\|\mathrm{d}\Phi\|\|^2-\left(b_2^2+\frac{b_3^2}{b_1}\right)C(m,n)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\\
&-\frac{b_2^2}{b_1}C(m,n)\|\Psi\|^2\ \|\nabla\Psi\|^2-\kappa\|\mathrm{d}\Phi\|^2-\left(1-\frac{1}{p+1}\right)b_1\|\mathrm{d}\Phi\|^4.
\end{aligned}
\tag{2.12}
$$

Since (Φ,Ψ) is Dirac-harmonic, by the Weitzenböck formula (3.10) in [2] and the Kato–Yau inequality (2.10), we have

$$
\begin{aligned}
\frac{1}{2}\Delta\|\Psi\|^4=\ &\|\Psi\|^2\Delta\|\Psi\|^2+\left\|\nabla\ \|\Psi\|^2\right\|^2\\
\geq\ &2\|\Psi\|^2\ \|\nabla\Psi\|^2+\left\|\nabla\ \|\Psi\|^2\right\|^2+\frac{S_M}{2}\|\Psi\|^4-(p-1)(n-1)b_2^2\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2\\
\geq\ &\|\Psi\|^2\ \|\nabla\Psi\|^2+\left(1+\frac{1}{m+1}\right)\left\|\nabla\ \|\Psi\|^2\right\|^2-\frac{m}{2}\kappa\|\Psi\|^4-b_2^2C(m,n)\|\Psi\|^4\ \|\mathrm{d}\Phi\|^2.
\end{aligned}
\tag{2.13}
$$

Now we fix the constant

$$
C_0:=C(m,n)
$$

in the above two inequalities (2.12) and (2.13), and set

$$
\tilde{e}:=\tfrac{\sqrt{C_0}b_2}{\sqrt{b_1}}\|\Psi\|^2,\qquad C_1:=\tfrac{\sqrt{C_0}b_2}{\sqrt{b_1}},\qquad e:=\sqrt{\|\mathrm{d}\Phi\|^2+\tilde{e}}=\sqrt{\|\mathrm{d}\Phi\|^2+C_1^2\|\Psi\|^4}.
$$

We have the following.

Lemma 2. *Suppose* $\mathrm{Ric}_M\geq-\kappa$, $-b_2\leq\sec_N\leq b_1$ *and* $\left\|\nabla R^N\right\|\leq b_3$, *where* $b_2\geq b_1>0$. *Denote*

$$
b_0:=\frac{b_2^3+b_2^4+b_3^2}{b_1}\quad\textit{and}
$$

$p=\min\{m,n\}\geq\max\operatorname{rank}\mathrm{d}\Phi$. *Then we have the following inequality*

$$
\Delta e\geq\frac{1}{m+1}\frac{\|\nabla e\|^2}{e}-\frac{m}{2}\kappa e-C(m,n)b_0\|\Psi\|^4e-\left(1-\frac{1}{p+1}\right)b_1\|\mathrm{d}\Phi\|^2e,
\tag{2.14}
$$

where $C(m,n)>0$ *is a constant depending only on* m *and* n.

Proof. Denote $\mu := \frac{1}{m+1}$, then by (2.12) and (2.13), we have

$$\frac{1}{2}\Delta e^2 \ge (1+\mu)\left(\|\nabla \|d\Phi\|\|^2 + \left\|\nabla\tilde{e}\right\|^2\right) - \frac{m}{2}\kappa e^2 - \left(b_2^2 + \frac{b_2^4+b_3^2}{b_1}\right) C(m,n)\|\Psi\|^4 \|d\Phi\|^2$$
$$- \left(1 - \frac{1}{p+1}\right) b_1 \|d\Phi\|^4 .$$

Independently,

$$\begin{aligned}\|\nabla e\|^2 &= \frac{\left\|\nabla e^2\right\|^2}{4e^2} = e^{-2}\left(\|d\Phi\|\nabla\|d\Phi\| + \tilde{e}\nabla\tilde{e}\right)^2\\ &\le e^{-2}\left(\|d\Phi\|^2\|\nabla\|d\Phi\|\|^2 + \tilde{e}^2\left\|\nabla\tilde{e}\right\|^2 + 2\|d\Phi\|\,\|\nabla\|d\Phi\|\|\,\tilde{e}\left\|\nabla\tilde{e}\right\|\right)\\ &\le e^{-2}\left(\|d\Phi\|^2\|\nabla\|d\Phi\|\|^2 + \tilde{e}^2\left\|\nabla\tilde{e}\right\|^2 + \|d\Phi\|^2\left\|\nabla\tilde{e}\right\|^2 + \|\nabla\|d\Phi\|\|^2\tilde{e}^2\right)\\ &= \|\nabla\|d\Phi\|\|^2 + \left\|\nabla\tilde{e}\right\|^2 .\end{aligned}$$

Therefore,

$$\Delta e \ge \mu\frac{\|\nabla e\|^2}{e} - \frac{m}{2}\kappa e - C(m,n)b_0\|\Psi\|^4 e - \left(1-\frac{1}{p+1}\right) b_1\|d\Phi\|^2 e. \quad \square$$

Denote by $B : N \longrightarrow \mathbb{R}^+$ a function which will be defined later, and denote $f = e/(B\circ\Phi)$. For any point $x_0 \in M$, we define a function on $B_a(x_0)$ by

$$F = \left(a^2 - r^2\right) f = \left(a^2 - r^2\right)\frac{e}{B\circ\Phi}, \tag{2.15}$$

where $r(x) = \text{dist}(x_0, x)$. It is easy to see that if $e \ne 0$, then F must achieve its maximum at some interior point x^*. We may assume that r is twice differentiable near x^* (cf. [17]). By the maximum principle, we have

$$\nabla F(x^*) = 0, \tag{2.16}$$
$$\Delta F(x^*) \le 0. \tag{2.17}$$

We recall the Laplace Comparison Theorem [18], for some constant $C(m) > 0$ depending only on m,

$$\Delta r^2 \le C(m)(1+\sqrt{\kappa}r), \tag{2.18}$$

where the constant $C(m)$ can be chosen as $2m$.

Lemma 3. *Set*

$$A = \frac{m}{2}\kappa + \frac{C(m)(1+\sqrt{\kappa}r)}{a^2-r^2} + \frac{8r^2}{\left(a^2-r^2\right)^2},$$

then at the point x^*, we have the following estimate

$$\frac{1}{m+1}\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi} - 4r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)B\circ\Phi}$$
$$- \left(1-\frac{1}{p+1}\right) b_1\|d\Phi\|^2 - A - C(m,n)b_0\|\Psi\|^4 \le 0. \tag{2.19}$$

Proof. We first have

$$\nabla f = \frac{\nabla e}{B\circ\Phi} - \frac{e\nabla(B\circ\Phi)}{(B\circ\Phi)^2},$$

and

$$\Delta f = \frac{\Delta e}{B\circ\Phi} - \frac{f\Delta(B\circ\Phi)}{B\circ\Phi} - \frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}.$$

By using (2.14) and setting $\mu = 1/(m+1)$, we get

$$\Delta f \ge \mu\frac{\|\nabla e\|^2}{eB\circ\Phi} - \frac{m}{2}\kappa f - C(m,n)b\|\Psi\|^4 f - \left(1-\frac{1}{p+1}\right) b_1\|d\Phi\|^2 f - \frac{f\Delta(B\circ\Phi)}{B\circ\Phi} - \frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}. \tag{2.20}$$

We also have that

$$
\begin{aligned}
-\frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi} &= -(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi} - 2\mu\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}\\
&= -(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi} - 2\mu\frac{\langle\nabla(B\circ\Phi),\nabla e\rangle}{(B\circ\Phi)^2} + 2\mu\frac{f\,\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}\\
&\geq -(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi} - \mu\frac{\|\nabla e\|^2}{eB\circ\Phi} + \mu\frac{f\,\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\Delta f}{f} \geq & -(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{fB\circ\Phi} + \mu\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi}\\
& -\frac{m}{2}\kappa - C(m,n)b_0\,\|\Psi\|^4 - \left(1-\frac{1}{p+1}\right)b_1\,\|d\Phi\|^2.
\end{aligned}
$$

At the point x^*, since F achieves its maximum, as a consequence $\nabla F(x^*) = 0$ and $\Delta F(x^*) \leq 0$. By the definition of F, one has that

$$
\frac{\nabla r^2}{a^2-r^2} = \frac{\nabla f}{f},
$$

and

$$
-\frac{\Delta r^2}{a^2-r^2} + \frac{\Delta f}{f} - \frac{2\langle\nabla r^2,\nabla f\rangle}{f(a^2-r^2)} \leq 0.
$$

It follows that

$$
\frac{\Delta f}{f} - \frac{\Delta r^2}{a^2-r^2} - \frac{2\left\|\nabla r^2\right\|^2}{\left(a^2-r^2\right)^2} \leq 0.
$$

Using $\left\|\nabla r^2\right\| = 2r$ and (2.18), we then have

$$
\begin{aligned}
0 &\geq \frac{\Delta f}{f} - \frac{C(m)(1+\sqrt{\kappa}r)}{a^2-r^2} - \frac{8r^2}{\left(a^2-r^2\right)^2}\\
&\geq -(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{fB\circ\Phi} + \mu\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi}\\
&\quad -\frac{m}{2}\kappa - C(m,n)b_0\,\|\Psi\|^4 - \left(1-\frac{1}{p+1}\right)b_1\,\|d\Phi\|^2 - \frac{C(m)(1+\sqrt{\kappa}r)}{a^2-r^2} - \frac{8r^2}{\left(a^2-r^2\right)^2}.
\end{aligned}
$$

However,

$$
\begin{aligned}
-(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{fB\circ\Phi} &= -(2-2\mu)2r\frac{\langle\nabla(B\circ\Phi),\nabla r\rangle}{(a^2-r^2)B\circ\Phi}\\
&\geq -(2-2\mu)2r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)B\circ\Phi},
\end{aligned}
$$

and we conclude that

$$
\mu\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi} - 4r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)B\circ\Phi} - \left(1-\frac{1}{p+1}\right)b_1\,\|d\Phi\|^2 - A - C(m,n)b_0\,\|\Psi\|^4 \leq 0. \quad \square
$$

Now we are in the position to give the following.

Proof of Theorem 1. We use the key Lemma 3 to prove this theorem. Choose

$$
B(y) = \sqrt{b_1}\cos(\sqrt{b_1}\rho(y)),
$$

where ρ is the distance function from the fixed point y_0 on N. Since $\Phi(M) \subset B_{y_0}(R)$, one gets that $B\circ\Phi > 0$. From the Hessian Comparison Theorem [18] and $\|\nabla\rho\| = 1$ we have

$$
\mathrm{Hess}B \leq -b_1^{3/2}\cos(\sqrt{b_1}\rho), \tag{2.21}
$$

$$
\|\nabla B\| = b_1\sin(\sqrt{b_1}\rho). \tag{2.22}
$$

It then follows that

$$\|\nabla(B\circ\Phi)\| \le \|(\nabla B)\circ\Phi\| \ \|\mathrm{d}\Phi\| = b_1 \sin(\sqrt{b_1}\rho\circ\Phi)\,\|\mathrm{d}\Phi\|,$$

and

$$\begin{aligned}\Delta(B\circ\Phi) &\le (\mathrm{Hess}B)\circ\Phi\,\|\mathrm{d}\Phi\|^2 + \|(\nabla B)\circ\Phi\|\ \|\tau(\Phi)\|\\ &\le -b_1^{3/2}\cos(\sqrt{b_1}\rho\circ\Phi)\,\|\mathrm{d}\Phi\|^2 + b_1 b_2 C(m,n)\,\|\Psi\|^2\ \|\mathrm{d}\Phi\|\\ &\le -b_1^{3/2}\cos(\sqrt{b_1}\rho\circ\Phi)\,\|\mathrm{d}\Phi\|^2 + \frac{1}{(p+1)(p+2)} b_1 B\circ\Phi\,\|\mathrm{d}\Phi\|^2 + \frac{1}{B\circ\Phi}(p+1)^2 C(m,n) b_1 b_2^2\,\|\Psi\|^4.\end{aligned}$$

Inserting this into (2.19), we have at x^*,

$$\frac{b_1}{p+2}\|\mathrm{d}\Phi\|^2 - \frac{4rb_1}{(a^2-r^2)B\circ\Phi}\ \|\mathrm{d}\Phi\| \le A + \frac{1}{(B\circ\Phi)^2}C(m,n)b\,\|\Psi\|^4,$$

where $b := b_1 b_0 = b_2^3 + b_2^4 + b_3^2$. In other words,

$$\frac{b_1}{p+2}e^2 - \frac{4rb_1}{(a^2-r^2)B\circ\Phi}e \le A + \frac{1}{(B\circ\Phi)^2}C(m,n)b\,\|\Psi\|^4.$$

Therefore,

$$\frac{b_1(B\circ\Phi)^2}{p+2}F^2(x^*) - 4rb_1F(x^*) \le \left(A + \frac{1}{(B\circ\Phi)^2}C(m,n)b\,\|\Psi\|^4\right)\left(a^2-r^2\right)^2. \tag{2.23}$$

For the RHS of the above inequality, we have the following estimate:

$$\begin{aligned}\left(a^2-r^2\right)^2\left(A + \frac{1}{(B\circ\Phi)^2}C(m,n)b\,\|\Psi\|^4\right) &\le a^2\left(\left(\frac{m}{2}\kappa + \frac{1}{(B\circ\Phi)^2}C(m,n)b\,\|\Psi\|^4\right)a^2 + C(m)(1+\sqrt{\kappa}a) + 8\right)\\ &\le \frac{C(m,n)b_1}{(B\circ\Phi)^2}a^2\left(1+\sqrt{\kappa}a + \sqrt{\frac{b}{b_1}}\,\|\Psi\|^2 a\right)^2.\end{aligned}$$

It is elementary that if $Ax^2 - Bx - C \le 0$ with A, B, C all positive, then

$$x \le \frac{B}{A} + \sqrt{\frac{C}{A}}.$$

From this and (2.23) we conclude that

$$F(x^*) \le \frac{C(m,n)a}{b_1\cos^2(\sqrt{b_1}R)}\left(1+\sqrt{\kappa}a + \sqrt{\frac{b}{b_1}}\sup_{B_{x_0}(a)}\|\Psi\|^2 a\right), \tag{2.24}$$

from which (1.2) follows. This proves Theorem 1. □

3. Liouville theorems for Dirac-harmonic maps

Using the gradient estimate for Dirac-harmonic maps, we can prove the Liouville property for Dirac-harmonic maps, Theorem 2.

Proof of Theorem 2. Since (Φ, Ψ) is Dirac harmonic, we can get from the Weitzenböck formula that

$$\Delta\,\|\Psi\| \ge \left(\frac{\epsilon}{4} - \frac{1}{2}\left(\min\{m,n\}-1\right)(n-1)b_2^2\,\|\mathrm{d}\Phi\|^2\right)\|\Psi\|.$$

It is obvious that this theorem is valid if M is compact. Now, we suppose M is noncompact. Suppose $\|\Psi\| \le \delta$, then according to Theorem 1,

$$\|\mathrm{d}\Phi\| \le \frac{C(m,n)\sqrt{b}\delta^2}{b_1\cos^2(\sqrt{b_1}R)}.$$

Thus,

$$\Delta\,\|\Psi\| \ge \left(\frac{\epsilon}{4} - \frac{C(m,n)b^2}{b_1^3\cos^4(\sqrt{b_1}R)}\delta^4\right)\|\Psi\|$$

for some constant $C(m,n) > 0$ depending only on m, n.

If one chooses δ such that

$$\frac{\epsilon}{4}-\frac{C(m,n)b^2}{b_1^3\cos^4(\sqrt{b_1}R)}\delta^4=0,$$

then $\|\Psi\|$ is a bounded subharmonic function on M.

Now we prove that $\Psi\equiv 0$. For every positive number c, let $u=(\|\Psi\|+c)^{-\frac{1}{2}}$, then

$$\begin{aligned}\Delta u &= -\frac{1}{2}(\|\Psi\|+c)^{-\frac{3}{2}}\Delta\|\Psi\|+3(\|\Psi\|+c)^{\frac{1}{2}}\|\nabla u\|^2\\ &\le -\frac{C_0}{2}(\|\Psi\|+c)^{-\frac{3}{2}}\|\Psi\|+3(\|\Psi\|+c)^{\frac{1}{2}}\|\nabla u\|^2.\end{aligned}\tag{3.3}$$

Since the Ricci curvature of M is nonnegative, the Omori–Yau maximum principle holds [17,7], that is, for every $\eta>0$, there exists a point $p\in M$ such that at p,

$$u<\inf u+\eta,\qquad \|\nabla u\|<\eta,\qquad \Delta u>-\eta.$$

It follows from (3.3) and $\|\Psi\|\le\delta$ that

$$\frac{C_0}{2}\|\Psi\|<\eta\left(\inf(\|\Psi\|+c)^{-\frac{1}{2}}+4\eta\right)(\|\Psi\|+c)^2<\eta\left(\delta^{-\frac{1}{2}}+4\eta\right)(\delta+c)^2.$$

Let $\eta\to 0$, we obtain

$$\sup\|\Psi\|=0.$$

Hence, $\mathrm{d}\Phi=0$ since (1.2), and Φ must be constant. □

Proof of Theorem 3. Choose a local orthonormal frame field $\{e_i\}$ such that Φ^*g^N is a diagonal matrix at the considered point, i.e. $\langle\Phi_*(e_i),\Phi_*(e_j)\rangle=\lambda_i\delta_{ij}$. Let q be the rank of Φ at the point, we may suppose that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_{q'}>0$ and $q'\le q$. By using Newton's inequality, we have

$$\begin{aligned}R^N(\Phi_*(e_i),\Phi_*(e_j),\Phi_*(e_i),\Phi_*(e_j)) &\le 2b_1\sum_{1\le i<j\le q'}\lambda_i^2\lambda_j^2\\ &\le 2b_1\binom{q'}{2}\binom{q'}{1}^{-2}\left(\sum_{i=1}^{q'}\lambda_i^2\right)^2=\frac{q'-1}{q'}b_1\|\mathrm{d}\Phi\|^4\le\frac{q-1}{q}b_1\|\mathrm{d}\Phi\|^4.\end{aligned}$$

Applying the Kato–Yau inequality (2.8), i.e., choose $\epsilon=(\delta+1)/(m+\delta)\ge 1/m$, we have

$$\|\nabla\mathrm{d}\Phi\|^2\ge\frac{\delta}{1+\delta}\|\nabla\|\mathrm{d}\Phi\|\|^2+\frac{1}{m+\delta}\|\tau(\Phi)\|^2.$$

Then according to (2.1), we obtain

$$\frac{1}{2}\Delta\|\mathrm{d}\Phi\|^2-\mathrm{div}(\langle\tau(\Phi),\Phi_*(e_i)\rangle e_i)\ge\frac{\delta}{1+\delta}\|\nabla\|\mathrm{d}\Phi\|\|^2-\frac{m-1+\delta}{m+\delta}\|\tau(\Phi)\|^2+a\|\mathrm{d}\Phi\|^2-\frac{q-1}{q}b_1\|\mathrm{d}\Phi\|^4.$$

On the other hand, we have the following estimate

$$\begin{aligned}\|\tau(\Phi)\| &= \frac{1}{2}\left\|\sum_{\alpha\ne\beta,i}\langle\psi^\alpha,e_i\cdot\psi^\beta\rangle R^N(\theta_\alpha,\theta_\beta)\Phi_*(e_i)\right\|\\ &\le\frac{1}{2}b_2\sum_{\alpha\ne\beta}|\psi^\alpha|\left|\psi^\beta\right||\theta_\alpha|\left|\theta_\beta\right|\sum_{i=1}^{q}\lambda_i\\ &\le\frac{1}{2}b_2(n-1)\|\Psi\|^2\sqrt{q}\|\mathrm{d}\Phi\|.\end{aligned}\tag{3.4}$$

Hence, if for some $\delta>0$ such that (1.3) holds, then

$$\begin{aligned}\frac{1}{2}\Delta\|\mathrm{d}\Phi\|^2-\mathrm{div}(\langle\tau(\Phi),\Phi_*(e_i)\rangle e_i) &\ge\frac{\delta}{1+\delta}\|\nabla\|\mathrm{d}\Phi\|\|^2+\|\mathrm{d}\Phi\|^2\\ &\quad\times\left(a-\frac{q-1}{q}b_1\|\mathrm{d}\Phi\|^2-\frac{m-1+\delta}{4(m+\delta)}(n-1)^2qb_2^2\|\Psi\|^4\right)\\ &\ge\frac{\delta}{1+\delta}\|\nabla\|\mathrm{d}\Phi\|\|^2.\end{aligned}$$

The above inequality implies $\|d\Phi\|$ is constant, and consequently $d\Phi \equiv 0$. So, Φ is constant and Ψ is a harmonic spinor. It is obvious that the scalar curvature of M $S_M \geq ma$. Thus,

$$\Delta \|\Psi\|^2 \geq \frac{1}{2} S_M \|\Psi\|^2 \geq \frac{ma}{2} \|\Psi\|^2,$$

which implies $\Psi \equiv 0$ since M is compact. □

Now we give the following.

Proof of Theorem 4. We first show that Φ must be constant. Firstly, we have the following Kato inequality:

$$\|\nabla d\Phi\|^2 \geq \|\nabla \|d\Phi\|\|^2. \tag{3.5}$$

For any $\epsilon > 0$, we let $u = \sqrt{\|d\Phi\|^2 + \epsilon}$, then (2.1) and (3.5) imply that

$$\begin{aligned} u\Delta u - \operatorname{div}(\langle \tau(\Phi), \Phi_*(e_i)\rangle e_i) &= \frac{1}{2}\Delta \|d\Phi\|^2 - \left(\|d\Phi\|^2 + \epsilon\right)^{-1} \|d\Phi\|^2 \|\nabla \|d\Phi\|\|^2 - \operatorname{div}(\langle \tau(\Phi), \Phi_*(e_i)\rangle e_i) \\ &\geq \epsilon \left(\|d\Phi\|^2 + \epsilon\right)^{-1} \|\nabla \|d\Phi\|\|^2 - \|\tau(\Phi)\|^2 + \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle \\ &\quad - R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)) \\ &\geq -\|\tau(\Phi)\|^2 + \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle - R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)). \end{aligned} \tag{3.6}$$

Secondly, for any smooth function with compact support η, we have

$$\operatorname{div}(\eta^2 u \nabla u) = \eta^2 u\Delta u + 2\eta u \langle \nabla\eta, \nabla u\rangle + \eta^2 \|\nabla u\|^2.$$

By integrating on M, we have

$$\int_M \eta^2 u\Delta u - \eta^2 \operatorname{div}(\langle \tau(\Phi), \Phi_*(e_i)\rangle e_i) = -2\int_M \eta u\langle \nabla\eta, \nabla u\rangle - \int_M \eta^2 \|\nabla u\|^2 + 2\int_M \eta \langle \tau(\Phi), \Phi_*(\nabla\eta)\rangle. \tag{3.7}$$

Fix $x_0 \in M$, and choose η such that

$$\eta(x) = \begin{cases} 1, & x \in B_R(x_0); \\ 0, & x \notin B_{2R}(x_0), \end{cases}$$

and $0 \leq \eta \leq 1$, $\|\nabla\eta\| \leq C/R$, where C is a positive constant. We have the following estimate

$$\begin{aligned} -2\int_M \eta u\langle \nabla\eta, \nabla u\rangle - \int_M \eta^2 \|\nabla u\|^2 &\leq 2\left(\int_{B_{2R}(x_0)\setminus B_R(x_0)} \eta^2 \|\nabla u\|^2\right)^{\frac{1}{2}} \left(\int_{B_{2R}(x_0)\setminus B_R(x_0)} u^2 \|\nabla\eta\|^2\right)^{\frac{1}{2}} \\ &\quad - \int_{B_{2R}(x_0)\setminus B_R(x_0)} \eta^2 \|\nabla u\|^2 - \int_{B_R(x_0)} \eta^2 \|\nabla u\|^2, \end{aligned} \tag{3.8}$$

and for any $\delta \in (0, 1)$,

$$2\int_M \eta\langle \tau(\Phi), \Phi_*(\nabla\eta)\rangle \leq \frac{1}{\delta}\int_{B_{2R}(x_0)} \eta^2 \|\tau(\Phi)\|^2 + \delta \int_{B_{2R}(x_0)\setminus B_R(x_0)} u^2 \|\nabla\eta\|^2. \tag{3.9}$$

From (3.6)–(3.9), we have

$$\begin{aligned} &2\left(\int_{B_{2R}(x_0)\setminus B_R(x_0)} \eta^2 \|\nabla u\|^2\right)^{\frac{1}{2}} \left(\int_{B_{2R}(x_0)\setminus B_R(x_0)} u^2 \|\nabla\eta\|^2\right)^{\frac{1}{2}} - (1-\delta)\int_{B_{2R}(x_0)\setminus B_R(x_0)} \eta^2 \|\nabla u\|^2 - \int_{B_R(x_0)} \eta^2 \|\nabla u\|^2 \\ &\geq -\left(1 + \frac{1}{\delta}\right)\int_{B_{2R}(x_0)} \eta^2 \|\tau(\Phi)\|^2 + \int_{B_{2R}(x_0)} \eta^2 \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle \\ &\quad - \int_{B_{2R}(x_0)} \eta^2 R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)). \end{aligned} \tag{3.10}$$

Independently, for $\delta > 0$ in (1.5), we have

$$\begin{aligned} &-\left(1 + \frac{1}{\delta}\right)\|\tau(\Phi)\|^2 + \langle \Phi_*(\mathrm{Ric}^M(e_i)), \Phi_*(e_i)\rangle - R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)) \\ &\geq \|d\Phi\|^2 \left(a - \frac{q-1}{q} b_1 \|d\Phi\|^2 - \frac{1+\delta}{4\delta}(n-1)^2 q b_2^2 \|\Psi\|^4\right) \geq 0. \end{aligned}$$

Thus,

$$2\left(\int_{B_{2R}(x_0)\setminus B_R(x_0)}\eta^2\,\|\nabla u\|^2\right)^{\frac{1}{2}}\left(\int_{B_{2R}(x_0)\setminus B_R(x_0)}u^2\,\|\nabla\eta\|^2\right)^{\frac{1}{2}}-(1-\delta)\int_{B_{2R}(x_0)\setminus B_R(x_0)}\eta^2\,\|\nabla u\|^2-\int_{B_R(x_0)}\eta^2\,\|\nabla u\|^2\geq 0.$$

This inequality implies

$$(1-\delta)\int_{B_R(x_0)}\|\nabla u\|^2\leq\int_{B_{2R}(x_0)\setminus B_R(x_0)}u^2\,\|\nabla\eta\|^2\leq\frac{C^2}{R^2}\int_{B_R(x_0)}u^2. \tag{3.11}$$

Let ϵ go to 0,

$$(1-\delta)\int_{B_R(x_0)}\|\nabla\,\|d\Phi\|\,\|\leq\frac{C^2}{R^2}\int_{B_R(x_0)}\|d\Phi\|^2\,.$$

Since $\int_M\|d\Phi\|^2<\infty$, consequently, letting R go to infinity, we get that

$$\int_M\|\nabla\,\|d\Phi\|\,\|\leq 0.$$

Thus, $\|d\Phi\|$ must be constant. Since $\int_M\|d\Phi\|^2<\infty$, and any complete noncompact manifold with nonnegative Ricci curvature has infinite volume [19], Φ must be constant.

Next, we show that $\Psi\equiv 0$. Since Φ is constant, according to the Weitzenböck formula in [2],

$$\frac{1}{2}\Delta\,\|\Psi\|^2=\|\nabla\Psi\|^2+\frac{1}{4}S_M\,\|\Psi\|^2\geq 0.$$

Then $\|\Psi\|$ must be constant since there is no nonconstant nonnegative L^2 subharmonic function on any complete manifold M [19]. Thus $\Psi\equiv 0$ since $\int_M\|\Psi\|^4<\infty$ and M has infinite volume. □

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